Filomat 35:9 (2021), 2927–2949
https://doi.org/10.2298/FIL2109927C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Positive Solutions for the Nonlinear Fractional Boundary Value Problems with p-Laplacian

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Abstract. The monotone iterative technique, theory of fixed point index in a cone and the Leggett-Williams fixed point theorem are applied to investigate the existence and multiplicity of positive solutions for four boundary value problems of nonlinear fractional differential equations with a p-Laplacian point operator and infinite delay. Moreover, examples are presented to illustrate a vast applicability of our main results.

1. Introduction

Investigating existence of positive solutions is one of the most important features of boundary value problems based on a model. This has been one of the most crucial targets for researchers in recent years. In these studies, fixed point theorems are generally used and in some of the studies the lower and upper solution methods known as monotonous methods are also used.

With the acceleration of the studies in fractional derivative analysis, it has been applied to modelling boundary value problems in [3], [4], for physical phenomena, engineering and economic processes and concepts involving fractional derivatives. Recently, boundary value problems involving fractional order differential equations were considered in physics, chemistry, aerodynamics, polymer rheology, and many other fields [5]-[9], [27], [28]. This shows the extent to which fractional differential equations have attracted the attention of researchers working in various fields. The advantage of having fractional derivatives is that they provide more degrees of freedom in models and is thus more effective for modelling real-life events.

In paper [1], the nonlinear differential equation of fractional order

$$D^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,$$

subject to the boundary conditions

$$u(0) = 0, \quad D^{\beta}_{0^+}u(1) = aD^{\beta}_{0^+}u(\xi),$$

2020 *Mathematics Subject Classification*. Primary 34B10; Secondary 34B15.
Keywords. Riemann-Liouville derivative, p-Laplacian operator, Fixed point theorems, infinite delay, Boundary value problems.
Received: 23 July 2020; Revised: 31 July 2020; Accepted: 06 August 2020
Communicated by Maria Alessandra Ragusa
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was considered. In this problem, $D^{\alpha}_{0^+}$ and D^{β} are the standard Riemann-Liouville fractional order derivative, $1 < \alpha \leq 2,\ 0 \leq \beta \leq 1,\ 0 \leq a \leq 1,\ \xi \in (0,1),\ a\xi^{\alpha-\beta-2} \leq 1-\beta,\ 0 \leq \alpha-\beta-1$ and $f : [0,1]\times[0,\infty) \to [0,\infty)$ satisfies Carathéodory type conditions. The boundary value problem (BVP) was transformed to an equivalent integral equation. Fixed point theorems were applied to show the existence and multiplicity results of positive solutions of the BVP under consideration. Trivially, the Banach contraction mapping principle revealed that the operator considered has a unique fixed point which is a solution of the BVP. Incorporation with the Arzela-Ascoli theorem, the operation is completely continuous. Furthermore, some theorems were considered with some valid assumptions, implying that the BVP has multiple positive solutions.

In [2], the existence and multiplicity of positive solutions to m-point boundary value problem of nonlinear fractional differential equations with p-Laplacian operator

$$\begin{aligned} &D^{\beta}_{0^+}(\varphi_p(D^{\alpha}_{0^+}u(t))) + \varphi_p(\lambda)f(t,u(t)) = 0, \quad 0 < t < 1,\\ &u(0) = 0, \quad D^{\gamma}_{0^+}u(1) = \sum_{i=1}^{m-2}\xi_i D^{\gamma}_{0^+}u(\eta_i), \quad D^{\alpha}_{0^+}u(0) = 0, \end{aligned}$$

was considered, where $D^{\alpha}_{0^+}$, $D^{\beta}_{0^+}$ and $D^{\gamma}_{0^+}$ are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \leq 2,\ 0 < \beta,\gamma \leq 1$ such that $0 \leq \alpha-\gamma-1$ and $0 \leq \alpha-\beta-1,\ \lambda \in (0,+\infty)$, $0 < \xi_i, \eta_i < 1\,(i = 1,2\cdots,m-2)$ such that $\sum_{i=1}^{m-2}\xi_i\eta_i < 1,\ f \in C([0,1]\times[0,+\infty),[0,+\infty))$ and $\varphi_p(s) = |s|^{p-s}s,\ p > 1$ with $\varphi_p^{-1} = \varphi_q$ such that $\frac{1}{p}+\frac{1}{q} = 1$. They applied the monotone iterative technique and theory of fixed point index in a cone. The BVP is changed into an equivalent integral equation and the eigen value interval for existence of multiplicity of positive solution is considered. After showing that the operator is equicontinuous, the Arzela-Ascoli theorem proves complete continuity of the operator. Also, by the Schauder fixed point theorem, the operator is shown to have at least one fixed point and the BVP having a single positive solution. Furthermore, some theorems are proved to show that the BVP has multiple positive solutions. Of which, one of the BVP solutions is a minimal positive solution and the other a maximal positive solution.
As far as we know, there are handfuls of papers probing the existence of positive solutions for fractional differential equations with both a p-Laplacian operator and infinite delay. The applications of these are still at an initial stage.
Recently, a vast amount of studies in nonlinear fractional differential equations were considered (see [21]-[26]).

Motivated by the literature mentioned, in this paper we concentrate on the existence of positive solutions for a four point BVP of fractional differential equations with infinite delay

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}y(t))) = f(t,y_t), \quad \text{a.e} \quad t \in J = [0,1],\\ &y(0) = 0, \quad D^{\alpha}y(1) = aD^{\alpha}y(\xi), \quad D^{\alpha}y(0) = 0, \quad D^{\gamma}y(1) = bD^{\gamma}y(\eta),\\ &y(t) = \phi(t), \quad t \in (-\infty,0], \end{aligned} \tag{1}$$

where D^{α}, D^{β} and D^{γ} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha,\beta \leq 2,\ 0 < \gamma \leq 1$ such that $0 \leq \alpha-\gamma-1,\ 0 \leq a,b \leq 1,\ 0 < \xi,\eta < 1$ and $f : J\times B \to [0,+\infty)$ is a specified function satisfying certain assumptions to be stated in the next sections, $\phi \in B$ and B is a *phasespace*. For a function y and any $t \in [0,1]$, y_t denotes the element of B defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty,0]$, we assume that belonging to B are the histories y_t.
The essential part in the study of qualitative and quantitative theory in functional differential equations is characterised by the notion of phase space B, which entails seminormed space satisfying suitable axioms covered in detail by [10], [12] - [14].

In section 2, we will give some necessary definitions and lemmas which are used in the main results. For the sake of convenience, we also state the fixed point theorems.

In Section 3, we firstly consider the following nonlinear BVP

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}y(t))) = f(t, y(t)), \quad t \in (0,1), \\ y(0) = 0,\ &D^{\alpha}y(1) = aD^{\alpha}y(\xi),\ D^{\alpha}y(0) = 0,\ D^{\gamma}y(1) = bD^{\gamma}y(\eta) \end{aligned} \tag{2}$$

and we will give the existence results of this problem. To simplify BVP (2), we let $w = D^{\alpha}y$, and $v = \varphi_p(w)$, so BVP (2) becomes the following linear BVP

$$\begin{aligned} &D^{\beta}v(t) = g(t) \\ v(0) = 0 \quad &\text{and} \quad v(1) = a^{p-1}v(\xi), \end{aligned} \tag{3}$$

where $g \in L'[0,1]$ and $g \geq 0$.

In Section 4, we will give the multiplicity results for the BVP (1). In the last section, we will give some examples to illustrate our main results.

2. Basic Definitions and Preliminaries

We first introduce some necessary definitions and lemmas in this section.

Definition 2.1. *(see* [10], [15], [16]*) The integral*

$$I_a^{\alpha}g(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds,$$

defines the fractional (arbitrary) order integral of the function $g \in L^1([a,b), R_+)$ of the order $\alpha \in R_+$, where Γ is the gamma function. For $a = 0$, we have $I^{\alpha}g(t) = g(t) * \varphi_{\alpha}(t)$, where $\varphi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, when $t > 0$, $\varphi_{\alpha}(t) = 0$ for $t \leq 0$ and $\varphi_{\alpha}(t) \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2. *(see* [10], [15], [16]*) The αth Riemann-Liouville fractional-order derivative of g, $\alpha \in R_+$, for a function g given on the interval $[a,b]$ is defined as follows:*

$$D_{0^+}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^1 (t-s)^{n-\alpha-1} g(s)ds.$$

The following auxiliary Lemmas are necessary to illustrate the existence of solutions for problem (2).

