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Generalized Inverses – Idempotents and Projectors

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Abstract. In this paper, we present necessary and sufficient conditions for $\widetilde{X}$ to be idempotent and orthogonal idempotent, where $\widetilde{X} \in \{A^{\textcircled{\dagger}}, A^{D}, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$. Several characteristics when $\widetilde{X}$ is idempotent and orthogonal idempotent are derived by core-EP decomposition. Additionally, we give some equivalent conditions when matrix A is orthogonal idempotent, using the properties of some generalized inverses of A.

1. Introduction

Idempotent and orthogonal idempotent matrices are very important concepts in linear algebra, which have been widely used in matrix theory [16], physics [14], statistics and econometrics [18], or numerical analysis [10]. A similar statement can be made about the generalized inverses of matrices, which is a useful tool in areas such as cryptography [12], chemical equations [19], optimization theory [11] and so on. Recently, Baksalary and Trenkler studied characterizations of matrices whose Moore-Penrose is idempotent by the Hartwing-Spindelböck decomposition [2]. And some original features and new properties have been given in [2]. The present paper is devoted to investigating characterizations for some generalized inverses to be idempotent and orthogonal idempotent by utilizing the core-EP decomposition.

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. We denote the identity matrix of order n by I_n, range space, null space, conjugate transpose and rank of $A \in \mathbb{C}^{m\times n}$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $r(A)$, respectively. The index of $A \in \mathbb{C}^{n\times n}$ denoted by $\operatorname{ind}(A)$ is the smallest integer $k \geq 0$ such that $r(A^k) = r(A^{k+1})$. Let $\mathbb{C}^{n\times n}_k$ be the set consisting of $n \times n$ complex matrices with index k.

For the readers' convenience, we first recall the definitions of some types of generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose(MP) inverse of A is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations [16]: $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, $(A^\dagger A)^* = A^\dagger A$.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by A^D [7], is defined to be the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following equations :

$$XAX = X, \quad AX = XA, \quad XA^{k+1} = A^k.$$

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In particular, the Drazin inverse of A is called the group inverse of A which is denoted by $A^{\#}$ if $\mathrm{ind}(A) \le 1$. Recall that the existence of the group inverse is restricted to the matrices of index 1(known also as the core matrices). For results on Drazin inverse and idempotents, see [4, 5, 13].

In addition, in this paper we use some properties of core-EP inverse, DMP inverse, dual DMP inverse and weak group inverse. Definitions of these generalized inverses are listed below.

For a matrix $A \in \mathbb{C}_k^{n\times n}$, the unique solution $X \in \mathbb{C}^{n\times n}$ of the following equations

$$XAX = X, \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

is called the core-EP inverse of A written as $A^{\textcircled{\dagger}}$ [17].

The DMP inverse of $A \in \mathbb{C}_k^{n\times n}$ is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfying:

$$XAX = X, \quad XA = A^D A, \quad A^k X = A^k A^{\dagger}.$$

Such solution X is denoted by $A^{D,\dagger}$. Moreover, it was certified that $A^{D,\dagger} = A^D A A^{\dagger}$. Also, the dual DMP inverse of A is defined to be the matrix $A^{\dagger,D} = A^{\dagger} A A^D$ [15].

In 2018, Wang and Chen [21] defined the weak group inverse X of $A \in \mathbb{C}_k^{n\times n}$ satisfying:

$$AX^2 = X, \quad AX = A^{\textcircled{\dagger}} A,$$

denoted by $A^{\textcircled{w}}$. Moreover, it was proved that $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$.

For convenience, we adopt the following notations: $\mathbb{C}_n^{\mathrm{P}}$ and $\mathbb{C}_n^{\mathrm{OP}}$ will stand for the subsets of $\mathbb{C}^{n\times n}$ consisting of idempotent matrices and Hermitian idempotent matrices, respectively, i.e.,

- $\mathbb{C}_n^{\mathrm{P}} = \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A\}$;
- $\mathbb{C}_n^{\mathrm{OP}} = \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\} = \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A = A^{\dagger}\}$.

The present paper is organized as follows. In Section 2, some necessary and sufficient conditions for characterizing $\widetilde{X}$ as idempotent are given, where $\widetilde{X} \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$. In Section 3, some new properties of $\widetilde{X}$ are obtained, when $\widetilde{X}$ is orthogonal idempotent. In Section 4, we list some equivalent conditions when A is orthogonal idempotent, in terms of some generalized inverses of the matrix A.

2. Characterizations of matrices whose some generalized inverses are idempotent

In the section, some necessary and sufficient conditions for the idempotency of $A^{\textcircled{\dagger}}$, A^D, $A^{D,\dagger}$, $A^{\dagger,D}$ and $A^{\textcircled{w}}$ are investigated. We start with the core-EP decomposition.

Wang proposed a new decomposition of $A \in \mathbb{C}_k^{n\times n}$, which is referred to as core-EP decomposition [20]. It can be given in what follows.

Lemma 2.1. [20](*core-EP decomposition*) *Let $A \in \mathbb{C}_k^{n\times n}$. Then A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(a) $A_1 \in \mathbb{C}_n^{CM}$;

(b) $A_2^k = 0$;

(c) $A_1^* A_2 = A_2 A_1 = 0$.

Lemma 2.2. [20] *Let the core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ be as in Lemma* 2.1. *Then there exists a unitary matrix U such that:*

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{2.1}$$

where T is nonsingular, $r(T) = r(A^k) = t$ and N is nilpotent of index k. Furthermore, the core-EP inverse of A is

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{2.2}$$

The decomposition of A, $A = A_1 + A_2$, where A_1 and A_2 are given by (2.1), is unique [19, Theorem 2.4]. Matrices A_1 and A_2 are called core part and nilpotent part, respectively. It is easy to verify that $A_1 = AA^{\textcircled{\dagger}}A$.

