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# On Maps Preserving Skew Symmetric Operators

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**Abstract.** Given a conjugation $C$ on a separable complex Hilbert space $H$, a bounded linear operator $T$ on $H$ is said to be $C$-skew symmetric if $CTC = -T^*$. This paper describes the maps, on the algebra of all bounded linear operators acting on $H$, that preserve the difference of $C$-skew symmetric operators for every conjugation $C$ on $H$.

## 1. Introduction

The fundamental Hua's theorems of the geometry of matrices characterize the general form of bijective maps $\Phi$ on various spaces of matrices (Hermitian matrices, symmetric matrices, skew-symmetric matrices, etc.) that preserve adjacent matrices in both directions, that is $A - B$ is of rank one if and only if $\Phi(A) - \Phi(B)$ is of rank one for every $A, B$. Specially, the well-known Hua's theorem from [9] states that a bijective map $\Phi$ on $\mathcal{M}_n(\mathbb{K})$, the space of all $n \times n$ matrices over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, preserves adjacent matrices in both directions if and only if $\Phi$ has one of the following two forms

$$[a_{ij}] \mapsto P[\tau(a_{ij})]Q + R \quad \text{ or } \quad [a_{ij}] \mapsto P[\tau(a_{ij})]^{\mathrm{tr}}Q + R,$$

where $P, Q, R \in \mathcal{M}_n(\mathbb{K})$ with $P, Q$ invertible, $A^{\mathrm{tr}}$ denotes the transpose of $A$, and $\tau$ is an automorphism of $\mathbb{K}$. This beautiful theorem has many applications, for example in the theory of Jordan automorphisms and Lie automorphisms. For other applications of results on adjacency preserving maps, particularly in the theory of local homomorphisms, linear preserver problems, and graph theory, we refer to [3, 6, 10] and to Wan's book [18], where most of the known results on the geometry of matrices are collected.

Hua's theorem has motivated other researchers to consider general and similar situations in connection with the so-called *non-linear preservers problems* that require to describe the general structure of all maps on matrix algebras, linear bounded operators algebras or more generally Banach algebras, leaving invariant certain functions, subsets, or relations, without assuming in advance algebraic conditions such as linearity or additivity. While in the beginning, these problems were formulated for finite-dimensional vector spaces, and later more sophisticated non-linear preservers problems defined in the infinite-dimensional context have been investigated. In the meantime, the interested reader can find in [1–3, 5, 13], and the references within, more through results on such problems.

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Throughout this paper, $H$ is a separable complex Hilbert space of dimension at least four, and $\mathcal{B}(H)$ is used to refer to the algebra of all bounded linear operators acting on $H$. Recall that a *conjugation* $C$ on $H$ is a conjugate-linear operator satisfying $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in H$. An operator $T \in \mathcal{B}(H)$ is called *C-symmetric* (resp. *C-skew symmetric*) if $CTC = T^*$ (resp. $CTC = -T^*$) where $T^*$ denotes the adjoint of $T$, and it is called a *complex symmetric operator* (resp. *skew symmetric operator*) if there exists a conjugation $C$ for which $T$ is $C$-symmetric (resp. $C$-skew symmetric).

It is shown in [7] that an operator is skew symmetric if and only if it admits a skew symmetric matrix representation with respect to some orthonormal basis of $H$. Thus skew symmetric operators can be viewed as an infinite dimensional analogue of skew symmetric matrices. Skew symmetric operators have been studied for many years in the finite dimensional setting. Recently, there has been growing interest in skew symmetric operators in the infinite dimensional case, and some interesting results have been obtained in [12, 16, 17].

The concept of skew symmetric operators and complex symmetric operators has numerous applications in complex analysis, matrix theory, differential equations, function theory and even in quantum mechanics, see for instance [7, 8, 11, 12, 15–17]. The examples of complex symmetric operators are numerous and quite diverse. Besides the expected normal operators, we quote as complex symmetric operators: binormal operators, diagonal operators, quadratic operators, Hankel operators, truncated Toeplitz operators and many standard integral operators such as the Volterra integration operator. As for skew symmetric operators, the most examples are inspired from the previous ones; for instance the commutator of two $C$-symmetric operators is $C$-skew symmetric.

Recently, in [2] the authors considered a non-linear preserver problem involving complex symmetric operators. More precisely, they showed that if $\Phi$ is a map on $\mathcal{B}(H)$ satisfying

$$T - S \text{ is } C\text{-symmetric} \quad \Rightarrow \quad \Phi(T) - \Phi(S) \text{ is } C\text{-symmetric},$$

for every $T, S \in \mathcal{B}(H)$ and every conjugation $C$ on $H$, then it must have the following form

$$T \mapsto \alpha T + \beta T^* + f(T)I + \Phi(0),$$

where $\alpha, \beta \in \mathbb{C}$ and $f$ is a functional on $\mathcal{B}(H)$ that vanishes at 0.