Lemma 2.3. [11] *Let $\alpha > 0$ then the differential equation*

$$D^{\alpha}y(t) = 0$$

has solutions $y(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_Nt^{\alpha-N}$, for some $c_i \in \mathbb{R}$, where $\mathbb{N}$ is the smallest integer greater than or equal to α.

Lemma 2.4. [11] *Let $y \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$, then*

$$I^{\alpha}D^{\alpha}y(t) = y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_Nt^{\alpha-N},$$

for some $c_i \in \mathbb{R}$, where $\mathbb{N}$ is the smallest integer greater than or equal to α.

Lemma 2.5. *Let y be a continuous function and considering the BVP* (3)*. Problem* (3) *has a unique solution given by*

$$y(t) = \int_0^1 G(t,s)\rho(s)ds, \tag{4}$$

where

$$G(t,s) = G_1(t,s) + \frac{bt^{\alpha-1}}{d} G_2(\eta, s),$$

in which $d = 1 - b\eta^{\alpha-\gamma-1} > 0$,
where

$$\begin{aligned} G_1(t,s) &= \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \\ G_2(\eta,s) &= \begin{cases} \frac{((1-s)\eta)^{\alpha-\gamma-1}-(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & t \in [0,1], \\ \frac{((1-s)\eta)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & t \in [0,1], \end{cases} \end{aligned} \tag{5}$$

and

$$\rho(s) = \varphi_q\left(\int_0^1 H(s,\tau)g(\tau)d\tau\right),$$

in which

$$H(s,\tau) = H_1(s,\tau) + \frac{a^{p-1}s^{\beta-1}}{1-a^{p-1}\xi^{\beta-1}} H_2(\xi,\tau),$$

where

$$\begin{aligned} H_1(s,\tau) &= \begin{cases} \frac{s^{\beta-1}(1-\tau)^{\beta-1}-(s-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le \tau \le s \le 1, \\ \frac{s^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le \tau \le 1, \end{cases} \\ H_2(\xi,\tau) &= \begin{cases} \frac{((1-\tau)\xi)^{\beta-1}-(\xi-\tau)^{\beta-1}}{\Gamma(\beta)}, & s \in [0,1], \\ \frac{((1-\tau)\xi)^{\beta-1}}{\Gamma(\beta)}, & s \in [0,1], \end{cases} \end{aligned}$$

such that $a^{p-1}\xi^{\beta-1} < 1$.

Proof. From Lemma 2.4 and problem (3), we get

$$v(t) = c_1 t^{\beta-1} + c_2 t^{\beta-2} + I^\beta g(t).$$

Since $v(0) = 0$, we have $c_2 = 0$ and so

$$v(t) = c_1 t^{\beta-1} + I^\beta g(t). \tag{6}$$

Considering the boundary condition in problem (3), $v(1) = a^{p-1}v(\xi)$, we have

$$\begin{aligned} c_1 + \int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau &= a^{p-1}\left[c_1\xi^{\beta-1} + \int_0^\xi \frac{(\xi-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau\right] \\ c_1 &= \frac{1}{1-a^{p-1}\xi^{\beta-1}}\left[-\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau + a^{p-1}\int_0^\xi \frac{(\xi-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau\right]. \end{aligned} \tag{7}$$

Substituting for c_1 into (6), we get

$$v(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau + \frac{t^{\beta-1}}{1-a^{p-1}\xi^{\beta-1}}\left[a^{p-1}\int_0^{\xi} \frac{(\xi-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau - \int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau\right].$$

If

$$v(t) = -\nu(t),$$

then we have

$$\begin{aligned}
\nu(s) =& \frac{1}{\Gamma(\beta)}\int_0^s [(1-\tau)^{\beta-1}s^{\beta-1} - (s-\tau)^{\beta-1}]g(\tau)d\tau + \frac{1}{\Gamma(\beta)}\int_s^1 (1-\tau)^{\beta-1}s^{\beta-1}g(\tau)d\tau \\
&+ \frac{a^{p-1}s^{\beta-1}}{(1-a^{p-1}\xi^{\beta-1})\Gamma(\beta)}\left[\int_0^{\xi}((1-\tau)^{\beta-1}\xi^{\beta-1} - (\xi-\tau)^{\beta-1})g(\tau)d\tau + \int_{\xi}^1 (1-\tau)^{\beta-1}\xi^{\beta-1}g(\tau)d\tau\right] \\
=& \int_0^1 H_1(s,\tau)g(\tau)d\tau + \frac{a^{p-1}s^{\beta-1}}{1-a^{p-1}\xi^{\beta-1}}\int_0^1 H_2(\xi,\tau)g(\tau)d\tau \\
=& \int_0^1 H(s,\tau)g(\tau)d\tau,
\end{aligned}$$

where

$$H(s,\tau) = H_1(s,\tau) + \frac{a^{p-1}s^{\beta-1}}{1-a^{p-1}\xi^{\beta-1}} H_2(\xi,\tau), \tag{8}$$

in which

$$H_1(s,\tau) = \begin{cases} \frac{s^{\beta-1}(1-\tau)^{\beta-1} - (s-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le \tau \le s \le 1, \\ \frac{s^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le \tau \le 1 \end{cases}$$

and

$$H_2(\xi,\tau) = \begin{cases} \frac{((1-\tau)\xi)^{\beta-1} - (\xi-\tau)^{\beta-1}}{\Gamma(\beta)}, & s \in [0,1], \\ \frac{((1-\tau)\xi)^{\beta-1}}{\Gamma(\beta)}, & s \in [0,1]. \end{cases} \tag{9}$$

Noting that $D^{\alpha}y = w,\ w = \varphi_p^{-1}(v) = \varphi_q(v)$ from (3), we know that the solution of problem (2) satisfies

$$D^{\alpha}y(t) = -\varphi_q(\nu(t)). \tag{10}$$

Let

$$\rho(t) = \varphi_q(\nu(t)), \tag{11}$$

from Lemma 2.4, (10) and (11), we have

$$y(t) = -I^{\alpha}(\rho(t)) + c_1t^{\alpha-1} + c_2t^{\alpha-2}. \tag{12}$$

Since $y(0) = 0$ we get $c_2 = 0$.
Also, since $D^{\gamma}t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma}$ we get

$$\begin{aligned}
D^{\gamma}y(t) &= -I^{\alpha-\gamma}\rho(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1} \\
&= \frac{-1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\rho(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}. \qquad (13)
\end{aligned}$$

Using the boundary condition $D^\gamma y(1) = bD^\gamma y(\eta)$, we have

$$-\int_0^1 (1-s)^{\alpha-\gamma-1}\rho(s)ds + c_1\Gamma(\alpha) = b\left(-\int_0^\eta (\eta-s)^{\alpha-\gamma-1}\rho(s)ds + c_1\Gamma(\alpha)\eta^{\alpha-\gamma-1}\right)$$

and so

$$c_1 = \frac{1}{d\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\gamma-1}\rho(s)ds - b\int_0^\eta (\eta-s)^{\alpha-\gamma-1}\rho(s)ds\right], \tag{14}$$

where $d = 1 - b\eta^{\alpha-\gamma-1} > 0$.

