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Optimal Integrability for Some Integral System of Wolff Type

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Abstract. In the paper, we obtain the optimal integrability for positive solutions of the following integral system involving Wolff potentials:

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), & x \in R^n, \\ v(x) = W_{\beta,\gamma}(u^p)(x), & x \in R^n, \end{cases}$$

where $p, q > 0, \beta > 0, \gamma > 1$ and $0 < \beta\gamma < n$. Ma, Chen and Li [*Advances in Mathematics*, 226(2011), 2676-2699] developed the regularity lifting method and obtained the optimal integrability for $p > 1,\ q > 1$. Here, based on some new observations, we overcome the difficulty there, and derive the optimal integrability for the case of $p > 0, q > 0$ and $pq > 1$. This integrability plays a key role in estimating the asymptotic behavior of positive solutions.

1. Introduction

The Wolff potential is defined for any non-negative Borel measure μ:

$$W_{\beta,\gamma}\mu(x) = \int_0^\infty \left[\frac{\mu(B_t(x))}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

where $1 < \gamma < \infty$, $0 < \beta\gamma < n$ and $B_t(x)$ is the ball of radius t centered at point x.

If $d\mu = f dx$ with $f > 0$ and $f \in L^1_{loc}(R^n)$, we write(cf.[4]):

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}}\right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

It is easy to verify that $W_{1,2}(\cdot)$ is the well-know Newton potential and $W_{\frac{\alpha}{2},2}(\cdot)$ is the Riesz potential.

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The Wolff potentials are helpful to well understand the nonlinear PDEs (cf.[7],[10],[13]). For example, $W_{1,\gamma}(w)$ and $W_{\frac{2k}{k+1},k+1}(w)$ can be used to estimate the $\mathcal{A}$-superhamonic functions involving solutions of the γ-Laplace equation

$$-div(|\nabla u|^{\gamma-2}\nabla u) = w,$$

and the k-Hessian equation

$$F_K[-u] = w, \quad k = 1, 2, \cdots, n,$$

respectively. Here

$$F_K[u] = S_k(\lambda(D^2u)), \quad \lambda(D^2u) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$$

with λ_i being eigenvalues of the Hessian matrix (D^2u), and $S_k(\cdot)$ is the k-th symmetric function:

$$S_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Two special cases are $F_1[u] = \triangle u$ and $F_n[u] = det(D^2u)$.

In this paper, we consider the following system involving Wolff type

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), \ u > 0 \ in \ R^n, \\ v(x) = W_{\beta,\gamma}(u^p)(x), \ v > 0 \ in \ R^n, \end{cases} \tag{1}$$

with $p, q, \beta > 0$, $\gamma > 1$, $\beta\gamma < n$, $pq > 1$ and

$$\frac{1}{p+\gamma-1} + \frac{1}{q+\gamma-1} = \frac{n-\beta\gamma}{n(\gamma-1)}. \tag{2}$$

In particular, when $\beta = \frac{\alpha}{2}$ and $\gamma = 2$, system (1) reduces to

$$\begin{cases} u(x) = \int_{R^n} \frac{v^q(y)}{|x-y|^{n-\alpha}} dy, \quad v > 0 \ in \ R^n; \\ v(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy, \quad u > 0 \ in \ R^n. \end{cases} \tag{3}$$

The solutions (u, v) of (3) are critical points of the functional associated with the well-known hardy-Littlewood-Sobolev inequality (see [5])

$$\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}} dxdy \le C(n, s, \alpha)\|f\|_r\|g\|_s,$$

where $f \in L^r(R^n)$, $g \in L^s(R^n)$, $0 < \alpha < n$, $s, r > 1$ such that $\frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}$, and the best constant is given by

$$C(n, s, \alpha) = \max\left\{\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}} dxdy : \|f\|_r = \|g\|_s = 1\right\}.$$

Chen, Li and Ou [2] introduce the method of moving planes in integral forms to study the symmetry of the solutions for the HLS system (3). Jin and Li [6] thoroughly discussed the regularity of the the solutions of (3)(see also [3]). They found the optimal integrability intervals in the case of $p > 1, q > 1$ and established the smoothness for the integrable solutions. Furthermore, Onodera [12] obtain the optimal integrability intervals in the case of $0 < p, q < \infty$. Based on the results, [9] gave the asymptotic behavior of the integrable solutions when $|x| \to 0$ and $|x| \to \infty$.