In this paper, we propose an analogue study for skew symmetric operators. Our arguments are influenced by ideas from [2] and the approaches given therein, but the proofs of our main results require new ingredients. The fundamental results of this paper can be stated as follow:

**Theorem 1.1.** *Assume that $H$ is infinite dimensional, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a map. Then the following statements are equivalent:*

(i) *For every $T, S \in \mathcal{B}(H)$ and every conjugation $C$ on $H$,*

$$T - S \text{ is } C\text{-skew symmetric} \quad \Rightarrow \quad \Phi(T) - \Phi(S) \text{ is } C\text{-skew symmetric}.$$

(ii) *There exist two complex scalars $\alpha, \beta$ such that*

$$\Phi(T) = \alpha T + \beta T^* + \Phi(0) \quad \text{for all } T \in \mathcal{B}(H).$$

We say that a map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves strongly skew-symmetric operators* if, for every conjugation $C$ on $H$ and $T \in \mathcal{B}(H)$,

$$T \text{ is } C\text{-skew symmetric} \quad \Rightarrow \quad \Phi(T) \text{ is } C\text{-skew symmetric}.$$

In [1, Theorem 3], it was shown that a unital linear continuous map on $\mathcal{B}(H)$ preserves strongly skew-symmetric operators if and only if it is the identity map of $\mathcal{B}(H)$. We extend this result, as an immediate consequence of the previous theorem, to additive maps only.

**Corollary 1.2.** *Assume that $H$ is infinite dimensional, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be an additive map. Then $\Phi$ preserves strongly skew-symmetric operators if and only if there exist $\alpha, \beta \in \mathbb{C}$ such that $\Phi(T) = \alpha T + \beta T^*$ for all $T \in \mathcal{B}(H)$.*

Unfortunately, the approach used here does not allow us to obtain an analogue result of Theorem 1.1 in the setting of finite-dimensional Hilbert spaces. However, we focus to describe additive maps on $\mathcal{B}(\mathbb{C}^n)$, with $n \geq 4$, preserving strongly skew-symmetric operators as an extension of [1, Theorem 4].

**Theorem 1.3.** *Let $\Phi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ be an additive map. Then the following statements are equivalent:*

(i) *$\Phi$ preserves strongly skew-symmetric operators.*

(ii) *There exist $\alpha, \beta \in \mathbb{C}$ and an additive map $\Psi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ vanishing on $\mathrm{sl}_n(\mathbb{C})$, the subspace of all operators with trace zero, such that*

$$\Phi(T) = \alpha T + \beta T^* + \Psi(T) \quad \text{for all } T \in \mathcal{B}(\mathbb{C}^n).$$

In the next section, we shall establish some useful results to prove the above two theorems.

## 2. Proof of main results

Throughout the rest of the paper, $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a map satisfying the first assertion of Theorem 1.1. Moreover, as the map $\Phi - \Phi(0)$ satisfies the same property as $\Phi$, there is no loss of generality in assuming that $\Phi(0) = 0$. In this case, $\Phi$ preserves strongly skew symmetric operators.

For simplicity of notations, we make the following definition.

**Definition 2.1.** *Let $T, S \in \mathcal{B}(H)$. We will write $T \sim S$ if for every conjugation $J$ on $H$,*

$$T \quad \text{is } J\text{-skew symmetric} \quad \Rightarrow \quad S \quad \text{is } J\text{-skew symmetric}.$$

Note that our map $\Phi$ satisfies $A - B \sim \Phi(A) - \Phi(B)$ for every $A, B \in \mathcal{B}(H)$. In particular, since $\Phi(0) = 0$, we have also $A \sim \Phi(A)$ for every $A \in \mathcal{B}(H)$.

In what follows, the symbol $\oplus$ will always stand for an orthogonal sum. We reformulate the result of [1, Lemma 7] as follow.

**Lemma 2.2.** *Let $T, S \in \mathcal{B}(H)$ be such that $T = F \oplus 0$ with respect to an orthogonal decomposition of $H$ and $F$ is a skew symmetric operator for some conjugation $C$. If $T \sim S$, then $S = R \oplus 0$ with respect to the same decomposition of $H$ and $R$ is a $C$-skew symmetric operator.*

**Remark 2.3.** (i) *Let $T, S \in \mathcal{B}(H)$. If $JTJ = S^*$ for every conjugation $J$ on $H$, then $T = S = \mu I$ for some $\mu \in \mathbb{C}$. Moreover, if $T$ is skew symmetric, thus $T = 0$. Indeed, since $J'JT = TJ'J$ and every unitary operator is the product of two conjugations (see [8, Theorem 1]), we get that $T$ commutes with every unitary operator in $\mathcal{B}(H)$. This implies that $T = S = \mu I$ for some scalar $\mu$.*

(ii) *Let $T, S$ be bounded linear operators between Hilbert spaces. If $JTJ' = S$ for every conjugations $J, J'$, then $T = S = 0$. Indeed, as in the the previous assertion, we get that $UT = TV$ for every unitary operators $U$, $V$, and so $T = S = 0$.*

The next lemma will be needed throughout the paper.

