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A General Inequality for Pointwise Semi-Slant Warped Products in Nearly Kenmotsu Manifolds

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Abstract. In this paper, we prove that every pointwise semi-slant warped product submanifold $M = N^T \times_f N^\theta$ in a nearly Kenmotsu manifold $\tilde{M}$ satisfies the following inequality: $\|h\|^2 \geq 2n_2\left(1 + \frac{10}{9}\cot^2\theta\right)\left(\|\hat{\nabla}(\ln f)\|^2 - 1\right)$, where $n_2 = \dim N^\theta$, $\hat{\nabla}(\ln f)$ is the gradient of $\ln f$ and $\|h\|$ is the length of the second fundamental form of M. The equality and special cases of the inequality are investigated.

1. Introduction

It was proved in [20] that every nearly Kenmotsu manifold is locally isometric to the warped product $\mathbb{R} \times_f \tilde{M}$ of a real line $\mathbb{R}$ and a nearly Kaehler manifold $\tilde{M}$. It was also proved that a normal nearly Kenmotsu manifold is a Kenmotsu manifold [20]. Nearly Kaehler manifolds were defined and studied by Gray in his series papers [22, 23]. Nearly Sasakian manifolds were introduced by Blair et al. [4]. Later, Olszak [29] studied nearly Sasakian non-Sasakian manifolds of dimension 5. In [19], Endo investigated the geometry of nearly cosymplectic manifolds. Later, Cappelletti Montano and Dileo studied nearly Sasakian manifolds for some other fundamental properties [7]. The geometry of nearly Kenmotsu manifolds was investigated in [34].

On the other hand, warped product manifolds introduced by Bishop on O'Neill to investigate the geometry of pseudo-Riemannian manifolds of negative curvature [2]. After a long gape, B.-Y. Chen introduced the notion of warped product submanifolds of Kaehler manifolds in his series papers [11, 12]. He investigate the geometry of CR-warped product submanifolds and proved that every CR-warped product $M = N^T \times_f N^\perp$ of a Kaehler manifold satisfies the following inequality

$$\|h\|^2 \geq 2q\|\hat{\nabla}(\ln f)\|^2, \quad q = \dim N^\perp \tag{1}$$

where $\|h\|^2$ is the squared norm of the second fundamental form h of M and $\hat{\nabla}(\ln f)$ is the gradient of $\ln f$. Later, this inequality known as Chen's first inequality for warped products and investigated for different

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kinds of warped product submanifolds of almost Hermitian as well as almost contact metric manifolds [1, 14, 17, 28, 33, 37, 38, 40, 43, 44].

In this paper, we study pointwise semi-slant warped product submanifolds of the form $N^T \times_f N^\theta$ of Kenmotsu manifolds where N^T and N^θ are invariant and proper pointwise slant submanifolds and obtain the following general inequality:

$$\|h\|^2 \geq 2n_2\Big(1 + \frac{10}{9}\cot^2\theta\Big)\Big(\|\hat{\nabla}(\ln f)\|^2 - 1\Big), \quad n_2 = \dim N^\theta$$

The equality case of this inequality is also investigated and a special case of this inequality is given for contact CR-warped products.

2. Preliminaries

An odd dimensional differentiable manifold $\tilde{M}$ endowed with a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g is called an almost contact metric manifold, if

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map and for any vector fields X, Y on $\tilde{M}$. The structure (φ, ξ, η, g) is called the almost contact metric structure on $\tilde{M}$ ([3], [4]). This structure also satisfies:

$$\eta(X) = g(X, \xi), \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, Y) = -g(X, \varphi Y).$$

In this paper, we refer to ξ as the structure vector field (Reeb vector field) and to η as the dual (Reeb form) of ξ.

An almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ is called a *nearly Kenmotsu manifold* [34], if

$$(\tilde{\nabla}_X\varphi)Y + (\tilde{\nabla}_Y\varphi)X = -\eta(Y)\varphi X - \eta(X)\varphi Y \tag{2}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ is the Lie algebra of the vector fields on $\tilde{M}$ and $\tilde{\nabla}$ is the Levi-Civita connection of g. Moreover, if $\tilde{M}$ satisfies

$$(\tilde{\nabla}_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{3}$$

then it is called a *Kenmotsu manifold* [27]. It was proved that every Kenmotsu manifold is a nearly Kenmotsu manifold but converse is not true in general [20]. The following useful result is proved in [20].