In the special case where $p = q = \frac{n+\alpha}{n-\alpha}$ and $u(x) = v(x)$, system (3) becomes the single integral equation

$$u(x) = \int_{R^n} \frac{u^{\frac{n+\alpha}{n-\alpha}}(y)}{|x-y|^{n-\alpha}} dy, \quad u > 0 \ in \ R^n.$$

and the equivalent PDE is the well-known family of semi-linear equations

$$(-\Delta)^{\alpha/2}u = u^{\frac{n+\alpha}{n-\alpha}}, \quad u > 0 \text{ in } R^n. \tag{4}$$

The classification of the solutions of (4) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem. It is also essential in deriving a priori estimates in many related nonlinear elliptic equations.

For the system of (1), Chen and Li [1] proved that the solutions u and v are radial symmetry and decreasing about some point x_0. Furthermore, Ma, Chen and Li thoroughly discussed the regularity of the solutions to (1) and obtained some nice results. Namely, in the case of $p > 1$ and $q > 1$, they found the optimal integrability intervals of the solutions, which is important to estimate the asymptotic rates of the solutions. Based on these results, Lei [8] obtained the decay rates of the integrable solutions when $|x| \to \infty$.

Proposition 1. ([1], Theorem 1.) *Let $1 < \gamma \le 2$. Assume that (u, v) is a pair of positive solutions of (1) with (2) and*

$$u \in L^{p+\gamma-1}(R^n), \quad v \in L^{q+\gamma-1}(R^n).$$

Then (u, v) must be radially symmetric and monotone decreasing about some point in R^n.

Proposition 2. ([11], Theorem 2.1.) *Let $(u, v) \in L^{p+\gamma-1}(R^n) \times L^{q+\gamma-1}(R^n)$ be a pair of positive solutions for system (1) in the case (2). Further assume $p > 1$, $q > 1$, and $1 < \gamma \le 2$. Without loss of generality, assume $p \le q$. Then $(u, v) \in L^r(R^n) \times L^s(R^n)$ when ever r and s are in the following rang:*

$$\left(\frac{1}{r}, \frac{1}{s}\right) \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right) \times \left(0, \min\left\{\frac{n-\beta\gamma}{n(\gamma-1)}, \frac{1}{\gamma-1}\frac{p+\gamma-1}{q+\gamma-1}\right\}\right).$$

The right end values are optimal in the sense that if $\frac{1}{r}$ or $\frac{1}{s}$ exceed the right end values, $\|u\|_r = \|v\|_s = \infty$.

For the case of $p, q > 0$, $pq > 1$ except $p > 1, q > 1$, there are some technical difficulty to derive the optimal integrability using the method in [11]. Roughly speaking, since one of the equations in (1) cannot use the smallness condition to obtain a contraction mapping which is essential for the regularity lifting method developed in [11]. In this paper, we find a way to deal with these problems and hence prove that Propositions 2 still hold for the cases $p = 1, q > 1$ or $q = 1, p > 1$, and $0 < p < 1, q > 1$ or $0 < q < 1, p > 1$. Together with the results in [11], we now know the optimal integrability for all cases $pq > 1$.

The following proposition will be used to derive the integrability intervals. The proof can be found in [11].

Let V be a topological vector space. Suppose there are two extended norms (i.e. the norm of an element in V might be infinity) defined on V,

$$\|\cdot\|_X, \ \|\cdot\|_Y : V \to [0, \infty].$$

Let

$$X := \{v \in V : \|v\|_X < \infty\} \quad \text{and} \quad Y := \{v \in V : \|v\|_Y < \infty\}.$$

Proposition 3. (Regularity lifting lemma) *Let T be a contraction map from X into itself and from Y into itself. Assume that $f \in X$, and that there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g$ in X. Then f also belongs to Z.*

Proposition 4. ([11], Corollary 2.1.) *Let $p, q > 1$, $\beta > 0$, $\gamma > 1$ and $\beta\gamma < n$, then there exists some positive constant C such that*

$$\|W_{\beta,\gamma}(f)\|_q \le C\|f\|_p^{\frac{1}{\gamma-1}}, \quad f \in L^p(R^n),$$

where $\frac{1}{p} - \frac{\gamma-1}{q} = \frac{\beta\gamma}{n}$ and $q > \gamma - 1$.