Substituting c_1 into (12) we get

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t ((1-s)^{\alpha-\gamma-1}t^{\alpha-1} - (t-s)^{\alpha-1})\rho(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-\gamma-1}t^{\alpha-1}\rho(s)ds \\ &+ \frac{bt^{\alpha-1}}{(1-b\eta^{\alpha-\gamma-1})\Gamma(\alpha)}\int_0^\eta [((1-s)\eta)^{\alpha-\gamma-1} - (\eta-s)^{\alpha-\gamma-1}]\rho(s)ds \\ &+ \frac{bt^{\alpha-1}}{(1-b\eta^{\alpha-\gamma-1})\Gamma(\alpha)}\int_\eta^1 ((1-s)\eta)^{\alpha-\gamma-1}\rho(s)ds \qquad (15) \\ =& \int_0^1 G_1(t,s)\rho(s)ds + \frac{bt^{\alpha-1}}{d}\int_0^1 G_2(\eta,s)\rho(s)ds \qquad (16) \\ =& \int_0^1 G(t,s)\rho(s)ds, \end{aligned}$$

where

$$G(t,s) = G_1(t,s) \quad + \quad \frac{bt^{\alpha-1}}{d}G_2(\eta,s), \tag{17}$$

in which

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1 \end{cases}$$

and

$$G_2(\eta,s) = \begin{cases} \frac{((1-s)\eta)^{\alpha-\gamma-1}-(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & t \in [0,1], \\ \frac{((1-s)\eta)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & t \in [0,1]. \end{cases} \tag{18}$$

The proof is complete. □

Lemma 2.6. *The functions $H(t,s)$ and $G(t,s)$ defined by* (8) *and* (17) *respectively satisfy the following conditions:*

1. $G(t,s) \ge 0, H(t,s) \ge 0, G(t,s) \le G(s,s)$ *and* $H(t,s) \le H(s,s)$ *for* $s,t \in [0,1]$,
2. $G(t,s) \ge t^{\alpha-1}G(1,s)$ *for all* $s,t \in [0,1]$,
3. *there exist positive functions* g_1 *and* $g_2 \in C(0,1)$ *such that*

$$\min_{\vartheta \le t \le \delta} G_1(t,s) \ge g_1(s)G_1(s,s) \text{ and } \min_{\vartheta \le t \le \delta} G_2(\eta,s) \ge g_2(s)G_2(s,s) \text{ for } s \in (0,1), \text{ where}$$

$$g_1(s) = \begin{cases} \frac{\delta^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(\delta-s)^{\alpha-1}}{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}, & \text{if } s \in [0,m_1], \\ (\frac{\vartheta}{s})^{\alpha-1}, & \text{if } s \in [m_1,1] \end{cases}$$

and

$$g_2(s) \quad = \quad \begin{cases} \frac{((1-s)\delta)^{\alpha-\gamma-1}-(\delta-s)^{\alpha-\gamma-1}}{((1-s)\eta)^{\alpha-\gamma-1}}, & \text{if } s \in [0, m_1], \\ (\frac{\vartheta}{s})^{\alpha-\gamma-1}, & \text{if } s \in [m_1, 1], \end{cases} \tag{19}$$

for $0 \le \vartheta < m_1 < \delta \le 1$.

4. $\max_{0 \le t \le 1} \int_0^1 G(t,s)ds = \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)} \left[1 + \frac{b}{d}\right]$.

Proof. The proof will be given in four parts. Part 1 and 2 are covered in [19] and [20] respectively. Here, we will prove Part 3 and Part 4. Considering $G_1(t,s)$ for $s \le t$, we define

$$L_{G_1}(t,s) = t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}$$

and we let

$$J_{G_1}(t,s) = t^{\alpha-1}(1-s)^{\alpha-\gamma-1} \text{ for } t \le s \le 1.$$

We also know that $L_{G_1}(t,s)$ is non-increasing for $s \le t$, and $J_{G_1}(t,s)$ to be non-decreasing for all $s \in [0,1]$ then

$$\begin{aligned} \min_{\vartheta \le t \le \delta} G_1(t,s) \quad &= \quad \frac{1}{\Gamma(\alpha)} \begin{cases} L_{G_1}(\vartheta, s), & s \in [0, m_1], \\ J_{G_1}(\delta, s), & s \in [m_1, 1], \end{cases} \\ &= \quad \frac{1}{\Gamma(\alpha)} \begin{cases} \vartheta^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (\vartheta - s)^{\alpha-1}, & s \in [0, m_1], \\ \delta^{\alpha-1}(1-s)^{\alpha-\gamma-1}, & s \in [m_1, 1], \end{cases} \end{aligned}$$

for $\vartheta \le m_1 \le \delta$ satisfies the equation

$$\vartheta^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (\vartheta - s)^{\alpha-1} = \delta^{\alpha-1}(1-s)^{\alpha-\gamma-1}.$$

By the monotonicity of L_{G_1} and J_{G_1}, we have

$$\max_{0 \le t \le 1} G_1(t,s) = G_1(s,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, \tag{20}$$

we assign $g_1(s)$ as in (19), we evidently see that

$$\min_{\vartheta \le t \le \delta} G_1(t,s) \ge g_1(s) G_1(s,s),$$

for all $s, t \in [0,1]$.
Using the same approach on $G_2(\eta, s)$ for $t, s \in [0,1] \times [0,1]$, we get

$$G_2(s,s) = \frac{(s(1-s))^{\alpha-\gamma-1}}{\Gamma(\alpha)} \text{ and } \max_{0 \le t \le 1} G_2(\eta, s) = \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)} \tag{21}$$

and assigning $g_2(s)$ as stated from (19), we see that for $s \in [0, m_1]$, $s \le t$ and $s \le \eta$,

$$\begin{aligned} g_2(s) G_2(s,s) \quad &= \quad \frac{((1-s)\delta)^{\alpha-\gamma-1} - (\delta - s)^{\alpha-\gamma-1}}{((1-s)\eta)^{\alpha-\gamma-1}} \times \frac{((1-s)s)^{\alpha-\gamma-1}}{\Gamma(\alpha)} \\ &\le \quad \frac{1}{\Gamma(\alpha)} [((1-s)\delta)^{\alpha-\gamma-1} - (\delta - s)^{\alpha-\gamma-1}], \end{aligned}$$

since $g_2(s)$ is non-increasing, for $\vartheta \le \delta$, we get

$$g_2(s) G_2(s,s) \le \frac{((1-s)\vartheta)^{\alpha-\gamma-1} - (\vartheta - s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}.$$

Therefore,

$$\min_{\vartheta\le\eta,t\le\delta} G_2(\eta,s) \ge g_2(s)G_2(s,s).$$

Also, since $g_2(s)$ is non-decreasing for $s\in[m_1,1]$, $\eta\le s$ and $\vartheta\le\delta$,

$$\begin{aligned} g_2(s)G_2(s,s) &= \left(\frac{\vartheta}{s}\right)^{\alpha-\gamma-1}\times\frac{1}{\Gamma(\alpha)}[(1-s)\eta]^{\alpha-\gamma-1} \\ &\le \frac{1}{\Gamma(\alpha)}[(1-s)\vartheta]^{\alpha-\gamma-1} \\ &\le \frac{1}{\Gamma(\alpha)}[(1-s)\delta]^{\alpha-\gamma-1}. \end{aligned}$$

Therefore,

$$\min_{\vartheta\le\eta,t\le\delta} G_2(\eta,s) \ge g_2(s)G_2(s,s) \text{ for all } s,t\in[0,1].$$

By the Beta integral $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt$, for $u,v\in\mathbb{R}$ and $B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$, using equation (20) and (21) we get

$$\begin{aligned} \max_{0\le t\le1}\int_0^1 G(t,s)ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-\gamma-1}ds+\frac{b}{d\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-\gamma-1}ds \\ &= \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]. \end{aligned} \tag{22}$$

This completes the proof. □

Lemma 2.7. *The function H defined by* (8) *satisfies the outlined conditions:*

1. $\max_{0\le t\le1}\int_0^1 H(t,s)ds=\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left[1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right]$,
2. *there exist a positive function* $g_H\in C(0,1)$ *such that*

$$\min_{\vartheta\le t\le\delta} H(t,s)\ge g_H(s)H(s,s),\ s\in(0,1),$$

where

$$g_H(s) = \begin{cases} \frac{\delta^{\beta-1}(1-s)^{\beta-1}-(\delta-s)^{\beta-1}}{t^{\beta-1}(1-s)^{\beta-1}}, & \text{if } s\in[0,m_1],\\ (\frac{\vartheta}{s})^{\beta-1}, & \text{if } s\in[m_1,1],\end{cases} \tag{23}$$

and $\vartheta<m_1<\delta$.

Proof. The proofs follow from Lemma 2.6, Part 3 and 4. We can easily see that

$$\max_{0\le t\le1}H_1(t,s)=H_1(s,s)=\frac{s^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)} \quad\text{and}\quad \max_{0\le t\le1}H_2(\xi,s)=H_2(s,s)=\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}. \tag{24}$$

Let $g_H(s)$ be defined as in (23). From (24), and the Beta integral function we get

$$\max_{0\le t\le1}\int_0^1 H(t,s)ds=\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left[1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right]. \tag{25}$$

The proof is complete. □

Also, we will use the following fixed point theorems and lemmas to give existence results.

