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Multivalued Hardy-Rogers Type $\mathfrak{Z}_\Theta$-Contraction and Generalized Simulation Functions

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Abstract. The purpose of this paper is to introduce the notion of multivalued Hardy-Rogers $\mathfrak{Z}_\Theta$-contraction in the sense of generalized simulation functions and to present the corresponding fixed point results with some examples. Moreover, we study the strict fixed point and well-posedness, data dependence, as well as, the Ulam-Hyres stability of the fixed point problem. As an application, we prove the existence of the solution for nonlinear fractional differential equation involving Caputo fractional derivative.

1. Introduction

A wide variety of mathematical and practical problems can be solved by reducing them to an equivalent fixed point problem. In fact, by introducing suitable operators, it is possible to solve an equilibrium problem by searching the fixed points of such operators. Moreover, the solutions of differential equations can be obtained in terms of fixed points of integro-differential operator, also the above solutions sets can be characterized by a stability analysis of fixed points sets. These facts are sufficient motivations to increase the interest of mathematicians to establishing extensions and generalizations of the celebrated Banach fixed point theorem [4], which is universally recognized as the fundamental result of metric fixed point theory, see also[15, 23, 24]. In this paper, we continue this study by stating existence of fixed point theorems for multivalued operators, in the setting of complete metric spaces. More precisely, we work with Hardy-Rogers type conditions which present one of the most interesting generalizations of Banach fixed point theorem. We combine the original idea of Hardy-Rogers [7] with the recent concept of Θ-contraction provided by Jleli and Samet [8], by involving generalized class of simulation functions [2, 11, 12, 14, 17, 25].

2. Perliminaries

Let $(\aleph, \wp)$ be a complete metric space. $\mathcal{P}(\aleph)$ denotes the family of all non-empty, $\mathcal{CL}(\aleph)$, the family of all non-empty, closed and $\mathcal{CB}(\aleph)$, the family of closed and bounded subsets of $\aleph$.
Let $\mathcal{A}$ be a non-empty subset of a metric space $\aleph$. For $x \in \aleph$, define

$$\mathcal{D}(x, \mathcal{A}) = \inf\{\wp(x, y);\ y \in \mathcal{A}\}.$$

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Suppose that $\mathcal{A}, \mathcal{B}$ are two subsets of $C\mathcal{B}(\aleph)$. We define the functional gap $\delta : \mathcal{P}(\aleph) \times \mathcal{P}(\aleph) \to \mathbb{R}_+$, by

$$\delta(\mathcal{A}, \mathcal{B}) = \sup\{\mathcal{D}(x, \mathcal{B});\ x \in \mathcal{A}\}.$$

The Pompeiu-Hausdorff functional, $\mathfrak{H} : \mathcal{P}(\aleph) \times \mathcal{P}(\aleph) \to \mathbb{R}_+$ is defined as

$$\mathfrak{H}(\mathcal{A}, \mathcal{B}) := \max\{\sup_{a \in \mathcal{A}} \mathcal{D}(a, \mathcal{B}), \sup_{b \in B} \mathcal{D}(b, \mathcal{A})\}.$$

Hardy and Rogers [7], proved the following important result:

Theorem 2.1. *[7] Let $(\aleph, \wp)$ be a complete metric space and $\Im$ a self mapping on $\aleph$ satisfying the following condition for $x, y \in \aleph$*

$$\wp(\Im x, \Im y) \leq \alpha\, \wp(x, y) + \beta\, \wp(x, \Im x) + \gamma\, \wp(y, \Im y) + \delta\, \wp(x, \Im y) + L\, \wp(y, \Im x), \tag{1}$$

where $\alpha, \beta, \gamma, \delta, L$ are nonnegative and $\alpha + \beta + \gamma + \delta + L < 1$. Then $\Im$ has a unique fixed point.

Jleli and Samet [8] introduced the following class of functions:

Definition 2.2. *[8] Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying*

(Θ_1) *Θ is increasing;*

(Θ_2) *for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$,*

$$\lim_{n\to\infty} \Theta(\alpha_n) = 1 \textit{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

(Θ_3) *there exist $k \in (0, 1)$ and $l \in (0, \infty)$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.*

The class of functions Θ satisfying $(\Theta_1 - \Theta_3)$ is denoted by Ψ.

Using the function Θ, Jleli and Samet in [8] defined that "A self map $\Im$ on a complete metric space along with a function $\Theta \in \Psi$ and $k \in (0, 1)$ satisfying for all $x, y \in \aleph$,

$$\wp(\Im x, \Im y) > 0 \Rightarrow \Theta(\wp(\Im x, \Im y)) \leq [\Theta(\wp(x, y))]^k$$

possesses a unique fixed point". Such mappings named as Θ-contractions. Later, Xin-dong Liu *et al.* [13] proved some fixed point theorems for Θ-type Suzuki contractions. In 2017, Ahmad *et al.* [1] extended the results of Jleli and Samet [8] by replacing (Θ_3) with

(Θ_3') Θ is continuous on $(0, \infty)$,

and proved some fixed point theorems for Suzuki-Berinde type Θ-contractions.
The class of functions Θ satisfying $(\Theta_1, \Theta_2, \Theta_3')$ will be denoted by Ω.

Khojasteh *et al.* [11] introduced a function $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, satisfying:

(η_1) $\eta(0, 0) = 0$;

(η_2) $\eta(t, s) < s - t$ for all $t, s > 0$;

(η_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} sup\ \eta(t_n, s_n) < 0.$$

Definition 2.3. *[11] Let $(\aleph, \wp)$ be a metric space, $\Im : \aleph \to \aleph$ a mapping and η a simulation function. Then $\Im$ is called a $\mathfrak{Z}$-contraction with respect to η if it satisfies*

$$\eta(\wp(\Im x, \Im y), \wp(x, y)) \geq 0 \ \textit{ for all } x, y \in \aleph.$$

Example 2.4. *[11] Let* $\eta_i : [0,\infty)\times[0,\infty) \to \mathbb{R},\ \ i = 1,2,3$ *be defined by*

(*i*) $\eta_1(t,s) = \lambda s - t$, *where* $\lambda \in (0,1)$;

(*ii*) $\eta_2(t,s) = s\varphi(s) - t$, *where* $\varphi : [0,\infty) \to [0,1)$ *is a mapping such that* $\lim\limits_{t\to r^+} sup\ \psi(t) < 1$ *for all* $r > 0$;

(*iii*) $\eta_3 = s - \psi(s) - t$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\psi(t) = 0$ *if and only if* $t = 0$.

Then η_i *for* $i = 1,2,3$ *are simulation functions.*

Rold*á*n-L*ó*pez-de-Hierro *et al.* [25] modified the notion of a simulation function by replacing (η_3) by (η_3'),

(η_3') if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$ and $t_n < s_n$, then

$$\lim_{n\to\infty} \sup \eta(t_n, s_n) < 0.$$

The class of functions $\eta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying $(\eta_1, \eta_2, \eta_3')$ is called simulation function in the sense of Rold*á*n-L*ó*pez-de-Hierro and we denote it by Δ.