Lemma 2.3. [9] *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1)*. Then*

$$A^D = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{bmatrix}U^*, \tag{2.3}$$

where $\widetilde{T} = \sum\limits_{j=0}^{k-1} T^j S N^{k-1-j}$. Furthermore, $\widetilde{T} = 0$ if and only if $S = 0$.

Lemma 2.4. [6] *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1)*. Then*

$$A^\dagger = U\begin{bmatrix} T^*\triangle & -T^*\triangle SN^\dagger \\ (I_{n-t} - N^\dagger N)S^*\triangle & N^\dagger - (I_{n-t} - N^\dagger N)S^*\triangle SN^\dagger \end{bmatrix}U^*, \tag{2.4}$$

where $\triangle = (TT^* + S(I_{n-t} - N^\dagger N)S^*)^{-1}$.

According to Lemma 2.2 and Lemma 2.3, a straightforward computation shows that [9]

$$A^{D,\dagger} = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^\dagger \\ 0 & 0 \end{bmatrix}U^*, \tag{2.5}$$

$$A^{\dagger,D} = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T} \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T} \end{bmatrix}U^*. \tag{2.6}$$

Lemma 2.5. [21] *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1)*. Then*

$$A^{\textcircled{w}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{2.7}$$

It's easy to prove that the group inverse of A is idempotent if and only if A is idempotent. In [1], the authors gave that the core inverse of A is idempotent if and only if A is idempotent. Baksalary and Trenkler have shown that, in general, the idempotency of a matrix is not inherited by its Moore-Penrose inverse(see [2]). These authors have given some equivalent conditions for $A^\dagger$ to be idempotent. The following results are given for $\widetilde{X}$ to be idempotent, where $\widetilde{X} \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{\textcircled{w}}\}$.

Theorem 2.6. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1)*, $X \in \{A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}, A^{\textcircled{w}}\}$. Then X is idempotent if and only if any of the following statements is satisfied:*

(a) $T = I_t$;

(b) $A^k = A^{k+1}$;

(c) $AX = X$;

(d) $AX^k = X^k$;

(e) $A^kX^k = X$;

(f) $XA^k = A^k$.

Proof. (a). By (2.2), (2.3), (2.5) and (2.7), it is easy to verify that $A^{\textcircled{\dagger}}$, A^{D}, $A^{D,\dagger}$ and $A^{\textcircled{w}}$ are idempotents if and only if $T = I_t$.

(b). By $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$, we have

$$A^k = U\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^*, \tag{2.8}$$

where $\widetilde{T} = \sum\limits_{j=0}^{k-1} T^j S N^{k-1-j}$. Thus, we get that

$$\begin{aligned} A^k = A^{k+1} \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{k+1} & T\widetilde{T} \\ 0 & 0 \end{bmatrix}U^* \\ &\Longleftrightarrow \quad T = I_t. \end{aligned}$$

(c). By (2.2), (2.3), (2.5) and (2.7), we have

$$X = U\begin{bmatrix} T^{-1} & X_1 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.9}$$

where $X_1 \in \{0,\ (T^{k+1})^{-1}\widetilde{T},\ (T^{k+1})^{-1}\widetilde{T}NN^{\dagger},\ T^{-2}S\}$, in the case when $X \in \{A^{\textcircled{\dagger}}, A^{D}, A^{D,\dagger}, A^{\textcircled{w}}\}$, respectively. Thus, we obtain that

$$\begin{aligned} AX = X \quad &\Longleftrightarrow \quad U\begin{bmatrix} I_t & TX_1 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & X_1 \\ 0 & 0 \end{bmatrix}U^* \\ &\Longleftrightarrow \quad T = I_t. \end{aligned}$$

(d) . By (2.9), it follows that

$$X^k = U\begin{bmatrix} T^{-k} & T^{-k+1}X_1 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.10}$$

where $X_1 \in \{0,\ (T^{k+1})^{-1}\widetilde{T},\ (T^{k+1})^{-1}\widetilde{T}NN^{\dagger},\ T^{-2}S\}$, in the case when $X \in \{A^{\textcircled{\dagger}}, A^{D}, A^{D,\dagger}, A^{\textcircled{w}}\}$, respectively. Since $AX^k = X^k$, it follows that

$$\begin{aligned} AX^k = X^k \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-k+1} & T^{-k+2}X_1 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-k} & T^{-k+1}X_1 \\ 0 & 0 \end{bmatrix}U^* \\ &\Longleftrightarrow \quad T = I_t. \end{aligned}$$

(e) and (f). These proofs are similar to that of (d). □

If $A^{\dagger,D}$ is idempotent, it can be verified that each of the statements (a), (b) in Theorem 2.6 holds. However, we can see that any of the four statements (c), (d), (e), (f) in Theorem 2.6 is not satisfied when $X = A^{\dagger,D}$ is idempotent. We now give the following example to illustrate it.

Example 2.7. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $\mathrm{ind}(A) = 2$, *and*

$$A^{\dagger,D} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (A^{\dagger,D})^2 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $AA^{\dagger,D} \neq A^{\dagger,D}$, $A(A^{\dagger,D})^2 \neq (A^{\dagger,D})^2$, $A^2(A^{\dagger,D})^2 \neq A^{\dagger,D}$ *and* $A^{\dagger,D}A^2 \neq A^2$.