Finally, we state the main result of this paper.

Theorem 1. *Let $(u,v) \in L^{p+\gamma-1}(R^n) \times L^{q+\gamma-1}(R^n)$ be a pair of positive solutions for system (1) in the case (2). Further assume $p, q > 0$, $pq > 1$, and $1 < \gamma \le 2$. Then $(u,v) \in L^r(R^n) \times L^s(R^n)$ when ever r and s are in the following rang:*
(i) when $p \le q$,

$$\left(\frac{1}{r}, \frac{1}{s}\right) \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right) \times \left(0, \min\left\{\frac{n-\beta\gamma}{n(\gamma-1)}, \frac{1}{\gamma-1}\frac{p+\gamma-1}{q+\gamma-1}\right\}\right);$$

(ii) when $p > q$,

$$\left(\frac{1}{r}, \frac{1}{s}\right) \in \left(0, \min\left\{\frac{n-\beta\gamma}{n(\gamma-1)}, \frac{1}{\gamma-1}\frac{q+\gamma-1}{p+\gamma-1}\right\}\right) \times \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right).$$

The right end values are optimal in the sense that if $\frac{1}{r}$ or $\frac{1}{s}$ exceed the right end values, $\|u\|_r = \|v\|_s = \infty$.

2. Proof of Theorem 1.

From Proposition 2, we can see that the case of $p > 1,\ q > 1$ is proved by Ma, Chen and Li. Therefore, in this section, we derive our result in two cases: the first step proves the case of $p = 1, q > 1$ and $q = 1, p > 1$, the second step proves the case of $0 < p < 1, q > 1$ and $0 < q < 1, p > 1$.

Case I. We prove the case of $p = 1, q > 1$ and $q = 1, p > 1$. Without loss of generality, we assume that $p = 1, q > 1$.

Step i. Estimate of v.
Set $r_0 = p + \gamma - 1 = \gamma$, $s_0 = q + \gamma - 1$, and let s satisfy

$$\frac{1}{s} \in \left(0, \frac{2}{s_0}\right). \tag{5}$$

Define

$$T_1 g := \int_0^\infty \left(\frac{\int_{B_t(x)} v^q dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} v_A^{q-1} g dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t},$$

and

$$T_2 f := \int_0^\infty \left(\frac{\int_{B_t(x)} u dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} f dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t},$$

where

$$v_A(x) = \begin{cases} v(x), & if\ v(x) \ge A \ or\ |x| \ge A; \\ 0, & otherwise. \end{cases} \tag{6}$$

For any $g \in L^s(R^n)$, we define

$$T_A g = T_2(T_1 g), \quad F = T_2(F_0),$$

with

$$F_0 := \int_0^\infty \left(\frac{\int_{B_t(x)} v^q dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} (v - v_A)^q dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t}.$$

Next, we estimate $T_1 g$ and $T_2 f$.
By the Hölder inequality, we have

$$|T_2 f| \le v^{2-\gamma} (T_2^0 f)^{\gamma-1},$$

where

$$T_2^0 f = \int_0^\infty \left(\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

Consequently,

$$\|T_2 f\|_s \le C\|v\|_{s_0}^{2-\gamma} \|T_2^0 f\|_{\bar{s}}^{\gamma-1},$$

with $\frac{1}{s} = \frac{\gamma-1}{\bar{s}} + \frac{2-\gamma}{s_0}$. Using Proposition 4, we obtain

$$\|T_2 f\|_s \le C\|v\|_{s_0}^{2-\gamma} \|f\|_{\frac{n\bar{s}}{n(\gamma-1)+\beta\gamma\bar{s}}}. \tag{7}$$

