A General Inequality for Pointwise Semi-Slant Warped Products in Nearly Kenmotsu Manifolds

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Abstract. In this paper, we prove that every pointwise semi-slant warped product submanifold \( M = N^T \times_f N^\theta \) in a nearly Kenmotsu manifold \( \tilde{M} \) satisfies the following inequality:
\[
\|h\|^2 \geq 2n_2 \left( 1 + \frac{10}{9} \cot^2 \theta \right) \left( \|\tilde{\nabla}(\ln f)\|^2 - 1 \right),
\]
where \( n_2 = \dim N^\theta \), \( \tilde{\nabla}(\ln f) \) is the gradient of \( \ln f \) and \( \|h\| \) is the length of the second fundamental form of \( M \). The equality and special cases of the inequality are investigated.

1. Introduction

It was proved in [20] that every nearly Kenmotsu manifold is locally isometric to the warped product \( \mathbb{R} \times_f M \) of a real line \( \mathbb{R} \) and a nearly Kaehler manifold \( \tilde{M} \). It was also proved that a normal nearly Kenmotsu manifold is a Kenmotsu manifold [20]. Nearly Kaehler manifolds were defined and studied by Gray in his series papers [22, 23]. Nearly Sasakian manifolds were introduced by Blair et al. [4]. Later, Olszak [29] studied nearly Sasakian non-Sasakian manifolds of dimension 5. In [19], Endo investigated the geometry of nearly cosymplectic manifolds. Later, Cappelletti Montano and Dileo studied nearly Sasakian manifolds for some other fundamental properties [7]. The geometry of nearly Kenmotsu manifolds was investigated in [34].

On the other hand, warped product manifolds introduced by Bishop on O’Neill to investigate the geometry of pseudo-Riemannian manifolds of negative curvature [2]. After a long gape, B.-Y. Chen introduced the notion of warped product submanifolds of Kaehler manifolds in his series papers [11, 12]. He investigate the geometry of CR-warped product submanifolds and proved that every CR-warped product \( M = N^T \times_f N^\perp \) of a Kaehler manifold satisfies the following inequality
\[
\|h\|^2 \geq 2q\|\tilde{\nabla}(\ln f)\|^2, \quad q = \dim N^\perp
\] (1)
where \( \|h\|^2 \) is the squared norm of the second fundamental form \( h \) of \( M \) and \( \tilde{\nabla}(\ln f) \) is the gradient of \( \ln f \). Later, this inequality known as Chen’s first inequality for warped products and investigated for different

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kinds of warped product submanifolds of almost Hermitian as well as almost contact metric manifolds [1, 14, 17, 28, 33, 37, 38, 40, 43, 44].

In this paper, we study pointwise semi-slant warped product submanifolds of the form $N^T \times_f N^0$ of Kenmotsu manifolds where $N^T$ and $N^0$ are invariant and proper pointwise slant submanifolds and obtain the following general inequality:

$$\|\eta\|^2 \geq 2n_2 \left( 1 + \frac{10}{9} \cot^2 \theta \right) (\|\nabla (\ln f)\|^2 - 1), \quad n_2 = \dim N^0$$

The equality case of this inequality is also investigated and a special case of this inequality is given for contact CR-warped products.

2. Preliminaries

An odd dimensional differentiable manifold $\tilde{M}$ endowed with a $(1,1)$ tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ is called an almost contact metric manifold, if

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map and for any vector fields $X, Y$ on $\tilde{M}$. The structure $(\varphi, \xi, \eta, g)$ is called the almost contact metric structure on $\tilde{M}$ ([3], [4]). This structure also satisfies:

$$\eta(X) = g(X, \xi), \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, Y) = -g(X, \varphi Y).$$

In this paper, we refer to $\xi$ as the structure vector field (Reeb vector field) and to $\eta$ as the dual (Reeb form) of $\xi$.

An almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ is called a nearly Kenmotsu manifold [34], if

$$(\tilde{\nabla}_X \varphi) Y + (\tilde{\nabla}_Y \varphi) X = -\eta(Y)\varphi X - \eta(X)\varphi Y$$ \quad (2)

for all $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ is the Lie algebra of the vector fields on $\tilde{M}$ and $\tilde{\nabla}$ is the Levi-Civita connection of $g$. Moreover, if $\tilde{M}$ satisfies

$$(\tilde{\nabla}_X \varphi) Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$ \quad (3)

then it is called a Kenmotsu manifold [27]. It was proved that every Kenmotsu manifold is a nearly Kenmotsu manifold but converse is not true in general [20]. The following useful result is proved in [20].