Definition 2.5. *[2] A mapping* $\mathfrak{G} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *is called a* $\mathfrak{C}$*-class function if* $\mathfrak{G}$ *is continuous and satisfies the following conditions:*

(1) $\mathfrak{G}(s,t) \le s$;

(2) $\mathfrak{G}(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$, *for all* $s,t \in [0,+\infty)$.

Definition 2.6. *[14] A* $\mathfrak{C}$*-class function* $\mathfrak{G} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *has the property* $\mathfrak{C}_{\mathfrak{G}}$, *if there exists* $\mathfrak{C}_{\mathfrak{G}} \ge 0$ *such that*

$(\mathfrak{G}_1)$ $\mathfrak{G}(s,t) > \mathfrak{C}_{\mathfrak{G}}$ *implies* $s > t$;

$(\mathfrak{G}_2)$**)** $\mathfrak{G}(s,t) \le \mathfrak{C}_{\mathfrak{G}}$, *for all* $s \in [0,+\infty)$.

Some examples of $\mathfrak{C}$-class functions that have property $\mathfrak{C}_{\mathfrak{G}}$ are as follows:

(a) $\mathfrak{G}(s,t) = s - t,\ \mathfrak{C}_{\mathfrak{G}} = r, r \in [0,+\infty)$;

(b) $\mathfrak{G}(s,t) = s - \frac{(2+t)t}{(1+t)}, \mathfrak{C}_{\mathfrak{G}} = 0$;

(c) $\mathfrak{G}(s,t) = \frac{s}{1+kt}, k \ge 1, \mathfrak{C}_{\mathfrak{G}} = \frac{r}{1+k}, r \ge 2$.

For more examples of $\mathfrak{C}$-class functions that have property $\mathfrak{C}_{\mathfrak{G}}$ see [3, 6, 14].

Definition 2.7. *[14] A* $\mathfrak{C}_{\mathfrak{G}}$ *simulation function is a mapping* $\eta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *satisfying the following conditions:*

(1) $\eta(t,s) < \mathfrak{G}(s,t)$ *for all* $t,s > 0$, *where* $\mathfrak{G} : [0,+\infty)^2 \to \mathbb{R}$ *is a* $\mathfrak{C}$*-class function;*

(2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,+\infty)$ *such that* $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, *and* $t_n < s_n$, *then* $\lim\limits_{n\to\infty} \sup \eta(t_n, s_n) < \mathfrak{C}_{\mathfrak{G}}$.

Some examples of simulation functions and $\mathfrak{C}_{\mathfrak{G}}$-simulation functions are:

(a) $\eta(t,s) = \frac{s}{s+1} - t$ for all $t,s > 0$.

(b) $\eta(t,s) = s - \phi(s) - t$ for all $t,s > 0$, where $\phi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\phi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and $\mathfrak{C}_{\mathfrak{G}}$-simulation functions see [3, 11, 14, 17, 25, 26].

3. Main results

We begin with the following definition:

Definition 3.1. *Let $(\aleph, \wp)$ be a metric space, $\eta \in \Delta$ and $\Theta \in \Omega$. A mapping*

(i) $\mathfrak{J} : \aleph \to \aleph$ *is called Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction if there is $k \in (0,1)$ such that for all $x, y \in \aleph$*

$$\eta(\Theta(\wp(\mathfrak{J}x, \mathfrak{J}y)), (\Theta(\mathfrak{R}(x,y)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{2}$$

where

$$\mathfrak{R}(x,y) = \alpha\, \wp(x,y) + \beta\, \wp(x, \mathfrak{J}x) + \gamma\, \wp(y, \mathfrak{J}y) + \delta\, \wp(x, \mathfrak{J}y) + L\, \wp(y, \mathfrak{J}x),$$

$\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ *and* $L \geq 0$.

(ii) $\mathfrak{J} : \aleph \to \mathcal{CB}(\aleph)$ *is called multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction if there is $k \in (0,1)$ such that for all $x, y \in \aleph$*

$$\eta(\Theta(\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y)), (\Theta(\mathfrak{R}(x,y)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{3}$$

where

$$\mathfrak{R}(x,y) = \alpha\, \wp(x,y) + \beta\, \mathcal{D}(x, \mathfrak{J}x) + \gamma\, \mathcal{D}(y, \mathfrak{J}y) + \delta\, \mathcal{D}(x, \mathfrak{J}y) + L\, \mathcal{D}(y, \mathfrak{J}x),$$

$\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ *and* $L \geq 0$.

Example 3.2. *Let $\aleph = [0,1]$ and $\wp(x,y) = |x-y|$. Define*

$$\mathfrak{J}x = \left[\frac{1-x}{2}, \frac{2-x}{2}\right] \tag{4}$$

for all $x, y \in \aleph$. Let $\eta(t,s) = \frac{15}{16}s - t$, $\mathfrak{G}(s,t) = s - t$ for all $s, t \in [0, \infty)$, $\mathfrak{C}_{\mathfrak{G}} = 0$ and $\Theta : (0,\infty) \to (1,\infty)$ is defined by $\Theta(t) = e^t$. Then for all $x, y \in \aleph$ and $k \in (0,1)$, we have that $\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y) = \frac{|x-y|}{2}$ and

$$\mathfrak{R}(x,y) = \frac{2\alpha\, |x-y| + \beta\, |3x-1| + \gamma\, |3y-1| + \delta\, |2x-y| + L\, |2y-x|}{2}.$$

This gives

$$\eta(\Theta(\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y)), (\Theta(\mathfrak{R}(x,y)))^k) = \frac{15}{16}\, (e^{\mathfrak{R}(x,y)})^k - e^{\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y)}$$

and

$$\mathfrak{G}((\Theta(\mathfrak{R}(x,y)))^k, \Theta(\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y))) = (e^{\mathfrak{R}(x,y)})^k - e^{\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y)}.$$

Taking nonnegative values of $\alpha, \beta, \gamma, \delta, L$ such that $\alpha + \beta + \gamma + 2\delta = 1, L \geq 0$ we get that

$$0 \leq \eta(\Theta(\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y)), \Theta(\mathfrak{R}(x,y)))^k) < \mathfrak{G}(\Theta(\mathfrak{R}(x,y)))^k, \Theta(\mathfrak{H}(\mathfrak{J}x, \mathfrak{J}y))). \tag{5}$$

Hence, from (5) *it is clear that $\mathfrak{J}$ is multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction.*

Theorem 3.3. *Let $(\aleph, \wp)$ be a complete metric space and $\mathfrak{J} : \aleph \to \mathcal{CB}(\aleph)$ be a multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction. Then $\mathfrak{J}$ possesses a fixed point.*

Proof. *Define a sequence $\{x_n\}$ in $\aleph$ by $x_{n+1} \in \mathfrak{J}x_n$ for all $n \geq 0$. If there exists an n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of $\mathfrak{J}$. Consequently, assume that $x_n \neq x_{n+1}$ for all n, then $\wp(x_n, x_{n+1}) > 0$, for all $n = 0, 1, \ldots$. Taking $x = x_{n-1}$ and $y = x_n$ in* (3), *we get*

$$\eta(\Theta(\mathfrak{H}(\mathfrak{J}x_{n-1}, \mathfrak{J}x_n)), (\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k) \geq \mathfrak{C}_{\mathfrak{G}}.$$