Now, the equivalent conditions when $A^{\dagger,D}$ is idempotent are given in what follows.

Theorem 2.8. *Suppose that* $A \in \mathbb{C}_k^{n\times n}$ *is given by* $A = A_1 + A_2$*, where* A_1 *and* A_2 *are given by* (2.1)*. Then* $A^{\dagger,D}$ *is idempotent if and only if any of the following statements is satisfied:*

(a) $T = I_t$;

(b) $A^k = A^{k+1}$;

(c) $A^{\dagger,D}A = A^{\dagger,D}$;

(d) $(A^{\dagger,D})^kA = (A^{\dagger,D})^k$;

(e) $(A^{\dagger,D})^kA^k = A^{\dagger,D}$;

(f) $A^kA^{\dagger,D} = A^k$.

Proof. (a). Since $(A^{\dagger,D})^2 = A^{\dagger}A^D$, we have that $A^{\dagger,D}$ is idempotent if and only if $A^{\dagger,D} = A^{\dagger}A^D$. Premultiplying $A^{\dagger}AA^D = A^{\dagger}A^D$ by A, we obtain that $AA^D = A^D$. By the point (b) of Theorem 2.6, we get $T = I_t$.

Conversely, if $T = I_t$, it can be directly checked that $A^{\dagger,D} = A^{\dagger}A^D$ from (2.3), (2.4) and (2.6).

(b). This follows similarly as in the point (b) of Theorem 2.6.

(c). If $A^{\dagger,D}$ is idempotent, then $(A^{\dagger,D})^*$ is also idempotent. It is noteworthy that $(A^{\dagger,D})^* = (A^*)^DA^*(A^*)^{\dagger} = (A^*)^{D,\dagger}$. Thus we now have $(A^*)^{D,\dagger}$ is idempotent, then it follows from condition (c) in Theorem 2.6 that $A^*(A^*)^{D,\dagger} = (A^*)^{D,\dagger}$. By taking the conjugate transpose of $A^*(A^*)^{D,\dagger} = (A^*)^{D,\dagger}$, we now obtain that $A^{\dagger,D}A = A^{\dagger,D}$. The above proof is completely reversible.

The proofs of the last three conditions are similar to point (c). □

Remark 2.9. *If* $A^{\dagger,D}$ *in Theorem* 2.8 *is replaced by* A^D*, Theorem* 2.8 *is still valid.*

We know that $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{P}}$ doesn't satisfy each of the four statements (c), (d), (e) and (f) in Theorem 2.6. Next theorem gives the necessary and sufficient conditions such that all four statements are satisfied.

Theorem 2.10. *Suppose that* $A \in \mathbb{C}_k^{n\times n}$ *is given by* $A = A_1 + A_2$*, where* A_1 *and* A_2 *are given by* (2.1)*. Then the following assertions are equivalent:*

(a) $T = I_t$ *and* $\mathcal{N}(N) \subseteq \mathcal{N}(S)$;

(b) $AA^{\dagger,D} = A^{\dagger,D}$;

(c) $A(A^{\dagger,D})^k = (A^{\dagger,D})^k$;

(d) $A^k(A^{\dagger,D})^k = A^{\dagger,D}$;

(e) $A^{\dagger,D}A^k = A^k$.

Proof. (a) ⇒ (b). Notice that $\mathcal{N}(N) \subseteq \mathcal{N}(S)$ is equivalent with $S(I_{n-t} - N^{\dagger}N) = 0$. If $T = I_t$, then the result can be directly checked by (2.6).

(b) ⇒ (c). It is evident.

(c) ⇒ (a). Note that $(A^{\dagger,D})^k = A^{\dagger}(A^D)^{k-1}$. By ($c$), we have $AA^{\dagger}(A^D)^{k-1} = A^{\dagger}(A^D)^{k-1}$. Thus, it follows from (2.3) and (2.4) that

$$\begin{bmatrix} T^{-k+1} & T^{-2k+1}\widetilde{T} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^*\triangle T^{-k+1} & T^*\triangle T^{-2k+1}\widetilde{T} \\ (I_{n-t} - N^{\dagger}N)S^*\triangle T^{-k+1} & (I_{n-t} - N^{\dagger}N)S^*\triangle T^{-2k+1}\widetilde{T} \end{bmatrix},$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$. Hence $T^*\triangle = I_t$, $(I_{n-t} - N^{\dagger}N)S^* = 0$, which implies $T = I_t$, $\mathcal{N}(N) \subseteq \mathcal{N}(S)$.

(b) ⇒ (d). Combining $AA^{\dagger,D} = AA^D$ with $AA^{\dagger,D} = A^{\dagger,D}$ immediately leads to the conclusion that $A^k(A^{\dagger,D})^k = (A^D)^kA^k = A^DA = AA^D = A^{\dagger,D}$.

(d) ⇒ (e). By (d) and the fact that $A^k(A^{\dagger,D})^k = AA^D$, we get that $A^{\dagger,D}A^k = A^k(A^{\dagger,D})^kA^k = AA^DA^k = A^k$.

(e) ⇒ (b). Postmultiplying $A^{\dagger,D}A^k = A^k$ by $(A^D)^k$ we have that $A^{\dagger,D} = AA^D = AA^{\dagger,D}$. □

Similarly, we can also deduce that $A^{\textcircled{\dagger}}$, $A^{D,\dagger}$ and $A^{\textcircled{w}}$ don't satisfy any of the four conditions (c), (d), (e) and (f) in Theorem 2.8 as will be shown in the next example:

Example 2.11. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

We have that $\operatorname{ind}(A) = 2$, *and*

$$A^{\textcircled{\dagger}} = A^{D,\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As in the Example 4.3, we can get that $X'A \neq X'$, $(X')^2A \neq (X')^2$, $(X')^2A^2 \neq X'$ *and* $A^2X' \neq A^2$ *for* $X' \in \{A^{\textcircled{\dagger}}, A^{D,\dagger}\}$.