**Proposition 2.1.** For a nearly Kenmotsu manifold, we have

$$g(\tilde{\nabla}_X \xi, Y) + g(X, \tilde{\nabla}_Y \xi) = 2g(\varphi X, \varphi Y),$$ \quad (4)

for any vector fields $X, Y \in \Gamma(T\tilde{M})$.

Now, we give the brief introduction of warped product manifolds.

Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian (or semi-Riemannian) manifolds and $f$ be a positive smooth function on $B$. The warped product of $B$ and $F$ is the Riemannian manifold

$$B \times_f F = (M = B \times F, g)$$

equipped with the warped metric $g = g_B + f^2 g_F$. The function $f$ is called the warping function and a warped product manifold $M$ is said to be trivial or simply a Riemannian product manifold of $B$ and $F$ if $f$ is constant (see, for instance, [2]).

Let $X$ be a vector field on $B$ and $Z$ be another vector field on $F$. Then, from Lemma 7.3 of [2], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z,$$ \quad (5)
where $\nabla$ denotes the Levi-Civita connection on $M$. Now for a smooth function $f$ on an $n$-dimensional manifold $M$, we have

$$
\|\hat{\nabla} f\|^2 = \sum_{i=1}^{m} (e_i(f))^2
$$

(6)

for the given orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ on $M$, where $\hat{\nabla} f$ is the gradient of $f$ defined by $g(\hat{\nabla} f, X) = X(f)$.

**Remark 2.2.** It is also important to note that for a warped product $M = B \times_f F$; $B$ is totally geodesic and $F$ is totally umbilical in $M$ [2, 11].

Now, if $M$ is a Riemannian manifold isometrically immersed in another Riemannian manifold $\tilde{M}$, then formulas of Gauss and Weingarten are respectively given by

$$
\hat{\nabla}_X Y = \nabla_X Y + h(X, Y),
$$

(7)

$$
\hat{\nabla}_X N = -A_X X + \nabla_X^N N,
$$

(8)

for any vector field $X$, $Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle, $h$ is the second fundamental form and $A$ is the shape operator of the submanifold. They are related by $g(h(X, Y), N) = g(A_X X, Y)$.

A submanifold $M$ is said to be totally geodesic if $h = 0$ and totally umbilical if $h(X, Y) = g(X, Y)H$, $\forall X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$ is the mean curvature vector of $M$. For any $x \in M$ and $\{e_1, \cdots, e_n, e_{2m+1}, \cdots, e_{2m+1}\}$ is an orthonormal frame of the tangent space $T_x M$ such that $e_1, \cdots, e_n$ are tangent to $M$ at $x$. Then, we set

$$
h'_{ij} = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \quad i, j \in \{1, \cdots, n\}, \quad r \in \{n+1, \cdots, 2m+1\}.
$$

(9)

B.-Y. Chen [9, 10] introduced a generalized class of holomorphic (invariant) and totally real (anti-invariant) submanifolds known as slant submanifolds in complex geometry. Later, A. Lotta [26] has extended Chen’s idea for contact metric manifolds.

A submanifold $M$ tangent to $\xi$ is said to be **slant** if for any $p \in M$ and any $X \in T_p M$, linearly independent to $\xi$, the angle between $\varphi X$ and $T_p M$ is a constant $\theta \in [0, \pi/2]$, called the **slant angle of $M$ in $\tilde{M}$**.

As natural extension of slant submanifolds, Etayo [21] introduced the notion of pointwise slant submanifolds. Later, these submanifolds were studied by Chen and Garay [15] for their characterizations and fundamental properties. They proved many interesting results and provided a method that how to construct non-trivial examples of such submanifolds. They defined pointwise slant submanifolds as follows:

A submanifold $M$ is called **pointwise slant** [15, 21, 42] if for any nonzero vector $X \in T_p M (p \in M)$, the angle $\theta(X)$ between $\varphi X$ and $T_p M$ is independent of the choice of $X \in T_p M$. In this case, $\theta$ defines a function on $M$, called the **slant function**. In particular, if the slant function $\theta$ is globally constant on $M$, then $M$ is said to be a slant submanifold or a $\theta$-slant submanifold.