Write

$$r = \frac{n\bar{s}}{n(\gamma-1)+\beta\gamma\bar{s}}. \tag{8}$$

Similarly, we have

$$|T_1 g| \le u^{2-\gamma} (T_1^0 g)^{\gamma-1},$$

where

$$T_1^0 g = \int_0^\infty \left(\frac{\int_{B_t(x)} v_A^{q-1} g dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

Therefore,

$$\|T_1 g\|_r \le C\|u\|_{r_0}^{2-\gamma} \|T_1^0 g\|_{\bar{r}}^{\gamma-1},$$

with $\frac{1}{r} = \frac{\gamma-1}{\bar{r}} + \frac{2-\gamma}{r_0}$. Using Proposition 4, we obtain

$$\begin{aligned} \|T_1 g\|_r \le& C\|u\|_{r_0}^{2-\gamma} \|v_A^{q-1} g\|_{\frac{n\bar{r}}{n(\gamma-1)+\beta\gamma\bar{r}}} \\ \le& C\|u\|_{r_0}^{2-\gamma} \|v_A\|_{s_0}^{q-1} \|g\|_s, \end{aligned} \tag{9}$$

where $\frac{\gamma-1}{\bar{r}} + \frac{\beta\gamma}{n} = \frac{q-1}{s_0} + \frac{1}{s}$ and $\frac{\gamma-1}{\bar{r}} < 1 - \frac{\beta\gamma}{n}$.

Combining (7) with (9), we derive

$$\|T_A g\|_s = \|T_2(T_1 g)\|_s \le C\|u\|_{r_0}^{2-\gamma} \|v\|_{s_0}^{2-\gamma} \|v_A\|_{s_0}^{q-1} \|g\|_s. \tag{10}$$

Noting that $u \in L^{r_0}(R^n)$ and $v \in L^{s_0}(R^n)$, we obtain a smallness condition

$$C\|u\|_{r_0}^{2-\gamma} \|v\|_{s_0}^{2-\gamma} \|v_A\|_{s_0}^{q-1} \le \frac{1}{2}$$

when A is sufficiently large.

Inserting this smallness condition into (10), we see that T_A is a contraction from $L^s(R^n)$ to $L^s(R^n)$. In addition, we can see that T_A is also a contraction from $L^{s_0}(R^n)$ to $L^{s_0}(R^n)$, since (5) holds. It is easy to verify that v solves the operator equation

$$g = T_A g + F.$$

Furthermore, according to the definition of F, we know that $F \in L^s(R^n)$. Take $X = L^{s_0}(R^n)$, $Y = Z = L^s(R^n)$ in Proposition 3. Thus, by regularity lifting lemma, we see that

$$v \in L^s(R^n), \quad \forall \frac{1}{s} \in \left(0, \frac{2}{s_0}\right). \tag{11}$$

Step ii. Estimate of u.

Once the integrability of v is obtained, we can use do similar discuss to the integral equation (1) to estimate the integrability of u.

Let

$$0 < \frac{1}{s} < \frac{2}{s_0}.$$

From (8), we have

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{r_0} - \frac{1}{s_0}. \tag{12}$$

Therefore, we can use Proposition 4 and Hölder inequality to obtain that

$$\begin{aligned}\|u\|_r \leq & C\|u\|_{r_0}^{2-\gamma}\|u\|_{\bar{r}}^{\gamma-1} \leq C\|u\|_{r_0}^{2-\gamma}\|v^q\|_{\frac{n\bar{r}}{n(\gamma-1)+\beta\gamma\bar{r}}}\\ \leq & C\|u\|_{r_0}^{2-\gamma}\|v\|_{s_0}^{q-1}\|v\|_s,\end{aligned}$$

where

$$\frac{1}{r} = \frac{2-\gamma}{r_0} + \frac{\gamma-1}{\bar{r}} \quad \mathit{and} \quad \frac{\gamma-1}{\bar{r}} + \frac{\beta\gamma}{n} = \frac{q-1}{s_0} + \frac{1}{s}.$$

Inserting (11) into (12), from the inequality above, we deduce that

$$u \in L^r(R^n), \quad \forall \frac{1}{r} \in \left(\frac{1}{r_0} - \frac{1}{s_0}, \frac{n-\beta\gamma}{n(\gamma-1)}\right). \tag{13}$$