Example 2.12. *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $\operatorname{ind}(A) = 2$, *and*

$$A^{\textcircled{w}} = (A^{\textcircled{w}})^2 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $A^{\textcircled{w}}A \neq A^{\textcircled{w}}$, $(A^{\textcircled{w}})^2A \neq (A^{\textcircled{w}})^2$, $(A^{\textcircled{w}})^2A^2 \neq A^{\textcircled{w}}$ *and* $A^2A^{\textcircled{w}} \neq A^2$.

The following theorems present some conditions such that $A^{\textcircled{\dagger}}$, $A^{D,\dagger}$ and $A^{\textcircled{w}}$ satisfy (c), (d), (e) and (f) of Theorem 2.8.

Theorem 2.13. *Suppose that* $A \in \mathbb{C}_k^{n\times n}$ *is given by* $A = A_1 + A_2$, *where* A_1 *and* A_2 *are given by* (2.1). *Then the following assertions are equivalent:*

(a) $T = I_t$ *and* $S = 0$;

(b) $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}$;

(c) $(A^{\textcircled{\dagger}})^kA = (A^{\textcircled{\dagger}})^k$;

(d) $(A^{\textcircled{\dagger}})^kA^k = A^{\textcircled{\dagger}}$;

(e) $A^kA^{\textcircled{\dagger}} = A^k$.

Proof. (b) $\Leftrightarrow$ (a). From (2.2), it follows that

$$\begin{aligned} A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}} &\Longleftrightarrow U\begin{bmatrix} I_t & T^{-1}S \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &\Longleftrightarrow T = I_t,\ S = 0. \end{aligned}$$

(c) $\Leftrightarrow$ (a). The proof is similar to (b) $\Leftrightarrow$ (a).

(d) $\Leftrightarrow$ (a). From (2.2) and (2.8), we obtain that

$$\begin{aligned}(A^{\textcircled{\dagger}})^k A^k = A^{\textcircled{\dagger}} &\iff U\begin{bmatrix} I_t & T^{-k}\widetilde{T} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T = I_t,\ \widetilde{T} = 0 (where\ \widetilde{T} = \sum_{j=0}^{k-1} SN^j) \\ &\iff T = I_t,\ \ S = 0.\end{aligned}$$

(e) $\Leftrightarrow$ (a). Similar as the part (d) $\Leftrightarrow$ (a). □

Theorem 2.14. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1). *Then the following assertions are equivalent:*

(a) $T = I_t$ *and* $\mathcal{N}(N^*) \subseteq \mathcal{N}(\widetilde{T})$;

(b) $A^{D,\dagger}A = A^{D,\dagger}$;

(c) $(A^{D,\dagger})^k A = (A^{D,\dagger})^k$;

(d) $(A^{D,\dagger})^k A^k = A^{D,\dagger}$;

(e) $A^k A^{D,\dagger} = A^k$.

where $\widetilde{T} = \sum_{j=0}^{k-1} SN^j$.

Proof. (a) $\Rightarrow$ (b). We know that $\mathcal{N}(N^*) \subseteq \mathcal{N}(\widetilde{T})$ is equivalent to $\widetilde{T}(I_{n-t} - NN^\dagger) = 0$. Thus the result can be directly verified by (2.5).

(b) $\Rightarrow$ (c). Evident.

(c) $\Rightarrow$ (a). Using (2.5), by $(A^{D,\dagger})^k A = (A^{D,\dagger})^k$, we get that

$$\begin{bmatrix} T^{-k+1} & T^k S + T^{-2k}\widetilde{T}N \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^{-k} & T^{-2k}\widetilde{T}NN^\dagger \\ 0 & 0 \end{bmatrix}.$$

Hence $T = I_t$, $\widetilde{T}(I_{n-t} - NN^\dagger) = 0$, which is equivalent to $T = I_t$, $\mathcal{N}(N^*) \subseteq \mathcal{N}(\widetilde{T})$.

(b) $\Rightarrow$ (d). Combining $A^{D,\dagger}A = A^{D,\dagger}$ with $A^{D,\dagger}A = A^D A$ immediately leads to the conclusion that $(A^{D,\dagger})^k A^k = AA^D = A^D A = A^{D,\dagger}$.

(d) $\Rightarrow$ (e). Since $(A^{D,\dagger})^k = (A^D)^{k-1}A^\dagger$. By ($d$), if $k = 1$, we get that $A^k A^{D,\dagger} = A^k(A^{D,\dagger})^k A^k = A^k(A^D)^{k-1}A^\dagger A^k = AA^\dagger A = A$. If $k \geq 2$, we have that $A^k A^{D,\dagger} = A^k(A^{D,\dagger})^k A^k = A^k(A^D)^{k-1}A^\dagger A^k = A^D A^2 A^\dagger A^k = A^k$.

(e) $\Rightarrow$ (b). Multiplying $A^k A^{D,\dagger} = A^k$ by $(A^D)^k$ we have that $A^{D,\dagger} = A^D A = A^{D,\dagger}A$. □

Theorem 2.15. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1). *Then the following assertions are equivalent:*

(a) $T = I_t$ *and* $SN = 0$;

(b) $A^{\textcircled{w}}A = A^{\textcircled{w}}$;

(c) $(A^{\textcircled{w}})^k A = (A^{\textcircled{w}})^k$;

(d) $(A^{\textcircled{w}})^k A^k = A^{\textcircled{w}}$;

(e) $A^k A^{\textcircled{w}} = A^k$.