Proposition 2.1. *For a nearly Kenmotsu manifold, we have*

$$g(\tilde{\nabla}_X\xi, Y) + g(X, \tilde{\nabla}_Y\xi) = 2g(\varphi X, \varphi Y), \tag{4}$$

for any vector fields $X, Y \in \Gamma(T\tilde{M})$.

Now, we give the brief introduction of warped product manifolds.

Let (B, g_B) and (F, g_F) be two Riemannian (or semi-Riemannian) manifolds and f be a positive smooth function on B. The *warped product* of B and F is the Riemannian manifold

$$B \times_f F = (M = B \times F,\ g)$$

equipped with the warped metric $g = g_B + f^2 g_F$. The function f is called the *w*arping function and a warped product manifold M is said to be *t*rivial or simply a Riemannian product manifold of B and F if f is constant (see, for instance, [2]).

Let X be a vector field on B and Z be an another vector field on F. Then, from Lemma 7.3 of [2], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{5}$$

where ∇ denotes the Levi-Civita connection on M. Now for a smooth function f on an n-dimensional manifold M, we have

$$\|\hat{\nabla} f\|^2 = \sum_{i=1}^{m} (e_i(f))^2 \tag{6}$$

for the given orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ on M, where $\hat{\nabla} f$ is the gradient of f defined by $g(\hat{\nabla} f, X) = X(f)$.

Remark 2.2. *It is also important to note that for a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [2, 11].*

Now, if M is a Riemannian manifold isometrically immersed in an another Riemannian manifold $\tilde{M}$, then formulas of Gauss and Weingarten are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{7}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{8}$$

for any vector field $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the normal connection in the normal bundle, h is the second fundamental form and A is the shape operator of the submanifold. They are related by $g(h(X, Y), N) = g(A_N X, Y)$

A submanifold M is said to be totally geodesic if $h = 0$ and totally umbilical if $h(X, Y) = g(X, Y)H,\ \forall\, X,\ Y \in \Gamma(TM)$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ is the mean curvature vector of M. For any $x \in M$ and $\{e_1, \cdots, e_n, \cdots, e_{2m+1}\}$ is an orthonormal frame of the tangent space $T_x\tilde{M}$ such that $e_1, \cdots, e_n$ are tangent to M at x. Then, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \quad i, j \in \{1, \cdots, n\}, \quad r \in \{n+1, \cdots, 2m+1\}. \tag{9}$$

B.-Y. Chen [9, 10] introduced a generalized class of holomorphic (invariant) and totally real (anti-invariant) submanifolds known as slant submanifolds in complex geometry. Later, A. Lotta [26] has extended Chen's idea for contact metric manifolds.

A submanifold M tangent to ξ is said to be *slant* if for any $p \in M$ and any $X \in T_pM$, linearly independent to ξ, the angle between φX and T_pM is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of M in $\tilde{M}$.

As natural extension of slant submanifolds, Etayo [21] introduced the notion of pointwise slant submanifolds. Later, these submanifolds were studied by Chen and Garay [15] for their characterizations and fundamental properties. They proved many interesting results and provided a method that how to construct non-trivial examples of such submanifolds. They defined pointwise slant submanifolds as follows:

A submanifold M is called *pointwise slant* [15, 21, 42] if for any nonzero vector $X \in T_pM\,(p \in M)$, the angle $\theta(X)$ between φX and T_pM is independent of the choice of $X \in T_pM$. In this case, θ defines a function on M, called the *slant function*. In particular, if the slant function θ is globally constant on M, then M is said to be a *slant submanifold* or a *θ-slant submanifold*.

Anti-invariant submanifolds are pointwise slant submanifolds with slant function $\theta = \frac{\pi}{2}$ everywhere on M. A pointwise slant submanifold is called *proper* if $0 < \theta < \frac{\pi}{2}$. See [**? ?**] for non-trivial examples of pointwise slant submanifolds.

For any vector field $X \in \Gamma(TM)$, we have

$$\varphi X = TX + FX, \tag{10}$$

where TX and FX are the tangential and normal components of φX, respectively.

We recall the following useful characterization from [42].

Proposition 2.3. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with $\xi \in \Gamma(TM)$. Then M is pointwise slant if and only if*

$$T^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{11}$$

where θ is the slant function and I denotes the identity map on TM.