Step iii. To extend the left-end point of the interval in (13), we apply Proposition 4 to system (1). We have

$$\|u\|_r = \|W_{\beta,\gamma}(v^q)\|_r \leq C\|v^q\|_{\frac{nr}{n(\gamma-1)+\beta\gamma r}}^{\frac{1}{\gamma-1}} \tag{14}$$

provided

$$\frac{nr}{n(\gamma-1)+\beta\gamma r} > 1, \tag{15}$$

that is

$$\frac{1}{r} < \frac{n-\beta\gamma}{n(\gamma-1)}.$$

In order the right-hand side of (14) to be finite, we only need

$$0 < \frac{n(\gamma-1)+\beta\gamma r}{nqr} < \frac{2}{s_0} = \frac{2}{q+\gamma-1}.$$

and this is indeed true under conditions (15), since $\gamma - 1 < 1$, and $q > 1$. Thus, we deduce that

$$u \in L^r(R^n), \quad \forall \frac{1}{r} \in \left(0, \frac{n-\beta\gamma}{n(\gamma-1)}\right). \tag{16}$$

Similarly, applying proposition 4 to equation (1) with $p = 1$, we obtain

$$\|v\|_s = \|W_{\beta,\gamma}(u)\|_s \leq C\|u\|_{\frac{ns}{n(\gamma-1)+\beta\gamma s}}^{\frac{1}{\gamma-1}}.$$

This result, together with (16), implies

$$v \in L^s(R^n), \quad \forall \frac{1}{s} \in \left(0, \min\left\{\frac{n-\beta\gamma}{n(\gamma-1)}, \frac{1}{\gamma-1}\frac{\gamma}{q+\gamma-1}\right\}\right).$$

This is the integrability interval of v in Theorem 1 when $p = 1$.

Case II. We prove the case of $0 < p < 1, q > 1$ and $0 < q < 1, p > 1$. Without loss of generality, we assume that $0 < p < 1, q > 1$.

Step i. Since $pq > 1$, then there exists a $\rho > 0$ such that

$$1 < \frac{1}{p} < \rho < q.$$

Here ρ will be determined later.

Define

$$T_1^{\rho} g(x) := \int_0^{\infty} \left(\frac{\int_{B_t(x)} v^q dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} v_A^{q-\rho} g^{\rho} dy}{t^{n-\beta\gamma}} \right) \frac{dt}{t},$$

and

$$T_2^{\rho} f(x) := \int_0^{\infty} \left(\frac{\int_{B_t(x)} u^p dy}{t^{n-\beta\gamma}} \right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} u_A^{p-\frac{1}{\rho}} f^{\frac{1}{\rho}} dy}{t^{n-\beta\gamma}} \right) \frac{dt}{t},$$

where the definition of v_A, u_A is similar as (6).

Next, we estimate $T_1^{\rho} g(x)$ and $T_2^{\rho} f(x)$.

By the Hölder inequality, we have

$$|T_1^{\rho} g| \leq u^{2-\gamma} (T_1^{\rho,0} g)^{\gamma-1},$$

where

$$T_1^{\rho,0} g = \int_0^{\infty} \left(\frac{\int_{B_t(x)} v_A^{q-\rho} g^{\rho} dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

Consequently,

$$\|T_1^{\rho} g\|_r \leq C \|u\|_{r_0}^{2-\gamma} \|T_1^{\rho,0} g\|_{\bar{r}}^{\gamma-1},$$

where

$$\frac{1}{r} = \frac{2-\gamma}{r_0} + \frac{\gamma-1}{\bar{r}}. \tag{17}$$

Using Proposition 4 and the Hölder inequality, we deduce that

$$\begin{aligned} \|T_1^{\rho} g\|_r \leq & C \|u\|_{r_0}^{2-\gamma} \|v_A^{q-\rho} g^{\rho}\|_{\frac{n\bar{r}}{n(\gamma-1)+\beta\gamma\bar{r}}} \\ \leq & C \|u\|_{r_0}^{2-\gamma} \|v_A\|_{s_0}^{q-\rho} \|g\|_s^{\rho}, \end{aligned} \tag{18}$$