Lemma 2.8. ([2]) *Let E be a real Banach space, $C \subset E$ be a cone, $\Omega_r = \{y \in C : \|y\| \leq r\}$. Let the operator $T : C \cap \Omega_r \to C$ be completely continuous and satisfying $Tu \neq u$, $\forall u \in \partial\Omega_r$.*
Then

1. *If $\|Tu\| \leq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, C) = 1$,*
2. *If $\|Tu\| \geq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, C) = 0$.*

Letting $C \subset E$ be a cone in E and $(E, \|.\|)$ be a Banach space. We show a continuous mapping

$$\psi : C \to [0, \infty)$$

by a concave, positive and continuous functional ψ on C with
$\psi(\lambda x + (1 - \lambda)y) \geq \lambda\psi(x) + (1 - \lambda)\psi(y)$ for all $x, y \in C$ and $\lambda \in [0, 1]$. For $K, L, r \geq 0$ constants with C and ψ as above, we let

$$C_K = \{y \in C : \|y\| < K\}$$

and

$$C(\psi, L, K) = \{y \in C : \psi(y) \geq L \text{ and } \|y\| \leq K\}.$$

The current study is anchored on the fixed point theorem as presented by Leggett and Williams [18], see also [17], [10].

Theorem 2.9. *Let $C \subset E$ be a cone in E, which is a Banach space and $R > 0$ a constant. Suppose there exists a concave positive continuous functional on C with $\psi(y) \leq \|y\|$ for $y \in \overline{C}_R$ and let $N : \overline{C}_R \to \overline{C}_R$ be a continuous compact map. Assume that there are numbers r, L and K with $0 < r < L < K \leq R$:*

(A_1) *$\{y \in C(\psi, L, K) : \psi(y) > L, \|y\| \leq K\} \neq \emptyset$ and $\psi(N(y)) > L$ for $y \in C(\psi, L, K)$;*
(A_2) *$\|N(y)\| < r$ for $y \in \overline{C}_r$;*
(A_3) *$\psi(N(y)) > L$ for $y \in C(\psi, L, R)$ with $\|N(y)\| > K$.*
Then N has at least three fixed point y_1, y_2, y_3 in $\overline{C}_R$. Also we get

$$y_1 \in C_r,\ y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\}$$

and

$$y_3 \in \overline{C}_R - \{C(\psi, L, R) \cup \overline{C}_r\}.$$

Theorem 2.10. (Schauder-Tychonoff Fixed Point Theorem) *Let X be a Banach space. Assume that K is a closed, bounded, convex subset of X. If $T : K \to K$ is compact then T has a fixed point in K.*

3. Existence results for BVP (2)

We consider the Banach space $E = C([0, 1], \mathbb{R})$ endowed with the norm defined by
$\|y\| = \sup_{0 \leq t \leq 1} |y(t)|$. Let $C = \{y \in E | y(t) \geq 0\}$, then C is a cone in E. Define an operator $T : C \to C$ as

$$(Ty)(t) = \int_0^1 G(t, s)\varphi_q\left(\int_0^1 H(s, \tau) f(\tau, y(\tau))d\tau\right)ds. \tag{26}$$

Then, T has a solution if and only if the operator T has a fixed point.

Lemma 3.1. *If $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, then the operator $T : C \to C$ is completely continuous.*

Proof. From the continuity and non-negativeness of $G(t,s)$, $H(t,s)$ and $f(t,y(t))$, we see that $T : C \to C$ is continuous.
Let $\Omega \subset C$ be bounded. Then, for all $\tau \in [0,1]$ and $y \in \Omega$, there exists a positive constant M such that $|f(t,y(t))| \le M$. Thus, we get

$$\begin{aligned} |(Ty)(t)| &= |\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\right)ds| \\ &\le \int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau)f(\tau,y(\tau))d\tau\right)ds \\ &\le \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1}, \end{aligned}$$

which implies that $T(\Omega)$ is uniformly bounded.
Also, by the continuity of $G(t,s)$ and $H(t,s)$ on $[0,1]\times[0,1]$, we know that this is uniformly continuous on $[0,1]\times[0,1]$. Therefore, for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1,s) - G(t_2,s)| < \varphi_p\left[\frac{M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]\varepsilon.$$

Thus, for all $y \in \Omega$,

$$\begin{aligned} |(Ty)(t_2)-(Ty)(t_1)| &\le \int_0^1 |G(t_2,s)-G(t_1,s)|\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\right)ds \\ &\le \int_0^1 |G(t_2,s)-G(t_1,s)|\varphi_q\left(\int_0^1 H(\tau,\tau)f(\tau,y(\tau))d\tau\right)ds \\ &\le \varphi_q\left[\frac{M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]\int_0^1 |G(t_2,s)-G(t_1,s)|ds \\ &\le \varepsilon, \end{aligned}$$

which means that $T(\Omega)$ is equicontinuous and by the Arzella-Ascoli theorem, we obtain $T : C \to C$ is completely continuous. □

Theorem 3.2. *If $f \in C([0,1]\times[0,+\infty),[0,+\infty))$, $f(t,y)$ is non-decreasing in y, then BVP* (2) *has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{w}$ in B_r. In addition, $v_m(t) \to \overline{v}(t)$ and $w_m(t) \to \overline{w}(t)$ as $m \to \infty$ uniformly on* $[0,1]$, *where*

$$v_m(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(s,v_{m-1}(\tau))d\tau\right)ds \tag{27}$$

and

$$w_m(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(s,w_{m-1}(\tau))d\tau\right)ds. \tag{28}$$

Proof. Let

$$B_r = \{y \in C : \|y\| \le r\},$$

where

$$r \ge \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{M_1\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1}.$$

Step 1: Problem (2) has at least one solution.
For $y \in B_r$, there exists a positive constant M_1 such that $|f(t, y(t))| \leq M_1$,

$$\begin{aligned}|(Ty)(t)| &= |\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, y(\tau))d\tau\right)ds| \\ &\leq |\int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau)f(\tau, y(\tau))d\tau\right)ds| \\ &\leq \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{M_1\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1}.\end{aligned}$$

Therefore,

$$T : B_r \to B_r.$$

By Lemma 3.1, it is obvious that $T : B_r \to B_r$ is completely continuous. Thus, by the Schauder fixed point theorem, the operator T has at least one fixed point and BVP (2) has at least one solution in B_r.

Step 2: BVP (2) has a positive solution in B_r, which is a minimal positive solution. From (26) and (27), it can be seen that

$$v_m(t) = (Tv_{m-1})(t),\ t \in [0, 1],\ for\ m = 1, 2, 3, \ldots. \tag{29}$$

Also, since $f(t, y)$ is non-decreasing in y, we get

$$0 = v_0(t) \leq v_1(t) \leq \cdots v_m(t) \leq \cdots,\ t \in [0, 1],$$

we obtain that $\{v_m\}$ is a sequentially compact set because T is compact. As a result, there exists $\overline{v} \in B_r$ such that $v_m \to \overline{v}$ as $m \to \infty$.
Let $y(t)$ be any positive solution of BVP (2) in B_r. It is obvious that

$$0 = v_0(t) \leq y(t) = (Ty)(t).$$

Therefore,

$$v_m(t) \leq y(t)\ for\ m = 0, 1, 2, \cdots. \tag{30}$$

Taking limits as $m \to \infty$ in (30), we obtain $\overline{v} \leq y(t)$ for $t \in [0, 1]$.