Proof. (b) $\Leftrightarrow$ (a). From (2.7), it follows that

$$\begin{aligned}A^{\textcircled{w}}A = A^{\textcircled{w}} &\iff U\begin{bmatrix} I_t & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T = I_t,\ SN = 0.\end{aligned}$$

$(c) \Leftrightarrow (a)$. Similar as $(b) \Leftrightarrow (a)$.

$(d) \Leftrightarrow (a)$. From (2.7) and (2.8), it follows that

$$\begin{aligned}(A^{\textcircled{W}})^k A^k = A^{\textcircled{W}} &\iff U\begin{bmatrix} I_t & T^{-k}\widetilde{T} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{-1} = I_t,\ \widetilde{T} = S \\ &\iff T = I_t,\ SN = 0.\end{aligned}$$

$(e) \Leftrightarrow (a)$. Similar as $(d) \Leftrightarrow (a)$. □

Remark 2.16. *If the integer k in Theorems 2.6, 2.8, 2.10, 2.13, 2.14 and 2.15 is placed by $l (l \geq k)$, all the Theorems are still valid.*

3. Characterizations of matrices whose some generalized inverses are orthogonal idempotent

It is widely known that $\mathbb{C}_n^{\mathrm{OP}} \subseteq \mathbb{C}_n^{\mathrm{P}}$. Meanwhile, it follows from (2.2) that $A^{\textcircled{\dagger}} \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if $A^{\textcircled{\dagger}} \in \mathbb{C}_n^{\mathrm{P}}$. Therefore each of the six terms listed in Theorem 2.6 is equivalent to $A^{\textcircled{\dagger}} \in \mathbb{C}_n^{\mathrm{OP}}$. Then the main aim of this section is to investigate some characterizations for $A^{\textcircled{W}}$, A^D, $A^{D,\dagger}$ and $A^{\dagger,D}$ to be an orthogonal idempotent.

We will discuss some equivalent conditions for $A^{\textcircled{W}}$ and A^D to be an orthogonal idempotent.

Theorem 3.1. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by (2.1), $X_2 \in \{A^D, A^{\textcircled{W}}\}$. Then X_2 is orthogonal idempotent if and only if any of the following statements is satisfied:*

(a) $T = I_t$ *and* $S = 0$;

(b) $A^k = A^*A^k$;

(c) $AX_2 = X_2^*$;

(d) $X_2A = X_2^*$;

(e) $AX_2 = A^2A^{\textcircled{\dagger}}$;

(f) $X_2A = A^2A^{\textcircled{\dagger}}$;

(g) $A^kX_2^* = A^k$;

(h) $X_2^*A^k = A^k$;

(i) $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}$;

(j) $A^kA^{\textcircled{\dagger}} = A^k$;

(k) $(A^{\textcircled{\dagger}})^kA^k = A^{\textcircled{\dagger}}$;

(l) $(A^{\textcircled{\dagger}})^kA = (A^{\textcircled{\dagger}})^k$.

Proof. (a). By (2.3) we get that A^D is an orthogonal projector if and only if $T = I_t$ and $\widetilde{T} = 0$, i.e., $T = I_t$ and $S = 0$. Similarly, by (2.7) we have that $A^{\textcircled{W}} \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if $T = I_t$ and $S = 0$.

(b). Suppose $A^k = A^*A^k$. Using (2.8), it follows that

$$\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^*T^k & T^*\widetilde{T} \\ S^*T^k & S^*\widetilde{T} \end{bmatrix}.$$

Hence $T^* = I_t$ and $\widetilde{T} = 0$, which is equivalent to $T = I_t$ and $S = 0$. The sufficient condition can be easily checked.

(c). Assume $X_2 \in \mathbb{C}_n^{\mathrm{OP}}$, it's easy to verify that $AX_2 = (X_2)^*$.

On the contrary, from (2.3) and (2.7), we have

$$X_2 = U\begin{bmatrix} T^{-1} & W \\ 0 & 0 \end{bmatrix}U^*, \tag{3.1}$$

where $W \in \{(T^{k+1})^{-1}\widetilde{T},\ T^{-2}S\}$. If $AX_2 = (X_2)^*$, it follows from (3.1) that

$$\begin{bmatrix} I_t & TW \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T^{-1})^* & 0 \\ W^* & 0 \end{bmatrix},$$

where $W \in \{(T^{k+1})^{-1}\widetilde{T},\ T^{-2}S\}$. It implies $T = I_t$ and $S = 0$ since T is nonsingular.

(d), (e) and (f). These proofs are analogous to that of (c).

(g). By (2.8) and (3.1), it follows that

$$\begin{aligned} A^k(X_2)^* = A^k &\iff U\begin{bmatrix} T^k(T^{-1})^* + \widetilde{T}W^* & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^* \\ &\iff \widetilde{T} = 0,\ T^k(T^{-1})^* = T^k \\ &\iff S = 0,\ T = I_t. \end{aligned}$$

(h). By (2.8) and (3.1), we have that $X_2^*A^k = A^k$ is equivalent with,

$$U\begin{bmatrix} (T^{-1})^*T^k & (T^{-1})^*\widetilde{T} \\ W^*T^k & W^*\widetilde{T} \end{bmatrix}U^* = U\begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^*,$$

where $W \in \{(T^{k+1})^{-1}\widetilde{T},\ T^{-2}S\}$, which is equivalent with $T = I_t,\ S = 0$.