where

$$\frac{q-\rho}{s_0} + \frac{\rho}{s} = \frac{\gamma-1}{\bar{r}} + \frac{\beta\gamma}{n}, \tag{19}$$

and

$$\frac{\gamma-1}{\bar{r}} < 1 - \frac{\beta\gamma}{n}. \tag{20}$$

Similarly, we have

$$|T_2^{\rho} f| \leq v^{2-\gamma} (T_2^{\rho,0} f)^{\gamma-1},$$

where

$$T_2^{\rho,0} f = \int_0^{\infty} \left(\frac{\int_{B_t(x)} u_A^{p-\frac{1}{\rho}} f^{\frac{1}{\rho}} dy}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

By the Hölder inequality and Proposition 4, we derive

$$\begin{aligned}\|T_2^{\rho} f\|_s \leq & C\|v\|_{s_0}^{2-\gamma}\|T_2^{\rho,0} f\|_{\bar{s}}^{\gamma-1} \leq C\|v\|_{s_0}^{2-\gamma}\|u_A^{p-\frac{1}{\rho}} f^{\frac{1}{\rho}}\|_{\frac{n\bar{s}}{n(\gamma-1)+\beta\gamma\bar{s}}} \\ \leq & C\|v\|_{s_0}^{2-\gamma}\|u_A\|_{r_0}^{p-\frac{1}{\rho}}\|f\|_r^{\frac{1}{\rho}},\end{aligned} \tag{21}$$

where

$$\frac{1}{s} = \frac{2-\gamma}{s_0} + \frac{\gamma-1}{\bar{s}}, \tag{22}$$

$$\frac{p-\frac{1}{\rho}}{r_0} + \frac{\frac{1}{\rho}}{r} = \frac{\gamma-1}{\bar{s}} + \frac{\beta\gamma}{n}, \tag{23}$$

$$\frac{\gamma-1}{\bar{s}} < 1 - \frac{\beta\gamma}{n}. \tag{24}$$

Set $r_0 = p+\gamma-1$, $s_0 = q+\gamma-1$, then by (2), both conditions (19) and (23) become

$$\frac{1}{r} - \frac{1}{r_0} = \rho\left(\frac{1}{s} - \frac{1}{s_0}\right),$$

and the set of conditions (17)-(24) can now be simplified as

$$\frac{1}{r} = \frac{2-\gamma}{r_0} + \frac{\gamma-1}{\bar{r}}, \quad \frac{1}{s} = \frac{2-\gamma}{s_0} + \frac{\gamma-1}{\bar{s}}, \tag{25}$$

$$\frac{1}{r} - \frac{1}{r_0} = \rho\left(\frac{1}{s} - \frac{1}{s_0}\right), \tag{26}$$

$$\frac{\gamma-1}{\bar{r}} < 1 - \frac{\beta\gamma}{n}, \quad \frac{\gamma-1}{\bar{s}} < 1 - \frac{\beta\gamma}{n}. \tag{27}$$

In order to handle the smallness condition, we consider the following operators $T_1^{\rho,A}$, $T_2^{\rho,A}$:

$$T_1^{\rho,A} g(x) := T_1^{\rho} g(x) + \int_0^\infty \left(\frac{\int_{B_t(x)} v^q dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} (v-v_A)^q dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t},$$

$$T_2^{\rho,A} f(x) := T_2^{\rho} f(x) + \int_0^\infty \left(\frac{\int_{B_t(x)} u^p dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} (u-u_A)^p dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t}.$$

Clearly, we can see that

$$T_2^{\rho,A} T_1^{\rho,A} v = v \quad and \quad T_1^{\rho,A} T_2^{\rho,A} u = u. \tag{28}$$