Anti-invariant submanifolds are pointwise slant submanifolds with slant function $\theta = \frac{\pi}{2}$ everywhere on $M$. A pointwise slant submanifold is called **proper** if $0 < \theta < \frac{\pi}{2}$. See [? ? ] for non-trivial examples of pointwise slant submanifolds.

For any vector field $X \in \Gamma(TM)$, we have

$$
\varphi X = TX + FX,
$$

(10)

where $TX$ and $FX$ are the tangential and normal components of $\varphi X$, respectively.

We recall the following useful characterization from [42].
Proposition 2.3. Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$ with $\xi \in \Gamma(TM)$. Then $M$ is pointwise slant if and only if
\[ T^2 = \cos^2 \theta (-I + \eta \otimes \xi), \]  
where $\theta$ is the slant function and $I$ denotes the identity map on $TM$.

Following relations are straightforward consequence of (11)
\[ g(TX, TY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \]  
\[ g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \]  
for vector fields $X, Y \in \Gamma(M)$. Also, for pointwise slant submanifolds, we have
\[ tFX = \sin^2 \theta (-X + \eta(X)\xi), \quad fFX = -FTX, \quad X \in \Gamma(TM). \]  

A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a contact CR-submanifold [14] if there exist a pair of orthogonal distributions $D^T$ and $D^\perp$ such that
\[ TM = D^T \oplus D^\perp \oplus \langle \xi \rangle, \]  
where $D^T$ is $\phi$-invariant i.e., $\phi D^T \subseteq D^T$ and $D^\perp$ is anti-invariant i.e., $\phi D^\perp \subset T^\perp M$.

As a generalization of contact CR-submanifold, we define pointwise semi-slant submanifolds as follows:

Definition 2.4. Let $\tilde{M}$ be an almost contact metric manifold and $M$ be a submanifold of $\tilde{M}$ such that the structure vector field $\xi$ is tangent to $M$. Then $M$ is called a pointwise semi-slant submanifold of $\tilde{M}$ if there exists a pair of orthogonal distributions $D^T$ and $D^\theta$ on $M$ such that
\[ TM = D^T \oplus D^\theta \oplus \langle \xi \rangle \]  
where $D^T$ is $\phi$-invariant i.e., $\phi(D^T) \subseteq D^T$ and $D^\theta$ is a proper pointwise slant distribution with slant function $\theta$, $0 < \theta < \pi/2$.

A pointwise semi-slant submanifold $M$ is called proper if neither $\dim D^T = 0$ nor the slant function of $D^\theta$ is constant.

Clearly, semi-slant and contact CR-submanifolds are the pointwise semi-slant slant submanifolds with slant function $\theta$ is globally constant and $\theta = \pi/2$, repetitively.

The normal bundle of a pointwise semi-slant submanifold $M$ is decomposed as
\[ T^\perp M = F\mathcal{D}^\theta \oplus \mu \]  
where $\mu$ is the maximal $\phi$-invariant normal subbundle in $T^\perp M$.

3. Definition and a basic lemma

In this section, we give some preparatory results on pointwise semi-slant warped products. First, we define

Definition 3.1. A warped product of an $\phi$-invariant submanifold and a proper pointwise slant submanifold in an almost contact metric manifold $\tilde{M}$ is called a pointwise semi-slant warped product and it is denoted by $N^T \times_f N^\theta$ or $N^\theta \times_f N^T$, where $N^T$ and $N^\theta$ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively.