Since $\mathfrak{I}$ is multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction, we have

$$\begin{aligned}\mathfrak{C}_{\mathfrak{G}} &\leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1},\mathfrak{I}x_n)),(\Theta(\mathfrak{R}(x_{n-1},x_n))^k)\\ &< \mathfrak{G}(\Theta(\mathfrak{R}(x_{n-1},x_n))^k,\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1},\mathfrak{I}x_n))).\end{aligned}$$

Using $\mathfrak{G}_1$, we get

$$\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1},\mathfrak{I}x_n)) < (\Theta(\mathfrak{R}(x_{n-1},x_n)))^k. \tag{6}$$

Since, we have $\mathcal{D}(x_n,\mathfrak{I}x_n) \leq \mathfrak{H}(\mathfrak{I}x_{n-1},\mathfrak{I}x_n)$. Also by using (6), we obtain

$$\begin{aligned}\Theta(\mathcal{D}(x_n,\mathfrak{I}x_n)) &\leq \Theta(\mathfrak{H}(\mathfrak{I}x_{n-1},\mathfrak{I}x_n))\\
&< (\Theta(\mathfrak{R}(x_{n-1},x_n)))^k\\
&= (\Theta(\alpha\ \wp(x_{n-1},x_n)+\beta\ \mathcal{D}(x_{n-1},\mathfrak{I}x_{n-1})+\gamma\ \mathcal{D}(x_n,\mathfrak{I}x_n)\\
&\quad +\delta\ \mathcal{D}(x_{n-1},\mathfrak{I}x_n)+L\ \mathcal{D}(x_n,\mathfrak{I}x_{n-1})))^k\\
&\leq (\Theta(\alpha\ \wp(x_{n-1},x_n)+\beta\ \wp(x_{n-1},x_n)+\gamma\ \wp(x_n,x_{n+1})\\
&\quad +\delta\ \wp(x_{n-1},x_{n+1})+L\ \wp(x_n,x_n)))^k\\
&\leq (\Theta(\alpha\ \wp(x_{n-1},x_n)+\beta\ \wp(x_{n-1},x_n)+\gamma\ \wp(x_n,x_{n+1})\\
&\quad +\delta\ \{(\wp(x_{n-1},x_n)+\wp(x_n,x_{n+1})\}+L\ (0)))^k\\
&= (\Theta(\alpha\ \wp(x_{n-1},x_n)+\beta\ \wp(x_{n-1},x_n)+\gamma\ \wp(x_n,x_{n+1})\\
&\quad +\delta\ \{\wp(x_{n-1},x_n)+\wp(x_n,x_{n+1})\}))^k\\
&= (\Theta((\alpha+\beta+\delta)\wp(x_{n-1},x_n)+(\gamma+\delta)\wp(x_n,x_{n+1})))^k\\
&= (\Theta((1-\gamma-\delta)\wp(x_{n-1},x_n)+(\gamma+\delta)\wp(x_n,x_{n+1})))^k.\end{aligned} \tag{7}$$

We claim that $\wp(x_n,x_{n+1}) \leq \wp(x_{n-1},x_n)$. On contrary, suppose that

$$\wp(x_n,x_{n+1}) > \wp(x_{n-1},x_n).$$

Consequently, from (7) we have

$$\Theta(\mathcal{D}(x_n,\mathfrak{I}x_n)) < (\Theta((1-\gamma-\delta)\wp(x_{n-1},x_n)+(\gamma+\delta)\wp(x_n,x_{n+1})))^k,$$

so,

$$\Theta(\wp(x_n,x_{n+1})) < (\Theta((1-\gamma-\delta)\wp(x_{n+1},x_n)+(\gamma+\delta)\wp(x_n,x_{n+1})))^k,$$

this implies

$$\Theta(\wp(x_n,x_{n+1})) < (\Theta(\wp(x_{n+1},x_n)))^k, \tag{8}$$

a contradiction. Therefore, $\wp(x_n,x_{n+1}) \leq \wp(x_{n-1},x_n)$, $n \geq 1$. Hence, $\wp(x_{n-1},x_n)$ is a non-increasing sequence with positive terms. Thus, there exists $\mathcal{L} \geq 0$ such that

$$\lim_{n\to\infty} \wp(x_{n-1},x_n) = \mathcal{L}. \tag{9}$$

We claim that $\mathcal{L} = 0$. Suppose on contrary that $\mathcal{L} > 0$. Letting $s_n = \wp(x_{n+1},x_n)$ and $t_n = \wp(x_n,x_{n-1})$, using inequality (3) and Definition 2.7, we have

$$\mathfrak{C}_{\mathfrak{G}} \leq \lim_{n\to\infty}\sup \eta(\Theta(\mathcal{D}(x_{n+1},\mathfrak{I}x_n)),(\Theta(\wp(x_n,x_{n-1})))^k) < \mathfrak{C}_{\mathfrak{G}},$$

which is contradiction. Thus, $\mathcal{L} = 0$. We now prove that the sequence $\{x_n\}$ is Cauchy. If $\{x_n\}$ is not a Cauchy sequence in $\aleph$, then there exist $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ϵ^+ when $k \to +\infty$:

$$\wp(x_{m(k)},x_{n(k)}),\wp(x_{m(k)},x_{n(k)+1}),\wp(x_{m(k)-1},x_{n(k)}),\wp(x_{m(k)-1},x_{n(k)+1}),\wp(x_{m(k)+1},x_{n(k)+1}).$$

Particularly, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$\wp(x_{m(k)}, x_{n(k)}) > \frac{\epsilon}{2} > 0 \text{ and } \wp(x_{m(k+1)}, x_{n(k+1)}) > \frac{\epsilon}{2} > 0. \tag{10}$$

Since $\mathfrak{I}$ is multivalued Hardy-Rogers $\mathfrak{Z}^{\mathfrak{G}}_{\Theta}$-contraction with respect to η, together with (10), we get that

$$\begin{aligned} \mathfrak{C}_{\mathfrak{G}} &\leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_{m(k)}, \mathfrak{I}x_{n(k)})), (\Theta(\mathfrak{R}(x_{m(k)}, x_{n(k)})))^k) \\ &< \mathfrak{G}(\Theta(\mathfrak{R}(x_{m(k)}, x_{n(k)}))^k, \Theta(\mathfrak{H}(\mathfrak{I}x_{m(k)}, \mathfrak{I}x_{n(k)}))). \end{aligned} \tag{11}$$

Letting $k \to \infty$, using (9) *and* (10), *we get*

$$\lim_{k\to\infty} \wp(x_{m(k)}, x_{n(k)+1}) = \epsilon. \tag{12}$$

Using (9), (11), (12) and Definition 2.7, we have

$$\mathfrak{C}_{\mathfrak{G}} \leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_{m(k)}, \mathfrak{I}x_{n(k)})), (\Theta(\mathfrak{R}(x_{m(k)}, x_{n(k)})))^k) < \mathfrak{C}_{\mathfrak{G}},$$

which is a contradiction. As a consequence, $\{x_n\}$ is Cauchy. Since $(\aleph, \wp)$ is a complete metric space, there exists $x \in \aleph$ such that

$$\lim_{n\to\infty} \wp(x_n, x) = 0. \tag{13}$$

Suppose $x \notin \mathfrak{I}x$. It means that $x_n \notin \mathfrak{I}x_n$ for each $n \geq 0$, by taking $x = x_n$, $y = x$ in inequality (3) *and using Definition 2.7, we have*

$$\mathfrak{C}_{\mathfrak{G}} \leq \lim_{n\to\infty} sup \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_n, \mathfrak{I}x)), (\Theta(\mathfrak{R}(x_n, x))^k) < \mathfrak{C}_{\mathfrak{G}} . \tag{14}$$

Hence contradiction raised in (14). *Thus, we get $x \in \mathfrak{I}x$.* □

Now, we present an example of Theorem 3.3.