Step 3: BVP (2) has a positive solution in B_r, which is a maximal positive solution. Let $w_0(t) = r,\ t \in [0, 1]$ and $w_1(t) = Tw_0(t)$. From $T : B_r \to B_r$, we have $w_1 \in B_r$. Thus

$$0 \leq w_1(t) \leq r = w_0(t).$$

Also, since $f(t, y)$ in non-decreasing in y, we get

$$\cdots \leq w_m(t) \leq \cdots \leq w_1(t) \leq w_0(t),\ t \in [0, 1].$$

Using the same steps involved in Step 2, we see that

$$w_m(t) \to \overline{w}(t)\ as\ m \to \infty$$

and

$$\overline{w}(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, \overline{w}(\tau))d\tau\right)ds.$$

Let $y(t)$ be any positive solution of BVP (2) in B_r.
Trivially,

$$y(t) \leq w_0(t).$$

Therefore,

$$y(t) \leq w_m(t). \tag{31}$$

Taking limits as $m \to \infty$ in (31), we get $y(t) \leq \overline{w}(t)$ for $t \in [0,1]$. $\square$

We define

$$f^0 = \lim_{y\to 0^+} \sup_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_1\|y\|)}, \qquad f_0 = \lim_{y\to 0^+} \inf_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_2\|y\|)},$$
$$f^\infty = \lim_{y\to +\infty} \sup_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_3\|y\|)}, \qquad f_\infty = \lim_{y\to +\infty} \inf_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_4\|y\|)}.$$

Let

$$B = \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1}$$

and

$$B_1 = \int_0^1 G(1,s)\varphi_q\left(\int_\vartheta^\delta \min_{\vartheta\leq t\leq\delta} H(s,\tau)d\tau\right)ds.$$

Theorem 3.3. *Assume that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$, and the following conditions hold;*

(N_1) $f_0 = f_\infty = +\infty$.
(N_2) *There exists a constant $\rho_1 > 0$ such that $f(t,y) \leq \varphi_p(l_5\|y\|)$ for $t \in [0,1]$, $y \in [0,\rho_1]$.*
Then, BVP (2) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \rho_1 < \|y_2\|,$$

for

$$0 < \frac{1}{l_2B_1} < 1 < \frac{1}{l_5B} < +\infty \qquad \textit{and} \qquad 0 < \frac{1}{l_4B_1} < 1 < \frac{1}{l_5B} < +\infty. \tag{32}$$

Proof. Since

$$f_0 = \lim_{y\to 0^+} \inf_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_2\|y\|)} = +\infty,$$

there is $\rho_0 \in (0,\rho_1)$ such that

$$f(t,y) \geq \varphi_p(l_2\|y\|) \text{ for } t \in [0,1],\ y \in [0,\rho_0].$$

Let

$$\Omega_{\rho_0} = \{y \in C : \|y\| \leq \rho_0\}.$$

Then, for any $y \in \partial\Omega_{\rho_0}$, it follows from Lemma 2.6 that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\right)ds \\ &\geq \int_0^1 t^{\alpha-1}G(1,s)\varphi_q\left(\int_0^1 H(s,\tau)\varphi_p(l_2\|y\|)d\tau\right)ds \\ &\geq l_2\int_0^1 G(1,s)\varphi_q\left(\int_\vartheta^\delta \min_{\vartheta\leq t\leq\delta} H(s,\tau)d\tau\right)ds\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \geq l_2B_1\|y\|.$$

Considering also (32), we get

$$\|Ty\| \geq \|y\|,\ \forall\ y \in \partial\Omega_{\rho_0}.$$

By Lemma 2.8, we get

$$i(T,\Omega_{\rho_0},C) = 0. \tag{33}$$

Also,

$$f_\infty = \lim_{y\to\infty}\inf_{t\in[0,1]}\frac{f(t,y)}{\varphi_p(l_4\|y\|)} = +\infty,$$

there is ρ_0^*, $\rho_0^* > \rho_1$, such that

$$f(t,y) \geq \varphi_p(l_4\|y\|) \text{ for } t \in [0,1],\ y \in [\rho_0^*,+\infty).$$

Let

$$\Omega_{\rho_0^*} = \{y \in C : \|y\| \leq \rho_0^*\}.$$

Then, for any $y \in \partial\Omega_{\rho_0^*}$, it follows from Lemma 2.6 that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\right)ds \\ &\geq \int_0^1 t^{\alpha-1}G(1,s)\varphi_q\left(\int_0^1 H(s,\tau)\varphi_p(l_4\|y\|)d\tau\right)ds \\ &\geq l_4\int_0^1 G(1,s)\varphi_q\left(\int_\vartheta^\delta \min_{\vartheta\leq t\leq\delta} H(s,\tau)d\tau\right)ds\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \geq l_4B_1\|y\|.$$

Considering also (32), we get

$$\|Ty\| \geq \|y\|,\ \forall\ y \in \partial\Omega_{\rho_0^*}.$$

By Lemma 2.8, we get

$$i(T,\Omega_{\rho_0^*},C) = 0. \tag{34}$$

Finally, let $\Omega_{\rho_1} = \{y \in C : \|y\| \leq \rho_1\}$ for any $y \in \partial\Omega_{\rho_1}$, it follows from Lemma 2.6, 2.7 and (N_2) that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\right)ds\\ &\leq \int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau)\varphi_p(l_5\|y\|)d\tau\right)ds\\ &= l_5\frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1}\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \leq l_5B\|y\|.$$

Considering also (32), we get

$$\|Ty\| \leq \|y\|, \ \forall\ y \in \partial\Omega_{\rho_1}.$$

By Lemma 2.8, we get

$$i(T,\Omega_{\rho_1},C) = 1. \tag{35}$$

From (33)-(35) and $\rho_0 < \rho_1 < \rho_0^*$, we get

$$i(T,\Omega_{\rho_0^*}\setminus\overline{\Omega}_{\rho_1},C) = -1, \qquad i(T,\Omega_{\rho_1}\setminus\overline{\Omega}_{\rho_0},C) = 1.$$

Thus, T has a fixed point $y_1 \in \Omega_{\rho_1}\setminus\overline{\Omega}_{\rho_0}$ and a fixed point $y_2 \in \Omega_{\rho_0^*}\setminus\overline{\Omega}_{\rho_1}$. Trivially, $y_1,\ y_2$ are both positive solutions of BVP (2) and $0 < \|y_1\| < \rho_1 < \|y_2\|$.
This completes the proof. □

Similarly, we can get the following results;

Corollary 3.4. *Assume that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ and the following conditions hold:*

(N_1) $f^0 = f^\infty = 0$.

(N_2) *There exists a constant $\rho_2 > 0$ such that $f(t,y) \geq \varphi_p(l_6\|y\|)$ for $t \in [0,1],\ y \in [0,\rho_2]$. Then BVP (2) has at least two positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < \rho_2 < \|y_2\|$$

for

$$0 < \frac{1}{l_6B_1} < 1 < \frac{1}{l_3B} < +\infty \qquad \textit{and} \qquad 0 < \frac{1}{l_6B_1} < 1 < \frac{1}{l_1B} < +\infty.$$

4. Multiplicity result for BVP (1)

A solution to problem (1) is obtained by setting

$$B_1 = \{y : (-\infty,1] \to R : y|_{(-\infty,0]} \in B, y|_J \in C^2(J,R)\},$$

and let $\|.\|_1$ the semi norm in B_1 defined by:

$$\|y\|_1 = \|y_0\|_B + \sup\{|y(t)| : 0 \leq t \leq 1\}, y \in B_1.$$

Definition 4.1. *Problem (1) has a solution y, which is a function $y \in B_1$ that satisfies the equation $D^\beta(\varphi_p(D^\alpha y(t))) = f(t,y_t)$ on J and conditions $y(0) = 0,\ D^\alpha y(1) = aD^\alpha y(\xi)$, $D^\alpha y(0) = 0,\ D^\gamma y(1) = bD^\gamma y(\eta)$ and $y(t) = \phi(t),\ t \in (-\infty,0]$.*

The Banach space of all continuous functions from J into R is denoted by $C(J,R)$, with the norm:

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Now, we present axioms for definition of the phase space B.