We accept the following convention that $X, Y$ are vector fields on $N^T$ and $Z, W$ are the vector fields $N^\theta$ and for the simplicity we denote tangent spaces of $N^T$ and $N^\theta$ by the same $D^T$ and $D^\theta$, respectively.
On the other hand, we know that there is no proper warped product. Using (5), we obtain

\[ g(h(X,Y),\varphi Z) = 0; \]

for any \( X, Y \in \Gamma(\Sigma^T) \) and \( Z \in \Gamma(\Sigma^0) \), we have

\[ g(h(X,Z),FW) = -\frac{1}{2} (\eta(X) - X(\ln f)) g(TZ,W) - \varphi X(\ln f) g(Z,W). \]

Proof. From (4) and (5), for any \( Z \in \Gamma(\Sigma^0) \), we have

\[ 2\xi(\ln f) g(Z,W) = g(\tilde{\nabla}_Z \xi, W) + g(\tilde{\nabla}_W \xi, Z) = 2g(\varphi Z, \varphi W) = 2g(Z,W), \]

which gives (i). For the second part of the lemma, we have

\[ g(h(X,Y),FZ) = g(\tilde{\nabla}_Z \eta, \varphi Z - TZ) = g((\tilde{\nabla}_Z \eta)Y, Z) - g(\tilde{\nabla}_Z \eta, Z) + g(\varphi \tilde{\nabla}_Z Y, Z), \]

for any \( X, Y \in \Gamma(\Sigma^T) \) and \( Z \in \Gamma(\Sigma^0) \). Since \( \nabla_X Y \in \Gamma(\Sigma^T) \), the last two terms in r.h.s. of above equation are identically zero. Then, we find

\[ g(h(X,Y),FZ) = g((\tilde{\nabla}_Z \eta)Y, Z). \]

Hence, (ii) follows from (17) via polarization identity and (2). In the similar way, we have

\[ g(h(X,Z),FW) = g((\tilde{\nabla}_Z \eta)X, W) - g(\tilde{\nabla}_Z \eta, W) - g(\tilde{\nabla}_Z X, TW). \]

Using (5), we obtain

\[ g(h(X,Z),FW) = g((\tilde{\nabla}_Z \eta)X, W) - \varphi X(\ln f) g(Z,W) - X(\ln f) g(Z,TW). \]

On the other hand, we know that

\[ g(h(X,Z),FW) = g((\tilde{\nabla}_Z \eta)X, W) - g(\tilde{\nabla}_Z X, TW) - g(\tilde{\nabla}_Z FZ, W) - X(\ln f) g(Z,TW). \]

Using (8) and (5), we find

\[ g(h(X,Z),FW) = g((\tilde{\nabla}_Z \eta)X, W) + h(X,W), FZ). \]

Then, from (18) and (20) together with (2), we derive

\[ 2g(h(X,Z),FW) = -\eta(X) g(TZ,W) - \varphi X(\ln f) g(Z,W) + X(\ln f) g(TZ,W). + g(h(X,W), FZ). \]

Third relation immediately follows from above relation via polarization identity. Hence, the lemma is proved completely.

Now, if we interchange \( Z \) by \( TZ \) in (19) and using Proposition 2.3 and (5), then we find

\[ g(h(X,TZ),FW) = g((\tilde{\nabla}_Z \eta)TZ, W) + \cos^2 \theta X(\ln f) g(Z,W) - \sin 2\theta X(\theta) g(Z,W) + g(h(X,W), FTZ) - \cos^2 \theta X(\ln f) g(Z,W). \]

Since, \( \theta \) is the slant function on \( N^0 \) hence \( X(\theta) = 0, \forall X \in \Gamma(\Sigma^T) \). Then, Eq. (21) takes the form

\[ g(h(X,TZ),FW) = g((\tilde{\nabla}_Z \eta)TZ, W) + g(h(X,W), FTZ). \]
On the other hand, from (18), we have
\[ g(h(X, TZ), FW) = g((\nabla_{TZ}\varphi)X, W) - \varphi X(ln f)g(TZ, W) - \cos^2 \theta X(ln f)g(Z, W). \] (23)

Then, (22), (23) and (2) imply that
\[ 2g(h(X, TZ), FW) = g(h(X, W), FTZ) + \cos^2 \theta (\eta(X) - X(ln f)) g(Z, W) - \varphi X(ln f)g(TZ, W). \] (24)

Furthermore, from Lemma 3.3(iii), we have
\[ 2g(h(X, TZ), FW) = \frac{2}{3} \cos^2 \theta (\eta(X) - X(ln f)) g(Z, W) - 2\varphi X(ln f)g(TZ, W). \] (25)

Then, from (24) and (25), we derive
\[ g(h(X, W), FTZ) = -\frac{1}{3} \cos^2 \theta (\eta(X) - X(ln f)) g(Z, W) - \varphi X(ln f)g(TZ, W). \] (26)