Following relations are straightforward consequence of (11)

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{12}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)], \tag{13}$$

for vector fields $X, Y \in \Gamma(M)$. Also, for pointwise slant submanifolds, we have

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi)\,, \quad fFX = -FTX, \quad X \in \Gamma(TM). \tag{14}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a *contact CR-submanifold* [14] if there exist a pair of orthogonal distributions $\mathfrak{D}^T$ and $\mathfrak{D}^\perp$ such that

$$TM = \mathfrak{D}^T \oplus \mathfrak{D}^\perp \oplus \langle\xi\rangle,$$

where $\mathfrak{D}^T$ is φ-invariant i.e., $\varphi\mathfrak{D}^T \subseteq \mathfrak{D}^T$ and $\mathfrak{D}^\perp$ is anti-invariant i.e., $\varphi\mathfrak{D}^\perp \subset T^\perp M$.

As a generalization of contact CR-submanifold, we define pointwise semi-slant submanifolds as follows:

Definition 2.4. *Let $\tilde{M}$ be an almost contact metric manifold and M be a submanifold of $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then M is called a pointwise semi-slant submanifold of $\tilde{M}$ if there exists a pair of orthogonal distributions $\mathfrak{D}^T$ and $\mathfrak{D}^\theta$ on M such that*

$$TM = \mathfrak{D}^T \oplus \mathfrak{D}^\theta \oplus \langle\xi\rangle \tag{15}$$

where $\mathfrak{D}^T$ is φ-invariant, i.e., $\varphi(\mathfrak{D}^T) \subseteq \mathfrak{D}^T$ and $\mathfrak{D}^\theta$ is a proper pointwise slant distribution with slant function $\theta \neq 0, \frac{\pi}{2}$.

A pointwise semi-slant submanifold M is called *proper* if neither dim $\mathfrak{D}^T = 0$ nor the slant function of $\mathfrak{D}^\theta$ is constant.

Clearly, semi-slant and contact CR-submanifolds are the pointwise semi-slant slant submanifolds with slant function θ is globally constant and $\theta = \frac{\pi}{2}$, repetitively.

The normal bundle of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = F\mathfrak{D}^\theta \oplus \mu \tag{16}$$

where μ is the maximal φ-invariant normal subbundle in $T^\perp M$.

3. Definition and a basic lemma

In this section, we give some preparatory results on pointwise semi-slant warped products. First, we define

Definition 3.1. *A warped product of an φ-invariant submanifold and a proper pointwise slant submanifold in an almost contact metric manifold $\tilde{M}$ is called a pointwise semi-slant warped product and it is denoted by $N^T \times_f N^\theta$ or $N^\theta \times_f N^T$, where N^T and N^θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively.*

We accept the following convention that X, Y are vector fields on N^T and Z, W are the vector fields N^θ and for the simplicity we denote tangent spaces of N^T and N^θ by the same $\mathfrak{D}^T$ and $\mathfrak{D}^\theta$, respectively.

Remark 3.2. *Notice that in both the cases of proper pointwise semi-slant warped products (in Definition 3.1), the structure vector field ξ is tangent to the base manifold of the warped products otherwise from Proposition 2.1, we easily find that there is no proper warped product.*

Now, we have the following useful results for later use.

Lemma 3.3. *Let $M = N^T \times_f N^\theta$ be a pointwise semi-slant warped product of a nearly Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to N^T. Then, for any $X, Y \in \Gamma(\mathfrak{D}^T)$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$, we have*

(i) $\xi(\ln f) = 1$,
(ii) $g(h(X, Y), \varphi Z) = 0$;
(iii) $g(h(X, Z), FW) = -\frac{1}{3}\left(\eta(X) - X(\ln f)\right) g(TZ, W) - \varphi X(\ln f) g(Z, W)$.

Proof. From (4) and (5), for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$, we have

$$2\xi(\ln f) g(Z, W) = g(\tilde{\nabla}_Z \xi, W) + g(\tilde{\nabla}_W \xi, Z) = 2g(\varphi Z, \varphi W) = 2g(Z, W),$$

which gives (i). For the second part of the lemma, we have

$$g(h(X, Y), FZ) = g(\tilde{\nabla}_X Y, \varphi Z - TZ) = g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X \varphi Y, Z) + g(\varphi \nabla_X Y, Z),$$

for any $X, Y \in \Gamma(\mathfrak{D}^T)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$. Since $\nabla_X Y \in \Gamma(\mathfrak{D}^T)$ (see, Remark 2.2), the last two terms in r.h.s. of above equation are identically zero. Then, we find

$$g(h(X, Y), FZ) = g((\tilde{\nabla}_X \varphi)Y, Z). \tag{17}$$

Hence, (ii) follows from (17) via polarization identity and (2). In the similar way, we have

$$g(h(X, Z), FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - g(\tilde{\nabla}_Z X, TW).$$