(A_1) For every $t \in [0,1]$, if $y : (-\infty, 1) \to R,\ y_0 \in B$, then the following conditions hold:
  (a) $y_t \in B$,
  (b) There exists a positive constant $H : |y(t)| \le H\|y_t\|_B$;
  (c) There exist two functions $K(.), M(.) : R_+ \to R_+$, independent of y, with K continuous and M locally bounded:
  $$\|y_t\|_B \le K(t)\sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_B.$$
(A_2) y_t is a B-valued continuous function on $[0,1]$ for the function $y(.)$ in (A_1).
(A_3) The space B is complete. Denoted by

$$K = \sup\{K(t) : t \in [0,1]\} \text{ and } M = \sup\{M(t) : t \in [0,1]\}.$$

Let

$$\mu = \min_{t\in[\vartheta,\delta]}\{g_1(t), g_2(t), g_H(t)\} \qquad \text{and} \qquad \sigma = \max\{\vartheta^{\alpha-1}, \vartheta^{\beta-1}, \mu\}. \tag{36}$$

The following assumptions are necessary for the underlying theorem:

(H_1) f is a continuous function.
(H_2) There exists a function $q^* : [0,\infty) \to [0,\infty)$ which is continuous and non-decreasing and a function $h^* : [0,\infty) \to [0,\infty)$ which is continuous and non-increasing, $p_1 \in C(J, R_+)$ and $p_2 \in C(J, R_+)$ such that

$$p_2(t)h^*(\|u\|) \le f(t,u) \le p_1(t)q^*(\|u\|),$$

for each $(t,u) \in J \times B$.
(H_3) There exists a constant $r > 0$ such that

$$\left[q^*(Kr + M\|\phi\|_B)\|p_1\|_\infty \frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1 + \frac{a^{p-1}}{1 - a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1 + \frac{b}{d}\right] \le r.$$

(H_4) There exists a constant $L > r$ such that

$$\left[h^*(KL + M\|\phi\|_B)\ \|p_2\|_\infty\right]^{q-1} \times \left[\int_\vartheta^\delta G(t,s)\varphi_q\left(\int_\vartheta^\delta H(s,\tau)d\tau\right)ds\right] \ge L.$$

(H_5) There exists a constant R such that $0 < r < L \le \sigma R$ and

$$\left[q^*(KR + M\|\phi\|_B)\|p_1\|_\infty \frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1 + \frac{a^{p-1}}{1 - a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1 + \frac{b}{d}\right] \le R.$$

Theorem 4.2. *If* $(H_1) - (H_5)$ *are satisfied. Problem* (1) *has at least three positive solutions.*

Proof. Relying on Leggett-William fixed point Theorem Transform, we transform problem (1) into a fixed point problem. Considering the operator

$$N : B_1 \to B_1$$

defined as the following:

$$N(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \int_0^1 G(t,s)\varphi_q(\int_0^1 H(s,\tau)f(\tau, y_\tau)d\tau)ds, & t \in [0,1]. \end{cases}$$

$G(t,s)$ is defined in (17). Obviously, the fixed points of the operator N are solutions of problem (1), also $\rho(s, y_s)$ is defined in (11). We define $x(.) : (-\infty, 1] \to R$ be the function defined as:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ 0, & \text{if } t \in [0,1]. \end{cases}$$

Then, $x_0 = \phi$. For each $z \in B$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ z(t), & \text{if } t \in [0,1]. \end{cases}$$

Let $y(.)$ satisfy the integral equation

$$y(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, y_\tau)d\tau\right)ds.$$

We partition $y(.)$ into $y(t) = \overline{z}(t) + x(t)$, $0 \le t \le 1$, which makes $y_t = \overline{z}_t + x_t$, for every $t \in [0,1]$, and the function z(.) satisfies

$$z(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, \overline{z}_\tau + x_\tau)d\tau\right)ds.$$

Let $B_0 = \{z \in C([0,1], R) : z_0 = 0\}$ and $\|.\|_1$ be the seminorm in B_0 defined by

$$\|z\|_1 = \|z_0\|_B + \sup\{|z(s)| : 0 \le s \le 1\} = \|z\|_0.$$

B_0 is a Banach space with the norm $\|.\|_0$. We let the operator $P : B_0 \to B_0$ be defined by

$$P(z)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, \overline{z}_\tau + x_\tau)d\tau\right)ds. \tag{37}$$

It is easily seen that the operator N has a fixed point that is equivalent to the one P has, so we must prove that P has a fixed point. We now show that P is completely continuous:

Step 1: P is continuous.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_0. Then,

$$\begin{aligned} |P(z)(t)| &= \left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, \overline{z}_\tau + x_\tau)d\tau\right)ds\right| \\ &\le \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} \times \varphi_q[\|f(., \overline{z}_{(.)} + x_{(.)})\|] \end{aligned}$$

and

$$\begin{aligned} |P(z_n(t) - P(z)(t)| \quad &\le \quad \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} \\ &\quad \times[\varphi_q(\|f(., \overline{z}_{n_{(.)}} + x_{(.)})\|) - \varphi_q(\|f(., \overline{z}_{(.)} + x_{(.)})\|)]. \end{aligned}$$

Since f is continuous, we get: $\|P(z_n) - P(z)\|_0 \to 0$ as $n \to \infty$.
Step 2: P maps bounded sets into bounded sets in B_0.

It is sufficient to show that for any $\xi > 0$, there exists a positive constant l such that for each $z \in B_\xi = \{z \in B_0 : \|z\|_0 \le \varsigma^*\}$, one has $\|Pz\|_\infty \le l$ by (H_2) we have for each $t \in [0,1]$,

$$\begin{aligned}
|P(z)(t)| &= \int_0^1 \left| G(t,s)\varphi_q(\int_0^1 H(s,\tau) f(\tau, \overline{z}_\tau + x_\tau) d\tau) \right| ds \\
&\le \int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau) p_1(\tau) q(\|\overline{z}_\tau + x_\tau\|) d\tau\right) ds \\
&= \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\|p_1\|_\infty q(\varsigma^*)\right]^{q-1} =: l,
\end{aligned}$$

where

$$\begin{aligned}
\|\overline{z}_\tau + x_\tau\|_B \le& \|\overline{z}_\tau\|_B + \|x_\tau\|_B \\
\le& K(s)\sup\{|z(\tau)| : 0 \le \tau \le s\} + M(s)\|z_0\|_B \\
&+ K(s)\sup\{|x(\tau)| : 0 \le \tau \le s\} + M(s)\|x_0\|_B \\
\le& K \sup\{|z(\tau)| : 0 \le \tau \le s\} + M\|\phi\|_B \\
\le& K\xi + M\|\phi\|_B = \varsigma^*.
\end{aligned} \tag{38}$$

Step 3: P maps bounded sets into equicontinuous sets of B_0.
Let $t_1, t_2 \in [0,1]$, such that $t_1 < t_2$, let B_ξ be a bounded set of B_0 as in Step 2 and let $z \in B_\xi$. Then,

$$\begin{aligned}
|P(z)(t_2) - P(z)(t_1)| \le& \left| \int_0^1 G(t_1,s)\varphi_q\left(\int_0^1 H(s,\tau) f(\tau,\overline{z}_\tau + x_\tau) d\tau\right) ds \right. \\
&\left. - \int_0^1 G(t_2,s)\varphi_q\left(\int_0^1 H(s,\tau) f(\tau,\overline{z}_\tau + x_\tau) d\tau\right) ds \right| \\
\le& \left[\max_{s\in[0,1]}(|G_1(t_2,s) - G_1(t_1,s)|) + \frac{b(t_2^{\alpha-1} - t_1^{\alpha-1})}{d}\max_{s\in[0,1]}(|G_2(\eta,s)|)\right] \\
&\times \varphi_q\left(\|p\|_\infty q^*(\varsigma^*)\int_0^1 H(\tau,\tau) d\tau\right).
\end{aligned}$$

By the continuity of the G function, we get zero on the right hand side of the inequality, as $t_2 \to t_1$ and this show that $P(B(0,\xi))$ is equicontinuous in B_0. As a result of Steps 1 to 3 and the Ascoli-Arzela Theorem, we can conclude that the operator $P : B_0 \to B_0$ is completely continuous.
Let

$$\varrho = \{z \in B_0 : z(t) \ge 0 \quad \min_{t\in[\vartheta,\delta]} z(t) \ge \frac{\sigma}{3}\|z\|_0 \text{ for } t \in J\}$$

be a cone in B_0. We show that $P : \varrho \to \varrho$ is well defined. Let $z \in \varrho$, then it follows from Lemma 2.6 and (37) that

$$\begin{aligned}
\|P(z)\|_0 \le& \int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau) f(\tau,\overline{z}_\tau + x_\tau) d\tau\right) ds \\
\le& 3\left[\int_\vartheta^\delta G(s,s)\varphi_q\left(\int_\vartheta^\delta H(\tau,\tau) f(\tau,\overline{z}_\tau + x_\tau) d\tau\right) ds\right].
\end{aligned}$$