**Lemma 2.4.** *Consider an orthogonal decomposition $H = H_1 \oplus \cdots \oplus H_5$ where $\dim H_1 = \dim H_2$ and $\dim H_3 = \dim H_4$, and let $E_o, F_o \in \mathcal{B}(H)$ be the operators defined by*

$$E_o = I \oplus -I \oplus 0 \oplus 0 \oplus 0 \quad \text{and} \quad F_o = 0 \oplus 0 \oplus I \oplus -I \oplus 0.$$

*For every $A, B \in \mathcal{B}(H)$, if $A - B = \lambda E_o + \mu F_o$ for some $\lambda, \mu \in \mathbb{C}$, then there exist $\lambda', \mu' \in \mathbb{C}$ such that*

$$\Phi(A) - \Phi(B) = \lambda' E_o + \mu' F_o.$$

*Moreover, if $\mu$ is zero, then so is $\mu'$, and $\Phi(A) - \Phi(B) = \lambda''(A - B)$ for some $\lambda'' \in \mathbb{C}$.*

*Proof.* Let $A, B \in \mathcal{B}(H)$ be such that $A - B = \lambda E_o + \mu F_o$ for some $\lambda, \mu \in \mathbb{C}$, and set $F = \lambda I \oplus -\lambda I \oplus \mu I \oplus -\mu I$ with respect to the subspace $G := H_1 \oplus \cdots \oplus H_4$. Fix unitary operators $U : H_1 \to H_2$ and $V : H_3 \to H_4$. Now, take two arbitrary conjugations $J_1, J_3$ on $H_1, H_3$ respectively, and put

$$C = \begin{bmatrix} 0 & J_1U^* & 0 & 0 \\ UJ_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_3V^* \\ 0 & 0 & VJ_3 & 0 \end{bmatrix}.$$

It is easy to check that $C$ is a conjugation on $G$ for which the operator $F$ is skew symmetric. Since $A - B = F \oplus 0$ and $A - B \sim \Phi(A) - \Phi(B)$, it follows from Lemma 2.2 that

$$\Phi(A) - \Phi(B) = R \oplus 0$$

accordingly to the decomposition $H = G \oplus H_5$, where $R$ is a $C$-skew symmetric operator on $G$. Taking into account that $CRC = -R^*$, and by writing

$$R = \begin{bmatrix} T_1 & T_2 & S_1 & S_2 \\ T_3 & T_4 & S_3 & S_4 \\ R_1 & R_2 & Q_1 & Q_2 \\ R_3 & R_4 & Q_3 & Q_4 \end{bmatrix},$$

simple matrix calculations show that

$$\begin{bmatrix} J_1U^*T_4UJ_1 & J_1U^*T_3J_1U^* & J_1U^*S_4VJ_3 & J_1U^*S_3J_3V^* \\ UJ_1T_2UJ_1 & UJ_1T_1J_1U^* & UJ_1S_2VJ_3 & UJ_1S_1J_3V^* \\ * & * & J_3V^*Q_4VJ_3 & J_3V^*Q_3J_3V^* \\ * & * & VJ_3Q_2VJ_3 & VJ_3Q_1J_3V^* \end{bmatrix} = -\begin{bmatrix} T_1^* & T_3^* & R_1^* & R_3^* \\ T_2^* & T_4^* & R_2^* & R_4^* \\ * & * & Q_1^* & Q_3^* \\ * & * & Q_2^* & Q_4^* \end{bmatrix}.$$

Accordingly,

$$J_1U^*S_4VJ_3 = -R_1^*, \quad J_1U^*S_3J_3V^* = -R_3^*, \quad UJ_1S_2VJ_3 = -R_2^*, \quad UJ_1S_1J_3V^* = -R_4^*,$$

$$UJ_1T_2UJ_1 = -T_2^*, \quad J_1U^*T_3J_1U^* = -T_3^*, \quad J_3V^*Q_3J_3V^* = -Q_3^*, \quad VJ_3Q_2VJ_3 = -Q_2^*,$$

$$J_1U^*T_4UJ_1 = -T_1^*, \quad J_3V^*Q_4VJ_3 = -Q_1^*.$$

As $J_1$ and $J_3$ are arbitrary, we get easily from Remark 2.3 (ii) that $S_i = R_i = 0$ for $1 \leq i \leq 4$. On the other hand, considering the equality $UJ_1T_2UJ_1 = -T_2^*$, we see that $T_2U$ is skew-symmetric, and so $T_2 = 0$ by Remark 2.3 (i). In a similar way, we obtain that $T_3 = Q_3 = Q_2 = 0$. Furthermore, we get from $J_1U^*T_4UJ_1 = -T_1^*$ that $U^*T_4U$ is a scalar multiple of the identity, and hence so is $T_4$. Thus, $T_4 = -\lambda' I$ and $T_1 = \lambda' I$ for some $\lambda' \in \mathbb{C}$. Analogously, we have $Q_4 = -\mu' I$ and $Q_1 = \mu' I$ for some $\mu' \in \mathbb{C}$. Therefore,

$$\Phi(A) - \Phi(B) = R \oplus 0 = \lambda' I \oplus -\lambda' I \oplus \mu' I \oplus -\mu' I \oplus 0 = \lambda' E_o + \mu' F_o.$$

Finally, if $\mu = 0$, then $A - B$ is zero on $(H_1 \oplus H_2)^\perp$, and so is $\Phi(A) - \Phi(B)$ by the previous lemma because $(A - B)_{|H_1 \oplus H_2} = \lambda I \oplus -\lambda I$ is skew-symmetric with respect to the conjugation $C = \begin{pmatrix} 0 & J_1U^* \\ UJ_1 & 0 \end{pmatrix}$. Thus, $\mu' = 0$. □

Given two non-zero vectors $u, v \in H$, we denote by $u \otimes v$ the rank one operator given by $(u \otimes v)(x) = < x, v > u$ for all $x \in H$.