Using (5), we obtain

$$g(h(X, Z), FW) = g((\tilde{\nabla}_Z \varphi)X, W) - \varphi X(\ln f) g(Z, W) - X(\ln f) g(Z, TW). \tag{18}$$

On the other hand, we know that

$$g(h(X, Z), FW) = g((\tilde{\nabla}_X \varphi)Z, W) - g(\tilde{\nabla}_X TZ, W) - g(\tilde{\nabla}_X FZ, W) - X(\ln f) g(Z, TW). \tag{19}$$

Using (8) and (5), we find

$$g(h(X, Z), FW) = g((\tilde{\nabla}_X \varphi)Z, W) + g(h(X, W), FZ). \tag{20}$$

Then, from (18) and (20) together with (2), we derive

$$2g(h(X, Z), FW) = -\eta(X) g(TZ, W) - \varphi X(\ln f) g(Z, W) + X(\ln f) g(TZ, W). + g(h(X, W), FZ).$$

Third relation immediately follows from above relation via polarization identity. Hence, the lemma is proved completely. □

Now, if we interchange Z by TZ in (19) and using Proposition 2.3 and (5), then we find

$$\begin{aligned} g(h(X, TZ), FW) = {} & g((\tilde{\nabla}_X \varphi)TZ, W) + \cos^2\theta X(\ln f) g(Z, W) - \sin 2\theta X(\theta) g(Z, W) \\ & + g(h(X, W), FTZ) - \cos^2\theta X(\ln f) g(Z, W). \end{aligned} \tag{21}$$

Since, θ is the slant function on N^θ hence $X(\theta) = 0$, $\forall\ X \in \Gamma(\mathfrak{D}^T)$. Then, Eq. (21) takes the form

$$g(h(X, TZ), FW) = g((\tilde{\nabla}_X \varphi)TZ, W) + g(h(X, W), FTZ). \tag{22}$$

On the other hand, from (18), we have

$$g(h(X,TZ),FW) = g((\tilde{\nabla}_{TZ}\varphi)X,W) - \varphi X(\ln f)g(TZ,W) - \cos^2\theta X(\ln f)g(Z,W). \tag{23}$$

Then, (22), (23) and (2) imply that

$$2g(h(X,TZ),FW) = g(h(X,W),FTZ) + \cos^2\theta\,(\eta(X) - X(\ln f))\,g(Z,W) - \varphi X(\ln f)g(TZ,W). \tag{24}$$

Furthermore, from Lemma 3.3(iii), we have

$$2g(h(X,TZ),FW) = \frac{2}{3}\cos^2\theta\,(\eta(X) - X(\ln f))\,g(Z,W) - 2\varphi X(\ln f)g(TZ,W). \tag{25}$$

Then, from (24) and (25), we derive

$$g(h(X,W),FTZ) = -\frac{1}{3}\cos^2\theta\,(\eta(X) - X(\ln f))\,g(Z,W) - \varphi X(\ln f)g(TZ,W). \tag{26}$$

The following relations are easily obtained by interchanging X by φX, Z by TZ and W by TW in Lemma 3.3(iii) and (26).

$$g(h(\varphi X,W),FZ) = -\frac{1}{3}\varphi X(\ln f)\,g(TZ,W) + (X(\ln f) - \eta(X))\,g(Z,W), \tag{27}$$

$$g(h(X,TW),FZ) = \frac{1}{3}\cos^2\theta\,(\eta(X) - X(\ln f))\,g(Z,W) + \varphi X(\ln f)\,g(TZ,W), \tag{28}$$

$$g(h(X,TW),FTZ) = -\frac{1}{3}\cos^2\theta\,(\eta(X) - X(\ln f))\,g(TZ,W) - \cos^2\theta\varphi X(\ln f)\,g(Z,W), \tag{29}$$

$$g(h(\varphi X,TW),FZ) = -\frac{1}{3}\cos^2\theta\varphi X(\ln f)g(Z,W) - (X(\ln f) - \eta(X))\,g(TZ,W), \tag{30}$$

$$g(h(\varphi X,W),FTZ) = -\frac{1}{3}\cos^2\theta\varphi X(\ln f)g(Z,W) + (X(\ln f) - \eta(X))\,g(TZ,W), \tag{31}$$

$$g(h(\varphi X,TW),FTZ) = \frac{1}{3}\cos^2\theta\varphi X(\ln f)g(TZ,W) + \cos^2\theta\,(X(\ln f) - \eta(X))\,g(Z,W). \tag{32}$$

In particular, if $X = \xi$, then we find

$$g(h(\xi,W),FZ) = g(h(\xi,TW),FZ) = g(h(\xi,TW),FTZ) = g(h(\xi,W),FTZ) = 0. \tag{33}$$

4. Chen's first inequality for pointwise semi-slant warped products

In this section, we prove B.-Y Chen's first inequality for pointwise semi-slant warped product $N^T \times_f N^\theta$ in nearly Kenmotsu manifolds.