Example 3.4. *Let $\aleph = \{1, 3, 5, 7\}$ and $\wp(x, y) = |x - y|$. Define*

$$\mathfrak{I}x = \begin{cases} \{1,5\} & \text{if } x = 1 \\ \{1\} & \text{if } x = 3 \\ \{3,7\} & \text{otherwise} . \end{cases}$$

Let $\eta(t, s) = \frac{5}{6}s - t$, $\mathfrak{G}(s, t) = s - t$ for all $s, t \in [0, \infty)$, $\mathfrak{C}_{\mathfrak{G}} = 0$ and $\Theta : (0, \infty) \to (1, \infty)$ is defined by $\Theta(t) = e^t$. Now for any $x, y \in \aleph$ with $x \neq y$, we will discuss the following cases:

I. *for $x = 1$ and $y = 3$*

$$\wp(x, y) = 2, \mathcal{D}(x, \mathfrak{I}x) = 0, \mathcal{D}(y, \mathfrak{I}y) = 0$$
$$\mathcal{D}(x, \mathfrak{I}y) = 0, \mathcal{D}(y, \mathfrak{I}x) = 2, \mathfrak{H}(\mathfrak{I}x, \mathfrak{I}y) = 4$$

II. *for $x = 1$ and $y = 5$*

$$\wp(x, y) = 4, \mathcal{D}(x, \mathfrak{I}x) = 0, \mathcal{D}(y, \mathfrak{I}y) = 2$$
$$\mathcal{D}(x, \mathfrak{I}y) = 2, \mathcal{D}(y, \mathfrak{I}x) = 0, \mathfrak{H}(\mathfrak{I}x, \mathfrak{I}y) = 2$$

III. *for $x = 1$ and $y = 7$*

$$\wp(x, y) = 6, \mathcal{D}(x, \mathfrak{I}x) = 0, \mathcal{D}(y, \mathfrak{I}y) = 0$$
$$\mathcal{D}(x, \mathfrak{I}y) = 2, \mathcal{D}(y, \mathfrak{I}x) = 2, \mathfrak{H}(\mathfrak{I}x, \mathfrak{I}y) = 2$$

IV. *for $x = 3$ and $y = 5$*

$$\wp(x, y) = 2, \mathcal{D}(x, \Im x) = 2, \mathcal{D}(y, \Im y) = 2$$
$$\mathcal{D}(x, \Im y) = 0, \mathcal{D}(y, \Im x) = 4, \mathfrak{H}(\Im x, \Im y) = 2$$

V. *for $x = 3$ and $y = 7$*

$$\wp(x, y) = 4, \mathcal{D}(x, \Im x) = 2, \mathcal{D}(y, \Im y) = 0$$
$$\mathcal{D}(x, \Im y) = 0, \mathcal{D}(y, \Im x) = 6, \mathfrak{H}(\Im x, \Im y) = 2$$

VI. *for $x = 5$ and $y = 7$*

$$\wp(x, y) = 2, \mathcal{D}(x, \Im x) = 2, \mathcal{D}(y, \Im y) = 0$$
$$\mathcal{D}(x, \Im y) = 2, \mathcal{D}(y, \Im x) = 0, \mathfrak{H}(\Im x, \Im y) = 0.$$

Taking $\alpha = \gamma = 0.5, \beta = \delta = 0$, $L = 3$ and $k = 0.8$, we have that

$$\begin{aligned}
\mathfrak{G}((\Theta(\Re(1,3)))^k, \Theta(\mathfrak{H}(\Im 1, \Im 3))) &> \eta(\Theta(\mathfrak{H}(\Im 1, \Im 3)), (\Theta(\Re(1,3))^k) \\
&= \frac{5}{6}(\Theta(\Re(1,3)))^{0.8} - \Theta(\mathfrak{H}(\Im 1, \Im 3)) = 170.76 > 0 \\
\mathfrak{G}((\Theta(\Re(1,5)))^k, \Theta(\mathfrak{H}(\Im 1, \Im 5))) &> \eta(\Theta(\mathfrak{H}(\Im 1, \Im 5)), (\Theta(\Re(1,5))^k) \\
&= \frac{5}{6}(\Theta(\Re(1,5)))^{0.8} - \Theta(\mathfrak{H}(\Im 1, \Im 5)) = 1.79 > 0 \\
\mathfrak{G}((\Theta(\Re(1,7)))^k, \Theta(\mathfrak{H}(\Im 1, \Im 7))) &> \eta(\Theta(\mathfrak{H}(\Im 1, \Im 7)), (\Theta(\Re(1,7))^k) \\
&= \frac{5}{6}(\Theta(\Re(1,7)))^{0.8} - \Theta(\mathfrak{H}(\Im 1, \Im 7)) = 1108.80 > 0 \\
\mathfrak{G}((\Theta(\Re(3,5)))^k, \Theta(\mathfrak{H}(\Im 3, \Im 5))) &> \eta(\Theta(\mathfrak{H}(\Im 3, \Im 5)), (\Theta(\Re(3,5))^k) \\
&= \frac{5}{6}(\Theta(\Re(3,5)))^{0.8} - \Theta(\mathfrak{H}(\Im 3, \Im 5)) = 60934.64 > 0 \\
\mathfrak{G}((\Theta(\Re(3,7)))^k, \Theta(\mathfrak{H}(\Im 3, \Im 7))) &> \eta(\Theta(\mathfrak{H}(\Im 3, \Im 7)), (\Theta(\Re(3,7))^k) \\
&= \frac{5}{6}(\Theta(\Re(3,7)))^{0.8} - \Theta(\mathfrak{H}(\Im 3, \Im 7)) = 7405084.71 > 0 \\
\mathfrak{G}((\Theta(\Re(5,7)))^k, \Theta(\mathfrak{H}(\Im 5, \Im 7))) &> \eta(\Theta(\mathfrak{H}(\Im 5, \Im 7)), (\Theta(\Re(5,7))^k) \\
&= \frac{5}{6}(\Theta(\Re(5,7)))^{0.8} - \Theta(\mathfrak{H}(\Im 5, \Im 7)) = 0.86 > 0.
\end{aligned}$$

Hence, $\Im$ is an multivalued Hardy-Rogers $\mathfrak{Z}_\Theta^{\mathfrak{G}}$-contraction and all conditions of Theorem 3.3 are satisfied. Thus, $\Im$ has two fixed points 1 and 7 in $\aleph$.