Let $\mathcal{F}_2(H)$ denote the subset of $\mathcal{B}(H)$ of all operators of the form $e \otimes e - f \otimes f$ where $e, f \in H$ are linearly independent unit vectors.

**Remark 2.5.** *Each operator $A \in \mathcal{F}_2(H)$ can be represented by*

$$A = \alpha(I \oplus -I \oplus 0)$$

*with respect to an orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$ with $\dim H_1 = \dim H_2 = 1$ and $\alpha \in \mathbb{C}$. Indeed, since $A$ is a normal operator of trace zero, we can find orthonormal vectors $e_1, e_2$ in the range of $A$ and a non-zero $\alpha \in \mathbb{C}$ such that $A = \alpha(e_1 \otimes e_1 - e_2 \otimes e_2)$. Thus, it suffices to take $H_1 = \mathrm{Span}\{e_1\}$, $H_2 = \mathrm{Span}\{e_2\}$ and $H_3 = \mathrm{Span}\{e_1, e_2\}^\perp$.*

The next result gives us a partial information on the form of $\Phi$, and it will be extremely useful for other lemmas.

**Lemma 2.6.** *There exists a map $h : \mathbb{C} \to \mathbb{C}$ such that*

$$\Phi(\lambda A + \mu B) = h(\lambda)A + h(\mu)B,$$

*for all linearly independent operators $A, B \in \mathcal{F}_2(H)$ and all $\lambda, \mu \in \mathbb{C}$.*

*Proof.* First, we claim that there exists a map $h : \mathbb{C} \to \mathbb{C}$ such that $\Phi(\lambda A) = h(\lambda)A$ for every $A \in \mathcal{F}_2(H)$ and $\lambda \in \mathbb{C}$.

Let $A = u \otimes u - v \otimes v$ be an operator in $\mathcal{F}_2(H)$. According to the previous remark and Lemma 2.4, for every $\lambda \in \mathbb{C}$, there exists a unique scalar $h_{u,v}(\lambda)$ such that

$$\Phi(\lambda A) = h_{u,v}(\lambda)A.$$

Let us show that $h_{u,v}$ does not depend on $u$ and $v$. For this, we start by showing that we can replace in $h_{u,v}$ the vector $u$ (resp. $v$) by another unit vector linearly independent with $v$ (resp. $u$). Without loss of generality, we take a unit vector $w \in H$ such that $\{v, w\}$ is a linearly independent set, and we will establish that $h_{u,v} = h_{w,v}$. Observe that if $u$ and $w$ are linearly dependent, then $u = \alpha w$ for some unimodular $\alpha \in \mathbb{C}$ and so $h_{u,v} = h_{\alpha w,v} = h_{w,v}$. Hence, we may assume that $u$ and $w$ are linearly independent. Clearly

$$\lambda A - \lambda(w \otimes w - v \otimes v) = \lambda(u \otimes u - w \otimes w).$$

Hence, we get by Lemma 2.4 and Remark 2.5 that

$$\Phi(\lambda A) - \Phi(\lambda(w \otimes w - v \otimes v)) = c(u \otimes u - w \otimes w)$$

for some $c \in \mathbb{C}$. That is

$$h_{u,v}(\lambda)A - h_{w,v}(\lambda)(w \otimes w - v \otimes v) = c(u \otimes u - w \otimes w).$$

Hence, it may be concluded that

$$(h_{u,v}(\lambda) - c)\, u \otimes u + (h_{w,v}(\lambda) - h_{u,v}(\lambda))\, v \otimes v = (h_{w,v}(\lambda) - c)\, w \otimes w$$

Since $v$ and $w$ are linearly independent, we can find a vector $e$ satisfies $\langle e, v\rangle = 1$ and $\langle e, w\rangle = 0$. Applying the above operator equality to $e$, we get

$$(h_{u,v}(\lambda) - c)\langle e, u\rangle u + (h_{w,v}(\lambda) - h_{u,v}(\lambda))v = 0,$$

and so $h_{u,v}(\lambda) = h_{w,v}(\lambda)$.

Now let $u', v' \in H$ be two other linearly independent unit vectors, and let us show that $h_{u,v} = h_{u',v'}$. Obviously, either $\{u', u\}$ or $\{u', v\}$ is a linearly independent set. Without loss of generality, we may suppose that $\{u', v\}$ is a linearly independent set. Hence, it follows by what has already been proved that

$$h_{u,v} = h_{u',v} = h_{u',v'}.$$

Therefore, $\Phi(\lambda A) = h(\lambda)A$ for every $A \in \mathcal{F}_2(H)$ and $\lambda \in \mathbb{C}$.