Let $\tilde{M}$ be a $(2m+1)$-dimensional nearly Kenmotsu manifold and $M = N^T \times_f N^\theta$ be a n-dimensional warped product isometrically immersed in $\tilde{M}$. We denote the corresponding tangent spaces of N^T and $N^\perp$ by the same symbols $\mathfrak{D}^T$ and $\mathfrak{D}^\theta$, respectively. If $\dim N^T = n_1$ and $\dim N^\theta = n_2$, then we have

$$\mathfrak{D}^T = \mathrm{Span}\{\mathrm{e}_1, \cdots, \mathrm{e}_\mathrm{p}, \mathrm{e}_{\mathrm{p}+1} = \varphi\mathrm{e}_1, \cdots, \mathrm{e}_{2\mathrm{p}} = \varphi\mathrm{e}_\mathrm{p}, \mathrm{e}_{\mathrm{n}_1} = \mathrm{e}_{2\mathrm{p}+1} = \xi\},$$
$$\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{e}_{\mathrm{n}_1+1}, = \mathrm{e}_1^* \cdots, \mathrm{e}_{\mathrm{n}_1+\mathrm{q}} = \mathrm{e}_\mathrm{q}^*, \cdots, \mathrm{e}_{\mathrm{n}_1+\mathrm{q}+1} = \sec\theta\mathrm{Te}_1^*, \cdots, \mathrm{e}_\mathrm{n} = \sec\theta\mathrm{Te}_\mathrm{q}^*\}.$$

Then, the normal bundle $T^\perp M$ of M is spanned by

$$F\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{e}_{\mathrm{n}+1}, = \csc\theta\mathrm{Fe}_1^* \cdots, \mathrm{e}_{\mathrm{n}+\mathrm{q}} = \csc\theta\mathrm{Fe}_\mathrm{q}^*, \cdots, \mathrm{e}_{\mathrm{n}+\mathrm{q}+1} = \csc\theta\sec\theta\mathrm{FTe}_1^*, \cdots, \mathrm{e}_{\mathrm{n}+\mathrm{n}_2} = \csc\theta\sec\theta\mathrm{FTe}_\mathrm{q}^*\},$$
$$\mu = \mathrm{Span}\{\mathrm{e}_{\mathrm{n}+\mathrm{n}_2+1}, = \tilde{\mathrm{e}}_1 \cdots, \mathrm{e}_{2\mathrm{m}+1} = \tilde{\mathrm{e}}_{2\mathrm{m}+1-\mathrm{n}-\mathrm{n}_2}\}.$$

Now, we prove the main result of of this paper.

Theorem 4.1. *Let $M = N^T \times_f N^\theta$ be a pointwsie semi-slant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then, the second fundamental form h of M satisfies:*

$$\|h\|^2 \geq 2n_2\left(1 + \frac{10}{9}\cot^2\theta\right)\left(\|\hat{\nabla}(\ln f)\|^2 - 1\right), \tag{34}$$

where $\|\hat{\nabla}(\ln f)$ is the gradient of $\ln f$ and $n_2 = \dim N^\theta$.

Moreover, if the equality sign in (34) holds identically then N^T is a totally geodesic submanifold and N^θ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, M can not be mixed totally geodesic in $\tilde{M}$.