Taking $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, in Theorem 3.3, we obtain following result:

Corollary 3.5. *Let $(\aleph, \wp)$ be a complete metric space and let $\Im$ be a multivalued $\mathfrak{Z}_\Theta^{\mathfrak{G}}$-contraction. Assume that $\eta \in \Delta$ and $\Theta \in \Omega$ such that*

$$\eta(\Theta(\mathfrak{H}(\Im x, \Im y)), (\Theta(\wp(x, y)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{15}$$

for all $x, y \in \aleph$ and $\Im x \neq \Im y$. Then $Fix(\Im) \neq \emptyset$.

Further, putting $\alpha = \delta = L = 0$ and $\beta + \gamma = 1$ and $\beta \neq 0$, in Theorem 3.3 we obtain the following generalization of Kannan result:

Corollary 3.6. *Let $(\aleph, \wp)$ be a complete metric space and let $\Im$ be a multivalued Kannan $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction. Assume that $\eta \in \Delta$ and $\Theta \in \Omega$ such that*

$$\eta(\Theta(\mathfrak{H}(\Im x, \Im y)), (\Theta(\beta\ \mathcal{D}(x, \Im x) + \gamma\ \mathcal{D}(y, \Im y)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{16}$$

for all $x, y \in \aleph$ and $\Im x \neq \Im y$, where $\beta, \gamma \in [0, +\infty[$, $\beta + \gamma = 1, \gamma \neq 1$. Then $Fix(\Im) \neq \emptyset$.

Also, a version of the Chatterjee type fixed point theorem is obtained from Theorem 3.3, by putting $\alpha = \beta = \gamma = 0$ and $\delta = L = \frac{1}{2}$.

Corollary 3.7. *Let $(\aleph, \wp)$ be a complete metric space and let $\Im$ be a multivalued Chatterja $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction. Assume that $\eta \in \Delta$ and $\Theta \in \Omega$ such that*

$$\eta(\Theta(\mathfrak{H}(\Im x, \Im y)), (\Theta(\frac{1}{2}\ \mathcal{D}(x, \Im y) + \frac{1}{2}\ \mathcal{D}(y, \Im x)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{17}$$

for all $x, y \in \aleph, \Im x \neq \Im y$. Then $Fix(\Im) \neq \emptyset$.

Finely, if we choose $\delta = L = 0$, we obtain a Reich type theorem.

Corollary 3.8. *Let $(\aleph, \wp)$ be a complete metric space and let $\Im$ be an multivalued Reich $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction. Assume that $\eta \in \Delta$ and $\Theta \in \Omega$ such that*

$$\eta(\Theta(\mathfrak{H}(\Im x, \Im y)), (\Theta(\alpha\ \wp(x, y) + \beta\ \mathcal{D}(x, \Im x) + \gamma\ \mathcal{D}(y, \Im y)))^k) \geq \mathfrak{C}_{\mathfrak{G}}, \tag{18}$$

for all $x, y \in \aleph, \Im x \neq \Im y$, where $\alpha, \beta, \gamma \in [0, +\infty[$, $\alpha + \beta + \gamma = 1, \gamma \neq 1$. Then $Fix(\Im) \neq \emptyset$.

The case for a mapping to be self in Theorem 3.3, we can derive the following fixed point theorem:

Corollary 3.9. *Let $(\aleph, \wp)$ be a complete metric space and $\Im$ be a self Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction. Then $\Im$ has a unique fixed point.*

4. Strict fixed points and well-posedness

The set of fixed point of the mapping $\Im$ is defined as $Fix(\Im) := \{x \in \aleph : x \in \Im x\}$ and that of strict fixed point is defined as $SFix(\Im) := \{x \in \aleph : \{x\} = \Im x\}$. It is clear that $SFix(\Im) \subseteq Fix(\Im)$. We start the section with the following definition:

Definition 4.1. *(See [18, 24]) Let $\mathcal{Y} \in \mathcal{P}(\aleph)$ where $(\aleph, \wp)$ is a metric space and $\Im : \aleph \to C\mathcal{L}(\aleph)$ be a multivalued operator. Then the fixed point problem is well-posed for $\Im$ with respect to $\mathcal{D}$ if:*

(a_1) *$Fix(\Im) = \{x\}$;*

(a_2) *for a sequence $\{x_n\}$ in $\mathcal{Y}$, $\mathcal{D}(x_n, \Im x_n) \to 0$ as $n \to \infty$, then $\wp(x_n, x) \to 0$ as $n \to \infty$.*

Definition 4.2. *(See [18, 24]) Let $\mathcal{Y} \in \mathcal{P}(\aleph)$ where $(\aleph, \wp)$ is a metric space and $\Im : \aleph \to C\mathcal{L}(\aleph)$ be a multivalued operator. Then the fixed point problem is well-posed for $\Im$ with respect to $\mathfrak{H}$ if:*

(b_1) *$Fix(\Im) = \{x\}$;*

(b_2) *for a sequence $\{x_n\}$ in $\mathcal{Y}$, $\mathfrak{H}(x_n, \Im x_n) \to 0$ as $n \to \infty$, then $\wp(x_n, x) \to 0$ as $n \to \infty$.*

It is to be observe that a fixed point problem, which is well-posed for $\Im$ with respect to $\mathcal{D}$ then the it is well-posed for $\Im$ with respect to $\mathfrak{H}$.
Now we state main result of this section:

Theorem 4.3. *Let $(\aleph, \wp)$ be a complete metric space and $\Im : \aleph \to C\mathcal{L}(\aleph)$ be a multivalued operator. Suppose that $\Im$ is multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contractive operator with $L = 0$ and $SFix(\Im) \neq \emptyset$. Then $Fix(\Im) = SFix(\Im) = \{x\}$, and fixed point problem is well posed with respect to $\mathfrak{H}$.*

Proof. First we show that $Fix(\mathfrak{I}) = \{x\}$. Suppose that $u, x \in Fix(\mathfrak{I})$ with $u \neq x$.
Since $\mathfrak{I}$ is $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction, we obtain

$$\mathfrak{C}_{\mathfrak{G}} \leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}u, \mathfrak{I}x)), (\Theta(\mathfrak{R}(u,x))^k) < \mathfrak{G}((\Theta(\mathfrak{R}(u,x))^k, \Theta(\mathfrak{H}(\mathfrak{I}u, \mathfrak{I}x))),$$

where $k \in (0,1)$. *Using* $\mathfrak{G}_1$, *we have*

$$\Theta(\mathfrak{H}(\mathfrak{I}u, \mathfrak{I}x)) < (\Theta(\mathfrak{R}(u,x)))^k. \tag{19}$$