Now, let $\mu \in \mathbb{C}$, and let $B \in \mathcal{F}_2(H)$ be linearly independent with $A$. Since $\lambda A + \mu B - \lambda A = \mu B$ and $\lambda A + \mu B - \mu B = \lambda A$, we infer by Lemma 2.4 that

$$\Phi(\lambda A + \mu B) - \Phi(\lambda A) = \Phi(\lambda A + \mu B) - h(\lambda)A = cB$$

and

$$\Phi(\lambda A + \mu B) - \Phi(\mu B) = \Phi(\lambda A + \mu B) - h(\mu)B = c'A$$

for some $c, c' \in \mathbb{C}$. Consequently $h(\mu)B - h(\lambda)A = cB - c'A$, and so $h(\lambda) = c'$ and $h(\mu) = c$, which establishes the desired result. $\square$

Now, we describe explicitly the form of the map $h$ obtained in the previous lemma.

**Lemma 2.7.** *There exist two complex numbers $\alpha$ and $\beta$ such that*

$$h(z) = \alpha z + \beta\overline{z} \quad \text{for every } z \in \mathbb{C}. \tag{1}$$

*Proof.* We begin by proving that $h$ is additive. Let $z, z' \in \mathbb{C}$, and consider $A, B \in \mathcal{F}_2(H)$ given by $A = e_1 \otimes e_1 - e_2 \otimes e_2$ and $B = e_3 \otimes e_3 - e_4 \otimes e_4$ where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal set. On account of the previous lemma, we have

$$\Phi((z + z')A + zB) = h(z + z')A + h(z)B \quad \text{and} \quad \Phi(z'A) = h(z')A.$$

But, since $(z + z')A + zB - z'A = z(A + B) = zI \oplus (-zI) \oplus 0$ with respect to $H = \mathrm{Span}\{e_1, e_3\} \oplus \mathrm{Span}\{e_2, e_4\} \oplus \mathrm{Span}\{e_1, \ldots, e_4\}^{\perp}$, Lemma 2.4 implies that

$$\Phi((z + z')A + zB) - \Phi(z'A) = c(A + B)$$

for some $c \in \mathbb{C}$. This implies that

$$h(z + z')A + h(z)B - h(z')A = c(A + B),$$

and so $h(z + z') - h(z') = c = h(z)$. Thus, $h$ is additive as claimed.

Now, we establish the form of $h$. For this, we start by proving that $h$ restricted to real numbers is a scalar multiple of the identity. There is no loss of generality in assuming that $h(t_0)$ is non-zero for a certain real number $t_0$.

Let $t \in \mathbb{R}$ be non-zero, and choose two operators $F, K \in \mathcal{F}_2(H)$ linearly independent and having the same range. Under the fact that $t_0F + tK$ is a normal operator of rank two and trace zero, there exist two orthonormal vectors $e_1, e_2$ and a non-zero $c \in \mathbb{C}$ such that $t_0F + tK = c(e_1 \otimes e_1 - e_2 \otimes e_2)$. Hence,

$$\Phi(t_0F + tK) = h(t_0)F + h(t)K = h(c)(e_1 \otimes e_1 - e_2 \otimes e_2).$$

It follows that

$$\begin{aligned} h(c)t_0F + h(c)tK - h(t_0)cF - h(t)cK &= h(c)t_0F + h(c)tK - c\,(h(t_0)F + h(t)K) \\ &= h(c)t_0F + h(c)tK - h(c)c(e_1 \otimes e_1 - e_2 \otimes e_2) \\ &= h(c)t_0F + h(c)tK - h(c)(t_0F + tK) = 0, \end{aligned}$$

and so

$$\frac{h(t_0)}{t_0} = \frac{h(c)}{c} = \frac{h(t)}{t},$$

because $F, K$ are linearly independent. Consequently, for $\delta_0 = \frac{h(t_0)}{t_0}$, we get $h(t) = \delta_0 t$ for every $t \in \mathbb{R}$.

Note that the operator $t_0F + tK$ remains normal of rank two and trace zero if $t_0$ and $t$ are replaced by purely imaginary non-zero numbers. Accordingly, we obtain in the same way the existence of a complex number $\delta'$ satisfying $h(\mathrm{i}t) = \delta'\mathrm{i}t$ for every $t \in \mathbb{R}$. Therefore, by setting $\delta_1 = \delta'\mathrm{i}$, we obtain that $h(\mathrm{i}t) = \delta_1 t$ for every $t \in \mathbb{R}$.

Letting $z = a + \mathrm{i}b \in \mathbb{C}$ where $a$ and $b$ are real numbers, we get that

$$\begin{aligned} h(z) &= h(a) + h(\mathrm{i}b) = \delta_0 a + \delta_1 b \\ &= \frac{\delta_0 - \mathrm{i}\delta_1}{2}(a + \mathrm{i}b) + \frac{\delta_0 + \mathrm{i}\delta_1}{2}(a - \mathrm{i}b) \\ &= \alpha z + \beta\overline{z}, \end{aligned}$$

where $\alpha = 2^{-1}(\delta_0 - \mathrm{i}\delta_1)$ and $\beta = 2^{-1}(\delta_0 + \mathrm{i}\delta_1)$. This finishes the proof. $\square$

For an operator $T \in \mathcal{B}(H)$, write $\mathrm{ran}(T)$ for its range and $\ker(T)$ for its kernel.