Also, considering Lemma 2.6 and (36) this means that for any $t \in [\vartheta, \delta]$

$$\begin{aligned}
(Pz)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds \\
&\geq \int_\vartheta^\delta \left(g_1(s)G_1(s,s) + \frac{b\vartheta^{\alpha-1}}{d}g_2(s)G_2(s,s)\right)\varphi_q\left(\int_\vartheta^\delta g_H(\tau)H(\tau,\tau)f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds \\
&\geq \sigma\left[\int_\vartheta^\delta G(s,s)\varphi_q\left(\int_\vartheta^\delta H(\tau,\tau)f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\right] \\
&\geq \frac{\sigma}{3}\|Pz\|_0.
\end{aligned}$$

This implies that $P : \varrho \to \varrho$ is well defined. Using the assumptions $(H_1) - (H_2)$ and (H_5) $P : \overline{C}_R \to \overline{C}_R$ is well defined and completely continuous. Let $\psi : \varrho \to [0,\infty)$ is defined by

$$\psi(z) = \min_{t\in[\vartheta,\delta]} z(t).$$

It is evident that ψ is a non-negative concave continuous functional and

$$\psi(z) \leq \|z\|_0 \text{ for } z \in \overline{C}_R.$$

We are left to show that the hypotheses of Theorem 2.9 to be stated are satisfied.
We note that condition (A_2) of Theorem 2.9 is valid for $z \in \overline{C}_r$, and from (H_2), (H_3), and (38) we get

$$\begin{aligned}
\|P(z)\| &= \max_{0\leq t\leq 1} |P(z)(t)| \\
&\leq \max_{0\leq t\leq 1}\left\{\int_0^1 \left(|G_1(t,s)| + \frac{bt^{\alpha-1}}{d}|G_2(\eta,s)|\right)\varphi_q\left(\int_0^1 H(s,\tau)q^*(\|\overline{z}_\tau + x_\tau\|)|p_1(\tau)|d\tau\right)ds\right\} \\
&\leq \max_{0\leq t\leq 1}\left\{\int_0^1 |G(t,s)|\varphi_q\left(\int_0^1 H(s,\tau)q^*(K\|z\|_0 + M\|\phi\|_B)p_1(\tau)d\tau\right)ds\right\} \\
&\leq \int_0^1 G(s,s)\varphi_q\left(\int_0^1 H(\tau,\tau)q^*(K\|z\|_0 + M\|\phi\|_B)p_1(\tau)d\tau\right)ds \\
&\leq \varphi_q\left(q^*(Kr + M\|\phi\|_B)\|p_1\|_\infty \frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1 + \frac{a^{p-1}}{1 - a^{p-1}\xi^{\beta-1}}\right)\right)\frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1 + \frac{b}{d}\right] \\
&\leq r.
\end{aligned}$$

We now proceed to show that condition (A_1) of Theorem 2.9 is satisfied. Evidently, if $z \in C(\psi, L, \frac{L}{\sigma})$ then $L \leq z(s) \leq \frac{L}{\sigma}$, $s \in [\vartheta, \delta]$, and then $\{z \in C(\psi, L, \frac{L}{\sigma}),\ \psi(z) > L\} \neq \emptyset$. By condition (H_4) we have

$$\begin{aligned}
\psi(P(z)) &= \min_{\vartheta\leq t\leq\delta}\left\{\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\right\} \\
&\geq \min_{\vartheta\leq t\leq\delta}\left\{\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)h^*(\|\overline{z}_\tau + x_\tau\|)p_2(\tau)d\tau\right)ds\right\} \\
&\geq \varphi_p\left(h^*(KL + M\|\phi\|_B)\|p_2\|_\infty\right)\min_{\vartheta\leq t\leq\delta}\left\{\int_\vartheta^\delta G(t,s)\varphi_q\left(\int_\vartheta^\delta H(s,\tau)d\tau\right)ds\right\}.
\end{aligned}$$

Thus, condition (A_1) of Theorem 2.9 is satisfied.

Finally, we show that condition (A_3) of Theorem 2.9 is also satisfied. If $z \in C(\psi, L, R)$ and $\|Pz\| > \frac{L}{\sigma}$, we get

$$\begin{aligned}\psi(P(z)) =& \min_{\vartheta \le t \le \delta} \left\{ \int_0^1 G(t,s)\varphi_q \left(\int_0^1 H(s,\tau) f(\tau, \overline{z}_\tau + x_\tau) d\tau \right) ds \right\} \\ \ge& \sigma \|Pz\| \\ \ge& L.\end{aligned}$$

Thus, condition (A_3) holds. By the Leggett and William fixed point theorem this implies that N has at least three fixed points z_1, z_2, z_3 which are solutions to problem (1).
In addition, we have

$$z_1 \in C_r,\ z_2 \in \{z \in C(\psi, L, R) : \psi(z) > L\},\ z_3 \in C_R - \{(\psi, L, R) \cup C_r\}.$$

Once more, condition (A_3) of Theorem 2.9 is satisfied. By Theorem 2.9, there exist three positive solutions z_1, z_2, z_3 such that $\|z_1\| < r$, $L < \alpha(z_2(t))$, and $\|z_3\| > r$, with $\alpha(z_3(t)) < L$.
Finally, problem (1) has three positive solutions y_1, y_2, y_3 such that

$$y_i(t) = \begin{cases} \phi(t), & if\ t \in (-\infty, 0], \\ z_i(t), & if\ \ t \in [0,1], \end{cases} \qquad \text{for} \quad i \in \{1,2,3\}.$$

The proof is complete. □

5. Examples

In this section we give illustrative examples showing the necessity of the main results covered in previous sections.

Example 5.1. *Consider the following boundary value problem:*

$$D^{\frac{3}{2}}(\varphi_2(D^{\frac{3}{2}}y(t))) = \frac{t\pi|y(t)|}{1+|y(t)|},\ t \in (0,1), \tag{39}$$
$$y(0) = 0,\ D^{\frac{3}{2}}y(1) = \frac{1}{3}D^{\frac{3}{2}}y\left(\frac{1}{4}\right),\ D^{\frac{3}{2}}y(0) = 0, D^{\frac{1}{2}}y(1) = \frac{1}{2}D^{\frac{1}{2}}y\left(\frac{1}{4}\right),$$
$$where \quad \alpha = \frac{3}{2},\ \beta = \frac{3}{2},\ \gamma = \frac{1}{2},\ p = q = 2,\ a = \frac{1}{3},\ b = \frac{1}{2},\ \xi = \frac{1}{4},\ \eta = \frac{1}{4},\ \alpha - \gamma - 1 \ge 0$$

and $f \in C([0,1] \times [0,+\infty),\ [0,+\infty))$.
Therefore, $M_1 = \pi$ and $a^{p-1} = (\frac{1}{3})^{2-1} = \frac{1}{3}$. By computation we see that

$$r \ \ \ge\ \ \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[M_1\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} = \frac{14}{15}\pi.$$

Hence, by Theorem 3.2, BVP (39) has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{w}$ in B_r.

Example 5.2. *Consider the following boundary value problem:*

$$D^{\frac{7}{4}}(\varphi_{\frac{3}{2}}(D^{\frac{3}{2}}y(t))) = \frac{t}{8}[2(|y(t)|^{\frac{2}{3}} + 4\|y\|^{\frac{1}{3}}) + \|y\|], \quad t \in (0,1), \tag{40}$$
$$y(0) = 0,\ D^{\frac{3}{2}}y(1) = \frac{1}{16}D^{\frac{3}{2}}y\left(\frac{1}{3}\right),\ D^{\frac{3}{2}}y(0) = 0, D^{\frac{1}{2}}y(1) = \frac{3}{4}D^{\frac{1}{2}}y\left(\frac{1}{4}\right),$$

$$where\ \alpha = \tfrac{3}{2},\ \beta = \tfrac{7}{4},\ \gamma = \tfrac{1}{2},\ p = \tfrac{3}{2},\ q = 3,\ a = \tfrac{1}{16},\ b = \tfrac{3}{4},\ \xi = \tfrac{1}{3},\ \eta = \tfrac{1}{4},\ \alpha - \gamma - 1 \ge 0,\ d = \tfrac{1}{4},$$