The following relations are easily obtained by interchanging \( X \) by \( \varphi X, \ Z \) by \( TZ \) and \( W \) by \( TW \) in Lemma 3.3(iii) and (26).
\[ g(h(\varphi X, W), FZ) = -\frac{1}{3} \varphi X(ln f)g(TZ, W) + (X(ln f) - \eta(X)) g(Z, W), \] (27)
\[ g(h(X, TW), FZ) = \frac{1}{3} \cos^2 \theta (\eta(X) - X(ln f)) g(Z, W) + \varphi X(ln f)g(TZ, W), \] (28)
\[ g(h(X, TW), FTZ) = -\frac{1}{3} \cos^2 \theta (\eta(X) - X(ln f)) g(TZ, W) - \cos^2 \theta \varphi X(ln f)g(Z, W), \] (29)
\[ g(h(\varphi X, TW), FZ) = -\frac{1}{3} \cos^2 \theta \varphi X(ln f)g(Z, W) - (X(ln f) - \eta(X)) g(TZ, W), \] (30)
\[ g(h(\varphi X, TW), FTZ) = -\frac{1}{3} \cos^2 \theta \varphi X(ln f)g(TZ, W) + \cos^2 \theta (X(ln f) - \eta(X)) g(Z, W). \] (31)

In particular, if \( X = \xi \), then we find
\[ g(h(\xi, W), FZ) = g(h(\xi, TW), FZ) = g(h(\xi, TW), FTZ) = g(h(\xi, W), FTZ) = 0. \] (33)

4. Chen’s first inequality for pointwise semi-slant warped products

In this section, we prove B.-Y Chen’s first inequality for pointwise semi-slant warped product \( N^T \times_f N^\theta \) in nearly Kenmotsu manifolds.

Let \( \tilde{M} \) be a \((2m + 1)\)-dimensional nearly Kenmotsu manifold and \( M = N^T \times_f N^\theta \) be a \( n \)-dimensional warped product isometrically immersed in \( \tilde{M} \). We denote the corresponding tangent spaces of \( N^T \) and \( N^\perp \) by the same symbols \( T^T \) and \( T^\perp \), respectively. If \( \dim N^T = n_1 \) and \( \dim N^\theta = n_2 \), then we have
\[ T^T = \text{Span}\{e_1, \ldots, e_p, e_{p+1} = \varphi e_1, \ldots, e_{2p} = \varphi e_p, e_{n_1} = e_{2p+1} = \xi\}, \]
\[ T^\perp = \text{Span}\{e_{n_1+1}, = e_1', \ldots, e_{n+q} = e_q', \ldots, e_{n+n_q+1} = \sec \theta T\xi', \ldots, e_n = \sec \theta T\xi'_n\}. \]

Then, the normal bundle \( T^\perp M \) of \( M \) is spanned by
\[ F T^\perp = \text{Span}\{e_{n_1+1}, = \sec \theta T\xi', \ldots, e_{n+q} = \sec \theta T\xi', \ldots, e_{n+n_q+1} = \sec \theta T\xi', \ldots, e_{n+n_2} = \sec \theta T\xi'_n\}, \]
\[ \mu = \text{Span}\{e_{n+n_2+1} = \xi', \ldots, e_{2m+1} = e_{2m+1-n-n_2}\}. \]

Now, we prove the main result of of this paper.
Theorem 4.1. Let $M = N^T \times_f N^0$ be a pointwise semi-slant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then, the second fundamental form $h$ of $M$ satisfies:

$$
\|h\|^2 \geq 2n_2 \left(1 + \frac{10}{9} \cot^2 \theta \right) \left(\|\nabla (\ln f)\|^2 - 1\right),
$$

(34)

where $\nabla (\ln f)$ is the gradient of $\ln f$ and $n_2 = \dim N^0$.

Moreover, if the equality sign in (34) holds identically then $N^T$ is a totally geodesic submanifold and $N^0$ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, $M$ cannot be mixed totally geodesic in $\tilde{M}$.