(i) – (l). Note that X_2 is orthogonal idempotent if and only if $T = I_t$ and $S = 0$. Thus, these can be directly demonstrated by Theorem 2.13. □

Secondly, several sufficient and necessary conditions for $A^{D,\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$ are given in the following theorem.

Theorem 3.2. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1)*. Then $A^{D,\dagger}$ is orthogonal idempotent if and only if any of the following statements is satisfied:*

(a) $T = I_t$ *and* $SN = 0$;

(b) $AA^{D,\dagger} = (A^{D,\dagger})^*$;

(c) $A^{D,\dagger}A = A^{Ⓦ}$;

(d) $AA^{Ⓦ} = A^D$;

(e) $A^{Ⓦ}A = A^{Ⓦ}$;

(f) $A^kA^{Ⓦ} = A^k$;

(g) $(A^{Ⓦ})^kA^k = A^{Ⓦ}$;

(h) $(A^{Ⓦ})^kA = (A^{Ⓦ})^k$.

Proof. (a). By (2.5) it is easy to verify that $A^{D,\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if $T = I_t$ and $\widetilde{T}NN^\dagger = 0$, i.e., $T = I_t$ and $SN = 0$.

(b). By (2.5) we have that $AA^{D,\dagger} = (A^{D,\dagger})^*$ is equivalent with

$$\begin{bmatrix} I_t & T^{-k}\widetilde{T}NN^\dagger \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T^{-1})^* & 0 \\ ((T^{k+1})^{-1}\widetilde{T}NN^\dagger)^* & 0 \end{bmatrix},$$

which is further equivalent with $T = I_t$ and $SN = 0$.

(c). By (2.5) and (2.7), it follows that

$$\begin{aligned} A^{D,\dagger}A = A^{Ⓦ} &\iff U\begin{bmatrix} I_t & T^{-k}\widetilde{T} \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{-1} = I_t,\ T^{-k}\widetilde{T} = T^{-2}S \\ &\iff T = I_t,\ SN = 0. \end{aligned}$$

(d). The proof follows directly by (c).

(e) – (h). The proof follows by (a) and Theorem 2.15. □

Finally, some equivalent conditions for $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}}$ are given in the following theorem.

Theorem 3.3. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1). *Then the following assertions are equivalent:*

(a) $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}}$;

(b) $T = I_t$ *and* $\widetilde{T} = S(I_{n-t} - N^\dagger N)$;

(c) $T = I_t$ *and* $A^\dagger A^k = (A^{\dagger,D})^*$.

Proof. $(a) \Rightarrow (b)$. Since $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}} \subseteq \mathbb{C}_n^{\mathrm{P}}$, we have by Theorem 2.8 that $T = I_t$. It follows from (2.6) that

$$A^{\dagger,D} = U\begin{bmatrix} \triangle & \triangle\widetilde{T} \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle\widetilde{T} \end{bmatrix}U^*, \tag{3.2}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} SN^j$ and $\triangle = (I_t + S(I_{n-t} - N^\dagger N)S^*)^{-1}$. Since $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}}$, we get that $\widetilde{T} = S(I_{n-t} - N^\dagger N)$.

$(b) \Rightarrow (c)$. It follows by a direct calculations with the use of (2.4), (2.6) and (2.8).

$(c) \Rightarrow (a)$. Since $T = I_t$, we get

$$A^\dagger = U\begin{bmatrix} \triangle & -\triangle SN^\dagger \\ (I_{n-t} - N^\dagger N)S^*\triangle & N^\dagger - (I_{n-t} - N^\dagger N)S^*\triangle SN^\dagger \end{bmatrix}U^*, \quad A^k = U\begin{bmatrix} I_t & \widetilde{T} \\ 0 & 0 \end{bmatrix}U^*.$$

Thus, it follows from $A^\dagger A^k = (A^{\dagger,D})^*$ and (3.2) that

$$\begin{bmatrix} \triangle & \triangle\widetilde{T} \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle\widetilde{T} \end{bmatrix} = \begin{bmatrix} \triangle & \triangle S(I_{n-t} - N^\dagger N) \\ (\widetilde{T})^*\triangle & (\widetilde{T})^* S(I_{n-t} - N^\dagger N) \end{bmatrix}.$$

Hence $\widetilde{T} = S(I_{n-t} - N^\dagger N)$. Consequently, we have $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}}$. □

Corollary 3.4. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1), *$X_2 \in \{A^D, A^{Ⓦ}\}$. If $X_2 \in \mathbb{C}_n^{\mathrm{OP}}$, then any of the following statements is satisfied:*

(a) $A^{D,\dagger} \in \mathbb{C}_n^{\mathrm{OP}}$;

(b) $A^{\dagger,D} \in \mathbb{C}_n^{\mathrm{OP}}$.

Proof. It's evident from Theorems 3.1, 3.2 *and* 3.3. □

Remark 3.5. *If the integer k in Theorems* 3.1, 3.2 *and* 3.3 *is placed by $l(l \geq k)$, all the Theorems are still valid in the section.*

4. Further properties of orthogonal idempotent

In this section, we study equivalent conditions for a matrix A to be orthogonal idempotent in terms of some other generalized inverses, like core-EP, Drazin, DMP and dual DMP and weak group inverse.