So,

$$\begin{aligned}
\Theta(\mathcal{D}(\mathfrak{I}x, u)) = \Theta(\wp(x,u)) &\leq \Theta(\mathfrak{H}(\mathfrak{I}u, \mathfrak{I}x)) \\
&< (\Theta(\mathfrak{R}(u,x)))^k \\
&= (\Theta(\alpha\ \wp(u,x) + \beta\ \mathcal{D}(u, Tu) + \gamma\ \mathcal{D}(x, \mathfrak{I}x) + \delta\ \mathcal{D}(u, \mathfrak{I}x)))^k \\
&= (\Theta((\alpha+\delta)\wp(x,u)))^k \\
&< \Theta((\alpha+\delta)\wp(x,u))
\end{aligned}$$

therefore, we have

$$\wp(x,u) < (1-\beta-\delta)\wp(x,u),$$

that is

$$(\beta+\delta)\wp(u,x) < 0.$$

This implies $0 < \wp(x,u) < 0$. *Hence* $Fix(\mathfrak{I}) = \{x\}$.
Now, let $x \in SFix(\mathfrak{I})$ *and* $\{x_n\}_{n\in\mathbb{N}}$ *such that* $\mathcal{D}(x_n, \mathfrak{I}x_n) \to 0$, *as* $n \to \infty$. *Since* $\mathfrak{I}$ *is an multivalued Hardy-Rogers* $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$*-contractive operator with* $L = 0$, *therefore*

$$\mathfrak{C}_{\mathfrak{G}} \leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_n, \mathfrak{I}x)), (\Theta(\mathfrak{R}(x_n,x))^k) < \mathfrak{G}(\Theta(\mathfrak{R}(x_n,x))^k, \Theta(\mathfrak{H}(\mathfrak{I}x_n, \mathfrak{I}x))).$$

Using $\mathfrak{G}_1$ *we have,*

$$\Theta(\mathfrak{H}(\mathfrak{I}x_n, \mathfrak{I}x)) < (\Theta(\mathfrak{R}(x_n,x)))^k. \tag{20}$$

So,

$$\begin{aligned}
\Theta(\wp(x_n,x)) = \Theta(\mathcal{D}(x_n, \mathfrak{I}x)) &< (\Theta(\mathfrak{R}(x_n,x)))^k \\
&= (\Theta(\alpha\ \wp(x_n,x) + \beta\ \mathcal{D}(x_n, \mathfrak{I}x_n) + \gamma\ \mathcal{D}(x, \mathfrak{I}x) \\
&\quad +\delta\ \wp(x_n, \mathfrak{I}x))^k \\
&\leq (\Theta((\alpha+\delta)\ \wp(x_n,x) + \beta\ \mathcal{D}(x_n, \mathfrak{I}x_n)))^k \\
&< \Theta((\alpha+\delta)\ \wp(x_n,x) + \beta\ \mathcal{D}(x_n, \mathfrak{I}x_n))
\end{aligned} \tag{21}$$

implies

$$\wp(x_n,x) < (\alpha+\delta)\ \wp(x_n,x) + \beta\ \mathcal{D}(x_n, \mathfrak{I}x_n)$$

$$(1-\alpha-\delta)\wp(x_n,x) < \beta\ \mathcal{D}(x_n, \mathfrak{I}x_n)$$

$$\wp(x_n,x) < \frac{\beta}{1-\alpha-\delta}\ \mathcal{D}(x_n, \mathfrak{I}x_n).$$

Taking limit as $n \to \infty$, *we obtain that* $\wp(x_n,x) \to 0$. $\square$

5. Data dependence

This section is devoted to the study of data dependence of fixed point set for the multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction.

Theorem 5.1. *Let $\mathfrak{I}_1$, $\mathfrak{I}_2$ be two multivalued operators on a metric space $(\aleph, \wp)$, if*

1. *$\mathfrak{I}_i$ is multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contraction for $i \in \{1,2\}$.*
2. *$\mathfrak{H}(\mathfrak{I}_1 x, \mathfrak{I}_2 x)) \leq \omega'$ for all $x \in \aleph$, where $\omega' \in \mathbb{R}_+$.*

Then

1. *$Fix(\mathfrak{I}_i) \in \mathcal{CL}(\aleph)$, for $i = 1,2$,*
2. *$\mathfrak{I}_1$ and $\mathfrak{I}_2$ are multivalued Hardy-Rogers $\mathfrak{Z}_{\Theta}^{\mathfrak{G}}$-contractive operators and*

$$\mathfrak{H}(Fix(\mathfrak{I}_1), Fix(\mathfrak{I}_2))) \leq \frac{\omega'}{1 - max\{\upsilon_1, \upsilon_2\}}.$$

Proof. *From Theorem 3.3, we have $Fix(\mathfrak{I}_i) \neq \phi$ for $i = 1,2$, $Fix(\mathfrak{I})$ is closed. Choose a sequence $\{x_n\}$ in $Fix(\mathfrak{I})$ such that $x_n \to x$ as $n \to \infty$, for $x = x_{n-1}$ and $y = x_n$ we have*

$$\eta(\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1}, \mathfrak{I}x_n)), (\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k) \geq \mathfrak{C}_{\mathfrak{G}}. \tag{22}$$

From 2.7, we have,

$$\begin{aligned} \mathfrak{C}_{\mathfrak{G}} &\leq \eta(\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1}, \mathfrak{I}x_n)), (\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k) \\ &< \mathfrak{G}((\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k, \Theta(\mathfrak{H}(\mathfrak{I}x_{n-1}, \mathfrak{I}x_n))). \end{aligned}$$

Using $\mathfrak{G}_1$, we have

$$\Theta(\mathfrak{H}(\mathfrak{I}x_{n-1}, \mathfrak{I}x_n)) < (\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k. \tag{23}$$

Since $\Theta(\mathcal{D}(x_n, \mathfrak{I}x_n)) \leq \Theta(\mathfrak{H}(\mathfrak{I}x_{n-1}, \mathfrak{I}x_n))$. So

$$\begin{aligned} 1 < \Theta(\mathcal{D}(x_n, \mathfrak{I}x_n)) &< (\Theta(\mathfrak{R}(x_{n-1}, x_n)))^k \\ &= (\Theta(\alpha\ \wp(x_{n-1}, x_n) + \beta\ \mathcal{D}(x_{n-1}, \mathfrak{I}x_{n-1}) + \gamma\ \mathcal{D}(x_n, \mathfrak{I}x_n) \\ &\quad +\delta\ \mathcal{D}(x_{n-1}, \mathfrak{I}x_n) + L\ \mathcal{D}(x_n, \mathfrak{I}x_{n-1})))^k \\ &\leq (\Theta(\alpha\ \wp(x_{n-1}, x_n) + \beta\ \wp(x_{n-1}, x_n) + \gamma\ \wp(x_n, x_{n+1}) \\ &\quad +\delta\ \wp(x_{n-1}, \mathfrak{I}x_n) + L\ \wp(x_n, x_n)))^k \\ &\leq (\Theta(\alpha\ \wp(x_{n-1}, x_n) + \beta\ \wp(x_{n-1}, x_n) + \gamma\ \wp(x_n, x_{n+1}) \\ &\quad +\delta\ \{(\wp(x_{n-1}, x_n) + \wp(x_n, x_{n+1})\} + L\ (0)))^k \\ &= (\Theta(\alpha + \beta + \delta)\ \wp(x_{n-1}, x_n) + (\gamma + \delta)\ \wp(x_n, x_{n+1}))^k \\ &< (\Theta((\alpha + \beta + \delta)\wp(x_{n-1}, x_n) + (\gamma + \delta)\wp(x_n, x_{n+1}))). \end{aligned} \tag{24}$$