Proof. From (9), we have

$$
\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=1}^{2m+1} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2.
$$

Then from (16), the above equation decompose as

$$
\|h\|^2 = \sum_{r=1}^{n+n_2} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2 + \sum_{r=1}^{2m+1-n-n_2} \sum_{i,j=1}^{n} (g(h(e_i, e_j), \tilde{e}_r))^2.
$$

(35)

Leaving the last $\mu$-components term, we obtain

$$
\|h\|^2 \geq \sum_{r=1}^{n+n_2} \sum_{i,j=1}^{n} (g(h(e_i, e_j), Fe_r))^2 + \sum_{r=1}^{2m+1-n-n_2} \sum_{i,j=1}^{n} (g(h(e_i, e_j), Fe_r))^2 + \sum_{r=1}^{n+n_2} \sum_{i,j=1}^{n} (g(h(e_i, e_j), \tilde{e}_r))^2.
$$

(36)

First term in r.h.s. is identically zero by using Lemma 3.3(ii) and there is no relation for the third term, hence by leaving positive third term, we derive

$$
\|h\|^2 \geq 2 \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(e_i, e_j), Fe_r))^2 + 2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(e_i, e_j), Fe_r))^2
$$

$$
+ 2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(e_i, \tilde{e}_j), FTe_r))^2 + 2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(e_i, \tilde{e}_j), FTe_r))^2
$$

$$
+ 2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(\xi, e_j), FTe_r))^2 + 2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{p} (g(h(\xi, e_j), FTe_r))^2.
$$

Using Lemma 3.3, Eqs. (26)- (33), the above inequality takes the form

$$
\|h\|^2 \geq 4q \csc^2 \theta \sum_{i=1}^{p} (e_i(\ln f))^2 + \frac{4q}{9} \cot^2 \theta \sum_{i=1}^{p} (e_i(\ln f))^2
$$

$$
= 2n_2 \csc^2 \theta \sum_{i=1}^{p} (e_i(\ln f))^2 + \frac{2n_2}{9} \cot^2 \theta \sum_{i=1}^{p} (e_i(\ln f))^2 - 2n_2 \csc^2 \theta (\xi(\ln f))^2 - \frac{2n_2}{9} \cot^2 \theta (\xi(\ln f))^2.
$$

Then from (6) and Lemma 3.3(i), we get the required inequality (34). Now, for the equality case, from the excluded $\mu$-components terms in (35), we obtain

$$
h(X, Y) \perp \mu, \quad \forall X, Y \in \Gamma(TM).
$$

(37)
Also, from the vanishing first term of (36), we get
\[ h(\mathcal{D}^T, \mathcal{D}^T) \perp F \mathcal{D}^0. \] (38)

Then, from (37) and (38), we conclude that
\[ h(\mathcal{D}^T, \mathcal{D}^T) = 0. \] (39)

Hence, \( N^T \) is totally geodesic in \( \tilde{M} \) by using the fact that \( N^T \) being totally geodesic in \( M \) (see, Remark 2.2).

Furthermore, from the excluded third term of (36), we find
\[ h(\mathcal{D} \theta, \mathcal{D} \theta) \perp F \mathcal{D} \theta. \] (40)

Then, from (37) and (40), we obtain
\[ h(\mathcal{D} \theta, \mathcal{D} \theta) = 0. \] (41)

Moreover, from Lemma 3.3(iii), we find that
\[ h(\mathcal{D} T, \mathcal{D} \theta), \{0\}. \] (42)

Using the fact that \( N^0 \) is totally umbilical in \( M \) (see Remark 2.2) together with (41) and (42), we conclude that \( M \) can not be mixed totally geodesic in \( \tilde{M} \). Hence, the theorem is proved completely.

As a special case of Theorem 4.1, if \( \theta = \frac{\pi}{2} \) in (34), we have the following useful result.

**Theorem 4.2.** Let \( M = N^T \times_f N^\perp \) be a contact CR-warped product submanifold of a nearly Kenmotsu manifold \( \tilde{M} \). Then

(i) The second fundamental form \( h \) of the warped product immersion satisfies
\[ \|h\|^2 \geq 2n_2 \left( \|\nabla (\ln f)\|^2 - 1 \right), \quad n_2 = \dim N^\perp. \] (43)

(ii) If the equality sign holds in (43), then \( N^T \) is a totally geodesic submanifold and \( N^\perp \) is a totally umbilical submanifold of \( \tilde{M} \).

References