Next, we prove that, when $\rho > 1$, the mapping $T_2^{\rho,A} T_1^{\rho,A}$ becomes a contraction by taking A sufficiently large. By the sample fact that $(a+c)^{1/\rho} - (b+c)^{1/\rho} \leq (a)^{1/\rho} - (b)^{1/\rho}$ for $a \geq b \geq 0, c \geq 0$ and the Minkowski inequality, we see that

$$\begin{aligned}&| (T_1^{\rho,A} g_1(x))^{\frac{1}{\rho}} - (T_1^{\rho,A} g_2(x))^{\frac{1}{\rho}} | \\ &\leq \left(\int_0^\infty \left(\frac{\int_{B_t(x)} v^q dy}{t^{n-\beta\gamma}}\right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{\int_{B_t(x)} v_A^{q-1} |g_1 - g_2|^\rho dy}{t^{n-\beta\gamma}}\right) \frac{dt}{t}\right)^{\frac{1}{\rho}}.\end{aligned}$$

In view of the inequalities (18) and (21), it then follows that

$$
\begin{aligned}
&\|T_2^{\rho,A}T_1^{\rho,A}g_1(x) - T_2^{\rho,A}T_1^{\rho,A}g_2(x)\|_s \\
&\quad \le C\|v\|_{s_0}^{2-\gamma}\|u_A\|_{r_0}^{p-\frac{1}{\rho}}\|(T_1^{\rho,A}g_1(x))^{\frac{1}{\rho}} - (T_1^{\rho,A}g_2(x))^{\frac{1}{\rho}}\|_{\rho r} \\
&\quad \le C\|v\|_{s_0}^{2-\gamma}\|u_A\|_{r_0}^{p-\frac{1}{\rho}}\|u\|_{r_0}^{\frac{2-\gamma}{\rho}}\|v_A\|_{s_0}^{\frac{q-\rho}{\rho}}\|g_1 - g_2\|_s \\
&\quad \le \frac{1}{2}\|g_1 - g_2\|_s.
\end{aligned}
$$

Here the last inequality holds if A is sufficiently large.

Step ii. Since we assume $p \le q$, then $\frac{1}{r_0} - \frac{1}{s_0}$ is positive. We consider a co-ordinate plane with $\frac{1}{r}$ as its horizontal co-ordinate and $\frac{1}{s}$ as the vertical co-ordinate. Then (26) represents a line on this plane. Let $\mathfrak{L}$ denote part of this line which is diagonal to the open square

$$
\mathfrak{B} := \left(\frac{1}{r_0} - \frac{\rho}{s_0}, \frac{1}{r_0} + \frac{1}{s_0}\right) \times \left(0, \frac{\frac{1}{\rho}+1}{s_0}\right),
$$

here we take $\rho \le s_0/r_0$. Let

$$
\mathfrak{B}_1 := \left(\frac{1}{r_0} - \rho\frac{\gamma-1}{s_0}, \frac{1}{r_0} + \frac{\gamma-1}{s_0}\right) \times \left(\frac{2-\gamma}{s_0}, \frac{1}{s_0} + \frac{1}{\rho}\frac{\gamma-1}{s_0}\right)
$$

be a sub-square of $\mathfrak{B}$ with the same center.

Next, we will show that $v \in L^s(R^n)$ for any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{L}_1$, a diagonal of $\mathfrak{B}_1$ and a subset of $\mathfrak{L}$. Then we will extend this result to $\mathfrak{B}$ through $\mathfrak{L}$. Once we show that $\left(\frac{1}{r}, \frac{1}{s}\right)$ belongs to a diagonal, then we can immediately extend this result to the whole square by interpolations. Hence, in the following, we only need to show that $v \in L^s(R^n)$ when $\left(\frac{1}{r}, \frac{1}{s}\right)$ belongs to $\mathfrak{L}$.

For any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{L}_1$, one can find $\bar{r}$ and $\bar{s}$, so that all conditions (25)-(27) are met, hence $T_2^{\rho,A}T_1^{\rho,A}$ is a contraction. Since v satisfies Eq.(28), and

$$
\left(\frac{1}{r_0}, \frac{1}{s_0}\right) \in \mathfrak{L}_1.
$$

We take $X = L^{s_0}(R^n)$, $Y = Z = L^s(R^n)$ for any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{L}_1$, by the regularity lifting lemma (Proposition 3), we can obtain that $v \in L^s(R^n)$ for any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{L}_1$. Furthermore, by interpolations, v also belongs to $L^s(R^n)$ for any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{B}_1$.