As $n \to \infty$, we get that $1 \leq \Theta(\mathcal{D}(x, \mathfrak{I}x)) < 1$, so, $\Theta(\mathcal{D}(x, \mathfrak{I}x)) = 1$, this implies $\mathcal{D}(x, \mathfrak{I}x) = 0$. Since $\mathfrak{I}x \in \mathcal{CL}(\aleph)$ therefore $x \in \mathfrak{I}x$. Hence $x \in Fix(\mathfrak{I})$. Secondly, $\mathfrak{I}$ possesses a fixed point by the argument those given in Theorem 3.3. Let $\ell \in (1, +\infty)$ and choose an arbitrary $x_0 \in Fix(\mathfrak{I}_1)$. Then there exists $x_1 \in \mathfrak{I}_2 x_0$ such that $\wp(x_0, x_1) \leq \ell\mathfrak{H}(\mathfrak{I}_1 x_0, \mathfrak{I}_2 x_0)$. Now for $x_1 \in \mathfrak{I}_2 x_0$ there exists $x_2 \in \mathfrak{I}_2 x_1$ such that $\wp(x_1, x_2) \leq \ell\mathfrak{H}(\mathfrak{I}_2 x_0, \mathfrak{I}_2 x_1)$. Since $x_1 \in \mathfrak{I}_2 x_0, \mathcal{D}(x_1, \mathfrak{I}_2 x_0) = 0 \leq \wp(x_0, x_1)$. Therefore

$$\wp(x_1, x_2) \leq \ell\mathfrak{H}(\mathfrak{I}_2 x_0, \mathfrak{I}_2 x_1) \leq \ell\upsilon_2\ \wp(x_0, x_1).$$

In a similar fashion, a sequence of successive approximation for $\mathfrak{I}_2$ starting from x_0 can be obtained which satisfies

$$x_{n+1} \in \mathfrak{I}x_n \ \text{ and } \ \wp(x_n, x_{n+1}) \leq (\ell\upsilon_2)^2\ \wp(x_0, x_1),\ \forall\ \ n \geq 1.$$

Hence for all $n \geq \mathbb{N}$ and, $\rho \geq 1$

$$\begin{aligned} \wp(x_{n+\rho}, x_n)) &\leq \wp(x_n, x_{n+1}) + \wp(x_{n+1}, x_{n+2}) + ... + \wp(x_{n+\rho-1}, x_{n+\rho}) \\ &\leq (\ell\upsilon_2)^n \wp(x_0, x_1) + (\ell\upsilon_2)^{n+1}\wp(x_0, x_1) + ... + (\ell\upsilon_2)^{n+\rho-1}\wp(x_0, x_1) \\ &\leq \frac{(\ell\upsilon_2)^n}{1-\ell\upsilon_2}\wp(x_0, x_1). \end{aligned} \tag{25}$$

Choosing $1 < \ell < min\{\frac{1}{\upsilon_1}, \frac{1}{\upsilon_2}\}$ and taking limit as $n \to \infty$, the sequence $\{x_n\}$ is Cauchy in $(\aleph, \wp)$. From the completeness of $\aleph$, there exists $x^* \in \aleph$ such that $x_n \to x^*$ as $n \to \infty$. We show that x^* is a fixed point of $\Im_2$. Suppose on contrary that $x^* \notin \Im_2 x^*$ and $x_{n(k)} \notin \Im_2 x_{n(k)}$. Setting $x = x_{n(k)}$, $y = x^*$ in (3) and using Definition 2.7, we have

$$\mathfrak{C}_{\mathfrak{G}} \leq \lim_{n\to\infty} Sup[\eta(\Theta(\mathfrak{H}(\Im_2 x^*, \Im_2 x_{n(k)})), (\Theta(\Re(x^*, x_{n(k)})))^k)] < \mathfrak{C}_{\mathfrak{G}}, \tag{26}$$

a contradiction. Hence $x^* \in T_2x^*$. Thus $x^* \in Fix(\Im_2)$. Taking $\rho \to \infty$ in (25). We have $\wp(x^*, x_n) \leq \frac{(\ell\upsilon_2)^n}{1-\ell\upsilon_2}\wp(x_0, x_1)$ for each $n \in \mathbb{N}$. Then $\wp(x_0, x^*)) \leq \frac{1}{1-\ell\upsilon_2}\wp(x_0, x_1) \leq \frac{\ell\omega'}{1-\ell\upsilon_2}$. Similarly, for each $x_0^* \in Fix(\Im_2)$ there exists $x \in Fix(\Im_1)$ such that $\wp(x_0^*, x)) \leq \frac{1}{1-\ell\upsilon_2}]wp(x_0^*, x_1^*) \leq \frac{\ell\lambda'}{1-\ell\upsilon_2}$. Hence

$$\mathfrak{H}(Fix(\Im_1), Fix(\Im_2)) \leq \frac{\ell\omega'}{1 - max\{\ell\upsilon_1, \ell\upsilon_2\}}.$$

Taking $\ell \to 1$ we obtain the desire result. Moreover, we have that $\Im_i$ is $\frac{1}{1-\upsilon_i}$ operator, $i = 1, 2$. □

6. Ulam-Hyers Stability

This section describes the stability of fixed point inclusion.

Definition 6.1. *Let $(\aleph, \wp)$ be a metric space and $\Im : \aleph \to C\mathcal{L}(\aleph)$ be a multivalued operator. The fixed point inclusion*

$$x \in \Im(x), x \in \aleph \tag{27}$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous at 0 and $\Phi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $y* \in \aleph$ of the inequality

$$\mathcal{D}(y, \Im y) \leq \varepsilon \tag{28}$$

there exists a solution $x*$ of the fixed point inclusion (27) such that

$$\wp(x*, y*) \leq \Phi(\varepsilon).$$

If there exists $C > 0$ such that $\Phi(t) = C \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (27) is said to be generalized Ulam-Hyers stable.

Theorem 6.2. *Let $(\aleph, \wp)$ be a complete metric space and $\Im : \aleph \to C\mathcal{L}(\aleph)$ a multivalued mapping such that:*

1. *for any $x, y \in \aleph$ and $k \in (0, 1)$ operator $\Im$ satisfy*

$$\eta(\Theta(\mathfrak{H}(\Im x, \Im y)), (\Theta(\Re(x, y)))^k) \geq \mathfrak{C}_{\mathfrak{G}} \tag{29}$$

where

$$\Re(x, y) = \alpha\ \wp(x, y) + \beta\ \mathcal{D}(x, \Im x) + \gamma\ \mathcal{D}(y, \Im y) + \delta\ \mathcal{D}(x, \Im y),$$

$\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \in (0, 1)$.

2. *$SFix(\Im) \neq \emptyset$,*

then the fixed point problem is generalized Ulam-Hyers stable.