**Lemma 2.8.** *Consider an orthogonal decomposition* $H = H_1 \oplus \cdots \oplus H_5$ *where* $\dim H_1 = \dim H_2$ *and* $\dim H_3 = \dim H_4$. *Let* $E_o, F_o \in \mathcal{B}(H)$ *be defined by*

$$E_o = I \oplus -I \oplus 0 \oplus 0 \oplus 0 \quad \textit{and} \quad F_o = 0 \oplus 0 \oplus I \oplus -I \oplus 0.$$

*Then*

$$\Phi(\lambda E_o + \mu F_o) = h(\lambda)E_o + h(\mu)F_o \quad \textit{for every } \lambda, \mu \in \mathbb{C}.$$

*Proof.* Let $\lambda, \mu \in \mathbb{C}$, and put $A = \lambda E_o + \mu F_o$. By Lemma 2.4, we can write $\Phi(A) = \lambda' E_o + \mu' F_o$. Choose an orthonormal system $\{e_1, \ldots, e_4\}$ where $e_i \in H_i$, and set $F = e_1 \otimes e_1 - e_2 \otimes e_2$ and $K = e_3 \otimes e_3 - e_4 \otimes e_4$. Note that we can represent

$$A - (\lambda F + \mu K) = \lambda I \oplus -\lambda I \oplus \mu I \oplus -\mu I \oplus 0$$

with respect to the new decomposition $H = H_1' \oplus \cdots \oplus H_5'$ where $H_i' = H_i \ominus \mathrm{Span}\{e_i\}$ for $1 \leq i \leq 4$ and $H_5' = \mathrm{Span}\{e_1, \ldots, e_4\} \oplus H_5$. It follows by Lemma 2.6 that $\Phi(\lambda F + \mu K) = h(\lambda)F + h(\mu)K$, and by Lemma 2.4 that

$$(\Phi(A) - \Phi(\lambda F + \mu K))_{|H_5'} = (\Phi(A) - h(\lambda)F - h(\mu)K)_{|H_5'} = 0.$$

Computing for $e_1$ and $e_3$, we obtain that $\lambda' = h(\lambda)$ and $\mu' = h(\mu)$. □

In the sequel, $H$ is assumed to be infinite dimensional. Recall that an orthogonal projection is said to be a *proper projection* if its kernel and range are infinite dimensional subspaces. We denote by $\mathcal{P}_\infty(H)$ the set of all proper projections in $\mathcal{B}(H)$.

**Lemma 2.9.** *We have*

$$\Phi(\lambda P) = h(\lambda)P \quad \textit{for every } P \in \mathcal{P}_\infty(H) \textit{ and } \lambda \in \mathbb{C}.$$

*Proof.* Let $P \in \mathcal{P}_\infty(H)$ and $\lambda \in \mathbb{C}$. Then, $P = 0 \oplus I \oplus 0 \oplus 0 \oplus 0$ with respect to an orthogonal decomposition of $H$ into infinite-dimensional subspaces. Consider $Q, R \in \mathcal{P}_\infty(H)$ defined by

$$Q = 0 \oplus I \oplus I \oplus 0 \oplus 0 \quad \text{and} \quad R = 0 \oplus 0 \oplus 0 \oplus I \oplus 0.$$

Since $\lambda(Q - R) - \lambda P = \lambda(Q - R - P) = \lambda(0 \oplus 0 \oplus I \oplus -I \oplus 0)$, Lemma 2.4 ensures the existence of $c \in \mathbb{C}$ such that

$$\Phi(\lambda(Q - R)) - \Phi(\lambda P) = c(Q - R - P).$$

Note that, with respect to a suitable orthogonal decomposition of $H$ into infinite-dimensional subspaces, we may write $Q = I \oplus 0 \oplus 0$ and $R = 0 \oplus I \oplus 0$, so that $Q - R = I \oplus -I \oplus 0$. Thus, by Lemma 2.8, we have

$$\Phi(\lambda(Q - R)) = h(\lambda)(Q - R).$$

It may be concluded that

$$\Phi(\lambda P) = \Phi(\lambda(Q - R)) - c(Q - R - P) = cP + (h(\lambda) - c)Q + (c - h(\lambda))R.$$

Replacing $R$ by $R' = 0 \oplus 0 \oplus 0 \oplus 0 \oplus I$, we get in a similar way that

$$\Phi(\lambda P) = c'P + (h(\lambda) - c')Q + (c' - h(\lambda))R'$$

for some $c' \in \mathbb{C}$. Since $\{P, Q, R, R'\}$ is a linearly independent set, we infer that $c' = c = h(\lambda)$. Therefore, $\Phi(\lambda P) = h(\lambda)P$ as desired. □

In order to establish the form of $\Phi$ for a linear combination of proper projections, we start by the following special case.