In order to prove that $v \in L^s(R^n)$ for any $\left(\frac{1}{r}, \frac{1}{s}\right) \in \mathfrak{L}$, we apply Proposition 4 to derive

$$
\|v\|_{s^*} \le C\|u\|_{\frac{pns^*}{n(\gamma-1)+\beta\gamma s^*}}^{\frac{p}{\gamma-1}} = C\|u\|_r^{\frac{p}{\gamma-1}},
$$

where

$$
\frac{\gamma-1}{s^*} = \frac{p}{r} - \frac{\beta\gamma}{n}. \tag{29}
$$

Similarly, we have

$$
\|u\|_{r^*} \le C\|v\|_s^{\frac{q}{\gamma-1}}
$$

with condition

$$
\frac{\gamma-1}{r^*} = \frac{q}{s} - \frac{\beta\gamma}{n}. \tag{30}
$$

Condition (29) and (30) together with (26) are equivalent to

$$\frac{1}{s^*} - \frac{1}{s_0} = \frac{p}{\gamma - 1}\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{p\rho}{\gamma - 1}\left(\frac{1}{s} - \frac{1}{s_0}\right),$$

$$\frac{1}{r^*} - \frac{1}{r_0} = \frac{q}{\gamma - 1}\left(\frac{1}{s} - \frac{1}{s_0}\right) = \frac{q}{\rho(\gamma - 1)}\left(\frac{1}{r} - \frac{1}{r_0}\right).$$

Notice that both $\frac{p\rho}{\gamma-1}$ and $\frac{q}{\rho(\gamma-1)}$ are greater than 1, we can extend the range of $\frac{1}{r}$ and $\frac{1}{s}$ through the two equations above. Thus, we can extend the range where v "belongs" to from $\mathfrak{L}_1$ to $\mathfrak{L}$. Hence, we obtain

$$v \in L^s(R^n), \quad \forall \frac{1}{s} \in \left(0, \frac{\frac{1}{\rho} + 1}{s_0}\right). \tag{31}$$

Step iii. To extend the right-end point of the interval in (31), Applying proposition 4 to equation (1), we obtain

$$\|v\|_s = \|W_{\beta,\gamma}(u^p)\|_s \leq C\|u^p\|^{\frac{1}{\gamma-1}}_{\frac{ns}{n(\gamma-1)+\beta\gamma s}} \leq C\|u\|_r^{\frac{p}{\gamma-1}}, \tag{32}$$

where

$$\frac{\gamma - 1}{s} = \frac{p}{r} - \frac{\beta\gamma}{n}.$$

This result, together with

$$\frac{1}{r} \in \left(\frac{1}{r_0} - \frac{\rho}{s_0}, \frac{1}{r_0} + \frac{1}{s_0}\right),$$

which implies

$$0 < \frac{1}{s} < \frac{1}{\gamma - 1}\frac{p + \gamma - 1}{q + \gamma - 1}. \tag{33}$$

Furthermore, (32) provided

$$\frac{ns}{n(\gamma - 1) + \beta\gamma s} > 1$$

that is

$$\frac{1}{s} < \frac{n - \beta\gamma}{n(\gamma - 1)}. \tag{34}$$

Combining (33) with (34), we have

$$v \in L^s(R^n), \quad \forall \frac{1}{s} \in \left(0, \min\left\{\frac{n - \beta\gamma}{n(\gamma - 1)}, \frac{1}{\gamma - 1}\frac{p + \gamma - 1}{q + \gamma - 1}\right\}\right).$$

This is the integrability interval of v in Theorem 1.

Similarly, we have

$$u \in L^r(R^n), \quad \forall \frac{1}{r} \in \left(0, \frac{n - \beta\gamma}{n(\gamma - 1)}\right).$$

The proof of $\|u\|_r = \|v\|_s = \infty$ when $\frac{1}{r} \geq \frac{n-\beta\gamma}{n(\gamma-1)}$ or $\frac{1}{s} \geq \min\left\{\frac{n-\beta\gamma}{n(\gamma-1)}, \frac{1}{\gamma-1}\frac{\gamma}{q+\gamma-1}\right\}$ is the same as in [11]. □

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