Theorem 4.1. *Let $A \in \mathbb{C}_k^{n\times n}$ and $\widetilde{X} \in \{A^{†}, A^D, A^{D,\dagger}, A^{\dagger,D}, A^{Ⓦ}\}$. Then A is orthogonal idempotent if and only if $\widetilde{X} \in \mathbb{C}_n^{\mathrm{P}}$ and $A^l = A^*$, for some $l \in \mathbb{N}$, $l \geq k$.*

Proof. Suppose that A is given by $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$, it's clear that $A \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if $T = I_t$, $S = 0$ and $N = 0$. Thus we can easily conclude that $\widetilde{X} \in \mathbb{C}_n^{\mathrm{P}}$ and $A^l = A^*$.

Conversely, if $\widetilde{X} \in \mathbb{C}_n^{\mathrm{P}}$, we have $T = I_t$ by Theorem 2.6. We now obtain that

$$A = U\begin{bmatrix} I_t & S \\ 0 & N \end{bmatrix}U^*. \tag{4.1}$$

By $A^l = A^*$, it's easy to verify that $S = 0$ and $N = 0$. □

Theorem 4.2. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1) *and let $X \in \{A^{\textcircled{\dagger}}, A^{\textcircled{w}}, A^D\}$. Then A is orthogonal idempotent if and only if any of the following statements is satisfied:*

(a) $A^*X = A^*$;

(b) $XA^* = A^*$;

(c) $A^{D,\dagger}A^* = A^*$;

(d) $A^*A^{\dagger,D} = A^*$.

Proof. It is noteworthy that we just have to verify that each of the four conditions is equivalent to $T = I_t$, $S = 0$ and $N = 0$.

(a) and (b). According to (2.2), (2.3) and (2.7), it's not difficult to demonstrate that statement (a) and (b) are equivalent to $T = I_t$, $S = 0$ and $N = 0$.

(c). By (2.5), we obtain that

$$\begin{aligned} A^{D,\dagger}A^* = A^* \iff & U\begin{bmatrix} T^{-1}T^* + (T^{k+1})^{-1}\widetilde{T}NN^\dagger S^* & (T^{k+1})^{-1}\widetilde{T}NN^\dagger N^* \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ \iff & T^*T^{-1} = T^*,\ S^* = 0,\ N^* = 0 \\ \iff & T = I_t,\ S = 0,\ N = 0. \end{aligned}$$

(d). Suppose that $A^*A^{\dagger,D} = A^*$, it follows from (2.6) that

$$\begin{bmatrix} (T^*)^2\triangle & (T^*)^2\triangle T^{-k}\widetilde{T} \\ S^*T^*\triangle + N^*(I_{n-t} - N^\dagger N)S^*\triangle & S^*T^*\triangle T^{-k}\widetilde{T} + N^*(I_{n-t} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T} \end{bmatrix} = \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}.$$

Since T and $\triangle$ are nonsingular, we now deduce that $(T^*)^2\triangle = T^*$, $\widetilde{T} = 0$ and $N^* = 0$. Combining these three equations, we obtain that $T = I_t$, $S = 0$ and $N = 0$. The reverse is obvious. □

Notice that we can imply $A^*A^{D,\dagger} = A^*$ and $A^{\dagger,D}A^* = A^*$ if $A \in \mathbb{C}_n^{\mathrm{OP}}$ in Theorem 4.1. But, the converse is invalid. We present the following example to illustrate that.

Example 4.3. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $\mathrm{ind}(A) = 1$,

$$A^{D,\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger,D} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \end{bmatrix}.$$

It is easy to see that $A^*A^{D,\dagger} = A^*$, $A^{\dagger,D}A^* = A^*$ and $A^2 = A$, but $A^* \neq A$.

In the following theorem, we are going to give some new equivalent conditions such that the reverse is also true.

Lemma 4.4. [2] *Assume that $A \in \mathbb{C}^{n\times n}$. Then A is orthogonal idempotent if and only if both A and $A^\dagger$ are idempotent.*

Theorem 4.5. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1). *If $A^\dagger$ is idempotent, then A is orthogonal idempotent if and only if any of the following statements is satisfied:*

(a) $A^*A^{D,\dagger} = A^*$;

(b) $A^{\dagger,D}A^* = A^*$.

Proof. Combining Theorem 4.1 and Lemma 4.4, we just have to prove that each of the two statements is equivalent to the fact that A is idempotent, which is also equivalent to the requirement that $T = I_t$ and $N = 0$.

(a). From (2.5), it follows that

$$\begin{aligned} A^*A^{D,\dagger} = A^* &\iff U\begin{bmatrix} T^*T^{-1} & T^*(T^{k+1})^{-1}\widetilde{T}NN^\dagger \\ S^*T^{-1} & S^*(T^{k+1})^{-1}\widetilde{T}NN^\dagger \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ &\iff T^*T^{-1} = T^*,\ N^* = 0 \\ &\iff T = I_t,\ N = 0. \end{aligned}$$

(b). By (2.6), it follows that

$$\begin{aligned} A^{\dagger,D}A^* = A^* &\iff U\begin{bmatrix} T^*\triangle(T^* + T^{-k}\widetilde{T}S^*) & T^*T^{-k}\widetilde{T}N^* \\ (I_{n-t} - N^\dagger N)S^*\triangle(T^* + T^{-k}\widetilde{T}S^*) & (I_{n-t} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T}N^* \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ &\iff T^*\triangle(T^* + T^{-k}\widetilde{T}S^*) = T^*,\ N^* = 0 \\ &\iff T = I_t,\ N = 0. \end{aligned}$$

□

Theorem 4.6. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1), *$X_2 \in \{A^D, A^{ⓦ}\}$. Then the following assertions are equivalent:*

(a) A is idempotent and X_2 is orthogonal idempotent;

(b) A is idempotent and A is either Hermitian, EP, or normal ;

(c) A is core matrix and X_2 is orthogonal idempotent;

(d) A is orthogonal idempotent.