**Lemma 2.10.** *Let* $P, Q \in \mathcal{P}_\infty(H)$ *be such that* $\operatorname{codim} \overline{\operatorname{ran}(P) + \operatorname{ran}(Q)} = \infty$. *Then*

$$\Phi(\lambda P + \mu Q) = h(\lambda)P + h(\mu)Q \quad \textit{for every } \lambda, \mu \in \mathbb{C}.$$

*Proof.* Suppose first that $PQ = 0$. The fact that codim $\overline{\operatorname{ran}(P) + \operatorname{ran}(Q)} = \infty$ leads to $P = I \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$ and $Q = 0 \oplus I \oplus 0 \oplus 0 \oplus 0 \oplus 0$ on a suitable orthogonal decomposition of infinite-dimensional subspaces. As

$$\lambda P + \mu Q - (\lambda + \mu)Q = \lambda(P - Q),$$

Lemma 2.4 implies that

$$\Phi(\lambda P + \mu Q) - \Phi((\lambda + \mu)Q) = c(P - Q)$$

for some $c \in \mathbb{C}$. But, by the previous lemma $\Phi((\lambda + \mu)Q) = h(\lambda + \mu)Q$, and then

$$\Phi(\lambda P + \mu Q) = cP + (h(\lambda + \mu) - c)Q.$$

Consider $A = \lambda I \oplus \mu I \oplus \lambda I \oplus -\lambda I \oplus \mu I \oplus -\mu I$. Since

$$A - (\lambda P + \mu Q) = 0 \oplus 0 \oplus \lambda I \oplus -\lambda I \oplus \mu I \oplus -\mu I,$$

we infer by Lemma 2.4 that $(\Phi(A) - \Phi(\lambda P + \mu Q))_{|\operatorname{ran}(P)} = 0$, and thus $\Phi(A)_{|\operatorname{ran}(P)} = cI$. According to Lemma 2.8, we have

$$\Phi(A) = h(\lambda)I \oplus h(\mu)I \oplus h(\lambda)I \oplus -h(\lambda)I \oplus h(\mu)I \oplus -h(\mu)I,$$

that is $\Phi(A)_{|\operatorname{ran}(P)} = h(\lambda)I$. Therefore, $c = h(\lambda)$ and so $\Phi(\lambda P + \mu Q) = h(\lambda)P + h(\mu)Q$.

Now, for the general case, we can choose a proper projection $R$ such that $RP = RQ = 0$. Hence, Lemma 2.4 asserts that

$$\Phi(\lambda P + \mu Q) - \Phi(\lambda R + \mu Q) = c'(P - R)$$

for some $c' \in \mathbb{C}$. Taking into account the previous case, we get that

$$\Phi(\lambda P + \mu Q) = h(\lambda)R + h(\mu)Q + c'(P - R).$$

Since the choice of $R$ is arbitrary, we obtain necessary that $h(\lambda) = c'$, and thus

$$\Phi(\lambda P + \mu Q) = h(\lambda)P + h(\mu)Q.$$

This completes the proof. $\square$

**Lemma 2.11.** *Let $P_i \in \mathcal{P}_\infty(H)$, $1 \leq i \leq n$, be such that* codim $\overline{\operatorname{ran}(P_1) + \cdots + \operatorname{ran}(P_n)} = \infty$. *Then, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have*

$$\Phi\left(\sum_{i=1}^{n} \lambda_i P_i\right) = \sum_{i=1}^{n} h(\lambda_i) P_i. \tag{2}$$

*Proof.* The proof is by induction on $n$. According to Lemmas 2.9 and 2.10, we can assume that (2) holds for a fixed $n \geq 2$.

Let $\lambda_i \in \mathbb{C}$ and $P_i \in \mathcal{P}_\infty(H)$, $1 \leq i \leq n+1$, where codim $\overline{\operatorname{ran}(P_1) + \cdots + \operatorname{ran}(P_{n+1})} = \infty$. The fact that $h$ is additive allows to assume, without loss of generality, that $P_i \neq P_j$ for $i \neq j$.

Firstly, consider the particular case when $P_{n+1}P_n = P_{n+1}P_1 = 0$. Let $K \in \{P_1, P_n\}$. Clearly, the operator $P_{n+1} - K$ can be written as $I \oplus -I \oplus 0$ with respect to a suitable orthogonal decomposition of infinite-dimensional subspaces, and hence by Lemma 2.4 we have

$$\Phi\left(\sum_{i=1}^{n+1} \lambda_i P_i\right) - \Phi\left(\sum_{i=1}^{n} \lambda_i P_i + \lambda_{n+1} K\right) = c_K(P_{n+1} - K)$$

for some $c_K \in \mathbb{C}$. From induction hypothesis, we obtain

$$\Phi\left(\sum_{i=1}^{n} \lambda_i P_i + \lambda_{n+1} K\right) = \sum_{i=1}^{n} h(\lambda_i) P_i + h(\lambda_{n+1}) K.$$