We set $\vartheta = \frac{1}{3}$ *and* $\delta = \frac{2}{3}$. *By computation we see that*

$$B = \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} = 0.37744$$

and

$$\begin{aligned} B_1 =& \frac{1}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\gamma-1} - (1-s)^{\alpha-1} ds \right. \\ &\left. + \frac{b}{d}\int_0^{\frac{1}{3}} ((1-s)\eta)^{\alpha-\gamma-1} - (\eta-s)^{\alpha-\gamma-1} ds + \frac{b}{d}\int_{\frac{1}{3}}^1 ((1-s)\eta)^{\alpha-\gamma-1} ds\right] \\ &\times \left[\frac{1}{\Gamma(\beta)}\left(\int_\vartheta^\delta \delta^{\beta-1}(1-\tau)^{\beta-1}d\tau + \frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\int_\vartheta^\delta \xi^{\beta-1}(1-\tau)^{\beta-1}d\tau\right)\right]^{q-1} = 0.090137. \end{aligned}$$

Taking $\rho_1 = 8,\ l_5 = 2$, *we get*

$$f(t,y) \le \frac{1}{8}(2(4+8)+8) = 4 = \varphi_p(l_5\|y\|) = \varphi_{\frac{3}{2}}(8\times 2),\ \textit{for } t\in[0,1],\ y\in[0,\rho_1].$$

Therefore, condition (N_2) *is satisfied. It can be easily seen that condition* (N_1) *holds.*

$$\textit{Also, let } l_2 = 15 \textit{ and } l_4 = 12, \textit{we get } 0 < \frac{1}{l_2B_1} < 1 < \frac{1}{l_5B} < +\infty \textit{ and } 0 < \frac{1}{l_4B_1} < 1 < \frac{1}{l_5B} < +\infty.$$

Hence, by Theorem 3.3, BVP (40) has at least two solutions y_1 *and* y_2 *such that* $0 < \|y_1\| < 8 < \|y_2\|$ *for the given values of* $l_5,\ l_2$ *and* l_4.

Example 5.3. *Consider the functional differential equation:*

$$D^{\frac{3}{2}}(\varphi_2(D^{\frac{3}{2}}y(t))) = \frac{2\|y_t\|e^t}{3e^{\frac{\|y_t\|}{10}}\sqrt{4+t^2}}, \quad \textit{if } t\in J=[0,1], \tag{41}$$

$$y(0)=0,\ D^{\frac{3}{2}}y(1) = \frac{1}{3}D^{\frac{3}{2}}y\left(\frac{1}{4}\right),\ D^{\frac{3}{2}}y(0)=0,\ D^{\frac{1}{2}}y(1) = \frac{1}{2}D^{\frac{1}{2}}y\left(\frac{1}{4}\right),\ y(t)=\phi(t) \qquad \textit{if } t\in(-\infty,0],$$

where

$$\alpha=\frac{3}{2},\ \beta=\frac{3}{2},\ \gamma=\frac{1}{2},\ p=q=2,\ a=\frac{1}{3},\ b=\frac{1}{2},\ \xi=\frac{1}{4},\ \eta=\frac{1}{4},\ \alpha-\gamma-1\ge 0.$$

We set ϕ *such that* $\|\phi\| = \frac{1}{10}$, B_γ *to be defined by:*

$$B_\gamma = \{u\in C((-\infty,0],R) : \lim_{\theta\to-\infty} e^{\gamma\theta}u(\theta) \textit{ exists}\}$$

with the norm

$$\|u\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|u(\theta)|.$$

Let $u : (-\infty,1]\to R$ *be such that* $u_0\in B_\gamma$. *Then*

$$\begin{aligned} \lim_{\theta\to-\infty} e^{\gamma\theta}u(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta}u(t+\theta) \\ &= \lim_{\theta\to-\infty} e^{\gamma(\theta-t)}u(\theta) \\ &= e^{\gamma t}\lim_{\theta\to-\infty} e^{-\gamma\theta}u_0(\theta) < +\infty. \end{aligned}$$

Therefore, $u_t \in B_\gamma$. We now prove that

$$\|u_t\| \le K(t)\sup\{|u(s)| : 0 \le s \le t\} + M(t)\|y_0\|_\gamma,$$

where $K = M = 1$ and $H = 1$. we get $u(t) = u(t+\phi)$.
If $t + \theta \le 0$ we have

$$\|u_t(\theta)\| \le \sup\{|u(s)| : 0 \le s \le t\}.$$

Hence, for all $t + \theta \in [0,1]$, we get

$$\|u_t(\theta)\| \le \sup\{|u(s)| : -\infty \le s \le 0\} + \sup\{|u(s)| : 0 \le s \le t\}.$$

Therefore,

$$\|u_t\|_\gamma \le \|u\|_0 + \sup\{|u(s)| : 0 \le s \le t\}.$$

It is evident that $(B_\gamma, \|u\|_\gamma)$ is a Banach space, we conclude that B_γ is a phase space. Since

$$f(t,y) = \frac{2\|y_t\|e^t}{3e^{\frac{\|y_t\|}{10}}\sqrt{4+t^2}}, \quad (t,y) \in J \times B_\gamma.$$

We choose

$$q^*(y) = \frac{y}{3}, \quad p_1(t) = e^t, \quad h^*(y) = e^{\frac{-y}{10}}, \quad p_2(t) = \frac{2}{\sqrt{4+t^2}}, \quad y \ge 0, \quad t \in [0,1].$$

By the definitions of $f,\ q^*,\ p_1,\ h^*,\ p_2$, it follows that:

$$p_2(t)h^*(\|y\|) \le f(t,y) \le p_1(t)q^*(\|y\|).$$

By calculations, we obtain

$$\begin{aligned}
&\min_{\vartheta \le t \le \delta}\left[\int_\vartheta^\delta \left(G_1(t,s) + \frac{b}{d}G_2(\eta,s)\right)\left(\int_\vartheta^\delta H(s,\tau)d\tau + \frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\int_\vartheta^\delta H(\xi,\tau)d\tau\right)^{q-1} ds\right]\\
&=\left[\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{1}{3}}^{\frac{2}{3}} \delta^{\alpha-1}(1-s)^{\alpha-\gamma-1}ds + \frac{(\frac{1}{2})}{(\frac{1}{2})}\int_{\frac{1}{3}}^{\frac{2}{3}}(\eta(1-s))^{\alpha-\gamma-1}ds\right)\right]\\
&\times\left[\frac{1}{\Gamma(\beta)}\left(\int_{\frac{1}{3}}^{\frac{2}{3}} \delta^{\beta-1}(1-\tau)^{\beta-1}d\tau + \frac{(\frac{1}{3})}{1-(\frac{1}{3})(\frac{1}{4})^{\frac{3}{2}-1}}\int_{\frac{1}{3}}^{\frac{2}{3}}\xi^{\beta-1}(1-\tau)^{\beta-1}d\tau\right)\right]^{2-1}\\
&=0.18383.
\end{aligned}$$

Also,

$$\left[h^*(KL + M\|\phi\|_B)\|p_2\|_\infty\right]^{q-1} = h^*\left(L + \frac{1}{100}\right)$$

and then

$$\left[h^*(KL + M\|\phi\|_B)\|p_2\|_\infty\right]^{q-1} \times \left[\int_\vartheta^\delta G(t,s)\varphi_q\left(\int_\vartheta^\delta H(s,\tau)d\tau\right)ds\right] \ge L,$$

which gives

$$e^{-\frac{1}{10}(\frac{1}{100}-L)} \times 0.18383 \ge L \quad \textit{and we choose} \quad L = 0.15.$$

Also,

$$\left[q^*(Kr + M\|\phi\|_B)\|p_1\|_\infty\right]^{2-1} = \frac{1}{3}(r + \frac{1}{100})e^1$$

and

$$\frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right]\left[\frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} = 0.93333,$$

then

$$\left[q^*(Kr + M\|\phi\|_B)\|p_1\|_\infty \frac{\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{a^{p-1}}{1-a^{p-1}\xi^{\beta-1}}\right)\right]^{q-1} \frac{\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-\gamma)}\left[1+\frac{b}{d}\right] \le r,$$

which gives

$$\frac{1}{3}\left(r+\frac{1}{100}\right)e^1 \times 0.93333 \le r \textit{ and we choose } r = 0.10.$$

Also,

$$\frac{1}{3}\left(R+\frac{1}{100}\right)e^1 \times 0.93333 \le R \textit{ and we choose } R = 0.17.$$

Since all assumptions of Theorem 4.2 are satisfied, Problem (41) has three positive solutions y_1, y_2 and y_3.

Acknowledgements: The author are very appreciative of the constructive remarks by the referees.

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