Proof. From (9), we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(g(h(e_i, e_j), e_r)^2.$$

Then from (16), the above equation decompose as

$$\|h\|^2 = \sum_{r=n+1}^{n+n_2}\sum_{i,j=1}^{n}(g(h(e_i, e_j), e_r))^2 + \sum_{r=1}^{2m+1-n-n_2}\sum_{i,j=1}^{n}(g(h(e_i, e_j), \tilde{e}_r))^2. \tag{35}$$

Leaving the last μ-components term, we obtain

$$\|h\|^2 \geq \sum_{r=1}^{n_2}\sum_{i,j=1}^{2p+1}(g(h(e_i, e_j), Fe_r^*))^2 + 2\sum_{r=1}^{n_2}\sum_{i=1}^{2p+1}\sum_{j=1}^{n_2}(g(h(e_i, e_j^*), Fe_r^*))^2 + \sum_{r=1}^{n_2}\sum_{i,j=1}^{n_2}(g(h(e_i^*, e_j^*), Fe_r^*))^2. \tag{36}$$

First term in r.h.s. is identically zero by using Lemma 3.3(ii) and there is no relation for the third term, hence by leaving positive third term, we derive

$$\begin{aligned}
\|h\|^2 \geq\ & 2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2p}\sum_{j=1}^{q}(g(h(e_i, e_j^*), Fe_r^*))^2 + 2\csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2p}\sum_{j=1}^{q}(g(h(e_i, Te_j^*), Fe_r^*))^2\\
&+ 2\csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2p}\sum_{j=1}^{q}(g(h(e_i, e_j^*), FTe_r^*))^2 + 2\csc^2\theta\sec^4\theta\sum_{r=1}^{q}\sum_{i=1}^{2p}\sum_{j=1}^{q}(g(h(e_i, Te_j^*), FTe_r^*))^2\\
&+ 2\csc^2\theta\sum_{r=1}^{q}\sum_{j=1}^{q}(g(h(\xi, e_j^*), Fe_r^*))^2 + 2\csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{j=1}^{q}(g(h(\xi, Te_j^*), Fe_r^*))^2\\
&+ 2\csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{j=1}^{q}(g(h(\xi, e_j^*), FTe_r^*))^2 + 2\csc^2\theta\sec^4\theta\sum_{r=1}^{q}\sum_{j=1}^{q}(g(h(\xi, Te_j^*), FTe_r^*))^2.
\end{aligned}$$

Using Lemma 3.3, Eqs. (26)- (33), the above inequality takes the form

$$\begin{aligned}
\|h\|^2 &\geq 4q\csc^2\theta\sum_{i=1}^{2p}(e_i(\ln f))^2 + \frac{4q}{9}\cot^2\theta\sum_{i=1}^{2p}(e_i(\ln f))^2\\
&= 2n_2\csc^2\theta\sum_{i=1}^{2p+1}(e_i(\ln f))^2 + \frac{2n_2}{9}\cot^2\theta\sum_{i=1}^{2p+1}(e_i(\ln f))^2 - 2n_2\csc^2\theta\,(\xi(\ln f))^2 - \frac{2n_2}{9}\cot^2\theta\,(\xi(\ln f))^2.
\end{aligned}$$

Then from (6) and Lemma 3.3(i), we get the required inequality (34). Now, for the equality case, from the excluded μ-components terms in (35), we obtain

$$h(X, Y) \perp \mu, \quad \forall\ X, Y \in \Gamma(TM). \tag{37}$$

Also, from the vanishing first term of (36), we get

$$h(\mathfrak{D}^T, \mathfrak{D}^T) \perp F\mathfrak{D}^\theta. \tag{38}$$

Then, from (37) and (38), we conclude that

$$h(\mathfrak{D}^T, \mathfrak{D}^T) = \{0\}. \tag{39}$$

Hence, N^T is totally geodesic in $\tilde{M}$ by using the fact that N^T being totally geodesic in M (see, Remark 2.2). Furthermore, from the excluded third term of (36), we find

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta. \tag{40}$$

Then, from (37) and (40), we obtain

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) = \{0\}. \tag{41}$$

Moreover, from Lemma 3.3(iii), we find that

$$h(\mathfrak{D}^T, \mathfrak{D}^\theta) \neq \{0\}. \tag{42}$$

Using the fact that N^θ is totally umbilical in M (see Remark 2.2) together with (41) and (42), we conclude that N^θ is a totally umbilical submanifold of $\tilde{M}$. Also, from (42), we observe that M can not be mixed totally geodesic in $\tilde{M}$. Hence, the theorem is proved completely. □

As a special case of Theorem 4.1, if $\theta = \frac{\pi}{2}$ in (34), we have the following useful result.

Theorem 4.2. *Let $M = N^T \times_f N^\perp$ be a contact CR-warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then*

(i) The second fundamental from h of the warped product immersion satisfies

$$\|h\|^2 \geq 2n_2\left(\|\hat{\nabla}(\ln f)\|^2 - 1\right), \quad n_2 = \dim N^\perp. \tag{43}$$

(ii) If the equality sign holds in (43), then N^T is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$.

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