Proof. (a) ⇒ (b). Obviously, condition (a) in the theorem can be equivalently expressed as the conjunction $T = I_t$, $S = 0$ and $N = 0$. Therefore, the point (b) is apparently satisfied.

(b) ⇒ (c). We know that idempotency of A is equivalent with $T = I_t$ and $N = 0$. Then, A can be expressed in the following form

$$A = U\begin{bmatrix} I_t & S \\ 0 & 0 \end{bmatrix}U^*. \tag{4.2}$$

Thus, if A is either Hermitian, EP, or normal, we get $S = 0$. From Theorem 3.1, it follows that X_2 is orthogonal idempotent.

(c) ⇒ (d). Because A is core matrix, we get that $N = 0$. It can be verified directly by Theorem 3.1 that A is orthogonal idempotent.

(d) ⇒ (a). The proof is obvious. □

Corollary 4.7. *Suppose that $A \in \mathbb{C}_k^{n\times n}$ is given by $A = A_1 + A_2$, where A_1 and A_2 are given by* (2.1). *If $A \in \mathbb{C}_n^{\mathrm{P}}$, then A is orthogonal idempotent if and only if any of the following statements is satisfied:*

(a) $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}$; *(b)* $A^kA^{\textcircled{\dagger}} = A^k$;

(c) $(A^{\textcircled{\dagger}})^kA^k = A^{\textcircled{\dagger}}$; *(d)* $(A^{\textcircled{\dagger}})^kA = (A^{\textcircled{\dagger}})^k$.

Proof. From (2.1), it's easy to prove that $A \in \mathbb{C}_n^{\mathrm{P}}$ if and only if $T = I_t$, $N = 0$. By Theorem 2.13, we have that each of the four statements given in the theorem is equivalent with $S = 0$. Thus the corollary holds. □

Remark 4.8. *If the integer k in Corollary* 4.7 *is replaced by l(l ≥ k), Corollary* 4.7 *still holds.*

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References

[1] O.M. Baksalary and G. Trenkler. Core inverse of matrices. Linear and Multilinear Algebra, 58(2010): 681-697.
[2] O.M. Baksalary and G. Trenkler. On matrices whose Moore-Penrose inverse is idempotent, Linear and Multilinear Algebra, (2020): 1-13.
[3] A. Ben-Israel and T. N. E. Greville. Generalized Inverses: Theory and Applications, 2nd Edition. Springer Verlag, New York, 2003.
[4] D.S. Cvetković-Ilić. Expression of the Drazin and MP-inverse of partitioned matrix and quotient identity of generalized Schur complement, Applied Mathematics and Computation, 213(2009): 18-24.
[5] D.S. Cvetković-Ilić and C. Deng. Some results on the Drazin invertibility and idempotents, Journal of Mathematical Analysis and Applications, 359(2)(2009): 731-738.
[6] C.Y. Deng and H.K. Du. Representation of the Moore-Penrose inverse of 2 × 2 block operator valued matrices, Journal of the Korean Mathematical Society, 46(2009): 1139-1150.
[7] M.P. Drazin. Pseudo-inverses in associative rings and semigroups, American Mathematical Monthly, 65(7)(1958): 506-514.
[8] D.E. Ferreyra, F.E. Levis and N. Thome. Revisiting the core-EP inverse and its extension to rectangular matrices, Quaestiones Mathematicae, 41(2018): 1-17.
[9] D.E. Ferreyra, F.E. Levis and N. Thome. Characterizations of k-commutative egualities for some outer generalized inverse, Linear and Multilinear Algebra, 68(1)(2020): 177-192.
[10] A. Galántai. Projectors and projection methods, New York: Springer; 2004.
[11] S. Gigola, L. Lebtahi and N. Thome. The inverse eigenvalue problem for a Hermitian reflexive matrix and the optimization probiem, Journal of Computational and Applied Mathematics, 291(2016): 449-457.
[12] R.E. Hartwing. The weighted *-core-nilpotent decomposition, Linear Algebra and its Applications, 211(1994): 101-111.
[13] J.J. Koliha, D.S. Cvetković-Ilić and C. Deng. Generalized Drazin invertibility of combinations of idempotents, Linear Algebra and its Applications, 437(2012): 2317-2324.
[14] S. Lable and A. Perelomova. Dynamical projectors method in hydro and electrodynamic, Boca Raton: CRC Press; 2018.
[15] S.B. Malik and N. Thome. On a new generalized inverse for matrices of an arbitrary index, Applied Mathematics and Computation, 226(2014): 575-580.
[16] R.A. Penrose. A generalized inverse for matrices, Mathematical Proceedings of the Cambridge Philosophical Society, 51(03)(1955): 406-413.
[17] K.M. Prasad and K.S. Mohana. Core-EP inverse, Linear and Multilinear Algebra, 62(2014): 792-802.
[18] C.R. Rao and M.B. Rao. Matrix algebra and its applications to statistics and electrodynamics, Singapore: World Scientific; 1998.
[19] F. Soleimani, P.S. Stanimirović and F. Soleymani. Some matrix iterations for computing generalized inverses and balancing chemical equations, Algorithms, 8(2015): 982-998.
[20] H.X. Wang. Core-EP decomposition and its applications, Linear Algebra and its Applications, 508(2016): 289-300.
[21] H.X. Wang and J.L. Chen. Weak group inverse, Open Mathematics, 16(1)(2018): 1218-1232.
[22] K.Z. Zuo and Y.J. Cheng. The new rexisitation of core EP inverse of mateices, Filomat, 33(10)(2019): 3061-3072.