Proof. By Theorem 3.3 and 4.3, we have $SFix(\Im) = \{x*\}$. Let $\varepsilon > 0$ and $y*$ be a solution of (28). Then

$$\begin{aligned}\Theta(\wp(x*, y*)) = \Theta(\mathcal{D}(\Im x*, y*)) &< (\Theta(\Re(x*, y*)))^k \\ &= (\Theta(\alpha\ \wp(x*, y*) + \beta\ \mathcal{D}(x*, \Im x*) + \gamma\ \mathcal{D}(y*, \Im y*) \\ &\quad +\delta\ \mathcal{D}(x*, \Im y*)))^k \\ &\leq (\Theta((\alpha+\delta)\ \wp(x*, y*) + (\gamma+\delta)\ \mathcal{D}(y*, \Im y*)))^k \\ &< \Theta((\alpha+\delta)\ \wp(x*, y*) + (\gamma+\delta)\ \mathcal{D}(y*, \Im y*)), \end{aligned} \tag{30}$$

this implies

$$\wp(x*, y*) < (\alpha+\delta)\ \wp(x*, y*) + (\gamma+\delta)\ \mathcal{D}(y*, \Im y*)$$

and hence

$$\wp(x*, y*) < \frac{\gamma+\delta}{\beta+\gamma+\delta}\ \varepsilon.$$

Thus by taking $C = \frac{\gamma+\delta}{\beta+\gamma+\delta} > 0$, we derive that the fixed point inclusion is generalized Ulam-Hyers stable. □

7. Application to Fractional Calculus

First, we recall some notions (see[10]). For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$$^{C}D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s) ds \ \ (n-1 < \beta < n, n = [\beta]+1)$$

where $[\beta]$ denotes the integer part of real number β and Γ is gamma function.

In this section, we present an application of Corollary 3.9 to show the existence of the solution for nonlinear fractional differential equation:

$$^{C}D^{\beta}(x(t)) + f(t, x(t)) = 0 \ \ (0 \leq t \leq 1, \beta < 1) \tag{31}$$

via boundary conditions $x(0) = 0 = x(1)$, where $x \in C([0,1], \mathbb{R})$. $C([0,1], \mathbb{R})$ is the set of all continuous functions from $[0,1]$ into $\mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function (see[20]). Recall Green function associated to the problem (31) is given by

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Now we prove the following existence theorem:

Theorem 7.1. *Suppose that*

(i) *There exist a continuous function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $t > 1$ *such that*

$$|f(t,x) - f(t,y)| \leq e^{-t} \Re(x,y)$$

for all $t \in [0,1]$ *and* $x, y \in C([0,1], \mathbb{R})$. *Where*

$$\Re(x,y) = \alpha\ |x-y| + \beta\ |x - \Im x| + \gamma\ |y - \Im y| + \delta\ |x - \Im y| + L\ |y - \Im x|,$$

$\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ *and* $L \geq 0$.

(ii) *There exists* $x \in C([0,1], \mathbb{R})$ *such that for all* $t, s \in [0,1]$, *where* $\Im : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ *is defined by*

$$\Im x(t) = \int_0^1 G(t,s) f(s, x(s)) ds.$$

Then, problem (31) *has at least one solution.*

Proof. *First, let $x, y \in \aleph = C([0,1], \mathbb{R})$ a metric space defined as*

$$\wp(x,y) = \|x\|_\infty = \sup_{t\in[0,1]} |x(t) - y(t)|.$$

It is easy to see that $x \in \aleph$ is a solution of (31) *if and only if $x \in \aleph$ is a solution of equation $\Im x(t) = \int_0^1 G(t,s) f(s, x(s)) ds$ for all $t \in [0,1]$. Then the problem* (31) *is equivalent to finding $x^* \in \aleph$ which is fixed point of $\Im$. Now let $x, y \in \aleph$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By* (*i*) *and* (*ii*), *we have*

$$\begin{aligned}
|\Im x(t) - \Im y(t)| &= \left| \int_0^1 G(t,s) f(s,x(s)) ds - \int_0^1 G(t,s) f(s,y(s)) ds \right| \\
&\leq \int_0^1 |G(t,s)[f(s,x(s)) - f(s,y(s))]| ds \\
&\leq \left(\int_0^1 |G(t,s)|^q ds \right)^{\frac{1}{q}} \left(\int_0^1 |f(s,x(s)) - f(s,y(s))|^p ds \right)^{\frac{1}{p}} \quad \textit{(Holder's inequality)} \\
&\leq \left(\int_0^1 |G(t,s)|^q ds \right)^{\frac{1}{q}} \left(\int_0^1 (e^{-t} \Re(x(s), y(s)))^p ds \right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 |G(t,s)|^q ds \right)^{\frac{1}{q}} \left(\int_0^1 (e^{-t} (\alpha \sup_{t\in[0,1]} |x(s) - y(s)| + \beta \sup_{t\in[0,1]} |x(s) - \Im x(s)| + \right. \\
&\qquad \left. \gamma \sup_{t\in[0,1]} |y(s) - \Im y(s)| + \delta \sup_{t\in[0,1]} |x(s) - \Im y(s)| + L \sup_{t\in[0,1]} |y(s) - \Im x(s)|))^p ds \right)^{\frac{1}{p}} \\
&= \left(\int_0^1 |G(t,s)|^q ds \right)^{\frac{1}{q}} e^{-t} (\alpha \wp(x,y) + \beta \wp(x, \Im x) + \\
&\qquad \gamma \wp(y, \Im y) + \delta \wp(x, \Im y) + L \wp(y, \Im x)) \int_0^1 ds \\
&\leq e^{-t} \Re(x,y) \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q ds \right)^{\frac{1}{q}} \\
&\leq e^{-t} \Re(x,y),
\end{aligned}$$

where

$$\Re(x,y) = \alpha \wp(x,y) + \beta \wp(x - \Im x) + \gamma \wp(y, \Im y) + \delta \wp(x, \Im y) + L \wp(y, \Im x).$$

Thus for each $x, y \in \aleph$, we have

$$\wp(\Im x, \Im y) = \|\Im x - \Im y\|_\infty = \sup_{t\in[0,1]} |\Im x(t) - \Im y(t)| \leq e^{-t} \Re(x,y).$$

Let $\Theta(t) = e^{\sqrt{t}} \in \Omega$, $t > 0$, we have

$$e^{\sqrt{\wp(\Im x, \Im y)}} \leq e^{\sqrt{e^{-t}\Re(x,y)}} = [e^{\sqrt{\Re(x,y)}}]^k, \quad \forall\, x, y \in \aleph,$$

where $k = \sqrt{e^{-t}}$. Since $t > 1$, therefore $k \in (0,1)$. Then for $\eta(t,s) = \lambda\, s - t$, $\lambda \in (0,1)$ and $G(s,t) = s - t$ and $\mathfrak{C}_{\mathfrak{G}} = 0$, we have

$$0 < \eta(\Theta(\wp(\Im x, \Im y)), (\Theta(\Re(x,y)))^k) < \mathfrak{G}((\Theta(\Re(x,y)))^k, \Theta(\wp(\Im x, \Im y))) \tag{32}$$

where $x, y \in \aleph$*. So, it is proved that* $\mathfrak{J}$ *is an self Hardy-Rogers* $\mathfrak{Z}^{\mathfrak{S}}_{\Theta}$*-contraction. Hence all the conditions of Corollary 3.9 satisfied. Thus we concluded that there exists* $x^* \in \aleph$ *such that* $\mathfrak{J}x^* = x^*$ *and so* x^* *is a solution of the problem* (31)*. This completes the proof.* □

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