Consequently,

$$\Phi\left(\sum_{i=1}^{n+1}\lambda_i P_i\right)-\sum_{i=1}^{n}h(\lambda_i)P_i = h(\lambda_{n+1})K + c_K(P_{n+1}-K).$$

Writing this equality for $K = P_1$ and for $K = P_n$, we get that

$$h(\lambda_{n+1})P_1 + c_1(P_{n+1}-P_1) = h(\lambda_{n+1})P_n + c_n(P_{n+1}-P_n).$$

The multiplication by $P_{n+1}$ leads to $c_1 = c_n$, and so

$$(h(\lambda_{n+1})-c_1)P_1 = (h(\lambda_{n+1})-c_1)P_n.$$

As $P_1 \neq P_n$, we obtain $h(\lambda_{n+1}) = c_1$ and

$$\Phi\left(\sum_{i=1}^{n+1}\lambda_i P_i\right)=\sum_{i=1}^{n+1}h(\lambda_i)P_i.$$

Now, we consider the general case. As $\operatorname{codim}\overline{\operatorname{ran}(P_1)+\cdots+\operatorname{ran}(P_{n+1})}=\infty$, we can choose $Q\in\mathcal{P}_\infty(H)$ such that $QP_i = 0$ for every $1\leq i\leq n+1$. From Lemma 2.4, there is $c\in\mathbb{C}$ such that

$$\Phi\left(\sum_{i=1}^{n+1}\lambda_i P_i\right)-\Phi\left(\sum_{i=1}^{n}\lambda_i P_i+\lambda_{n+1}Q\right)=c(P_{n+1}-Q).$$

On the other hand, by the previous case, we have

$$\Phi\left(\sum_{i=1}^{n}\lambda_i P_i+\lambda_{n+1}Q\right)=\sum_{i=1}^{n}h(\lambda_i)P_i+h(\lambda_{n+1})Q.$$

Therefore,

$$\Phi\left(\sum_{i=1}^{n+1}\lambda_i P_i\right)=\sum_{i=1}^{n}h(\lambda_i)P_i+cP_{n+1}+(h(\lambda_{n+1})-c)Q.$$

Since $Q$ is not unique, we necessary get that $c = h(\lambda_{n+1})$, and so

$$\Phi\left(\sum_{i=1}^{n+1}\lambda_i P_i\right)=\sum_{i=1}^{n+1}h(\lambda_i)P_i,$$

as desired. $\square$

**Remark 2.12.** *Given two subspaces $M,N\subseteq H$ of infinite dimension and codimension, there exist two orthogonal subspaces $M_1,M_2$ of $M$ such that*

$$\dim M_i=\dim M\ominus M_i=\operatorname{codim}\ \overline{M_i+N}=\infty,\ \textit{for } i\in\{1,2\}.$$

*Indeed, take an orthonormal basis $\{u_n : n\geq 1\}$ of $N^\perp$ and write $u_n = x_n + y_n$ where $x_n\in M$ and $y_n\in M^\perp$. Choose a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ so that the subspace $K = M\ominus\overline{\operatorname{Span}\{x_{n_k} : k\geq 1\}}$ has infinite dimension. Then $u_{n_k}\in K^\perp\cap N^\perp$ for every $k\geq 1$. Therefore, every orthogonal infinite-dimensional subspaces $M_1$, $M_2$ of $K$ satisfy the requirement properties.*

**Lemma 2.13.** *Let $P_i\in\mathcal{P}_\infty(H)$, $1\leq i\leq n$, and let $Q_j\in\mathcal{P}_\infty(H)$, $1\leq j\leq m$, be such that* $\operatorname{codim}\overline{\operatorname{ran}(Q_1)+\cdots+\operatorname{ran}(Q_m)}=\infty$*. Then, for every $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_m\in\mathbb{C}$, we have*

$$\Phi\left(\sum_{i=1}^{n}\lambda_i P_i+\sum_{j=1}^{m}\mu_j Q_j\right)=\sum_{i=1}^{n}h(\lambda_i)P_i+\sum_{j=1}^{m}h(\mu_j)Q_j. \tag{3}$$

*Proof.* We proceed by induction on $n$. It is customary to use the standard convention that the sum over an empty set is zero, and hence the case $n = 0$ is proved in Lemma 2.11.

Suppose that (3) holds for $n \geq 0$. Let $P_i \in \mathcal{P}_\infty(H)$, $1 \leq i \leq n+1$, and let $Q_j \in \mathcal{P}_\infty(H)$, $1 \leq j \leq m$, be such that $\operatorname{codim} \overline{\operatorname{ran}(Q_1) + ... + \operatorname{ran}(Q_m)} = \infty$. Taking $M = \operatorname{ran}(P_{n+1})$ and $N = \overline{\operatorname{ran}(Q_1) + \cdots + \operatorname{ran}(Q_m)}$ in the previous remark, we can easily construct proper projections $R_1$ and $R_2$ that fulfill $\{R_1, R_2, P_{n+1}\}$ is a linearly independent set, $\operatorname{ran}(R_k) \subset \operatorname{ran}(P_{n+1})$ and

$$\dim \operatorname{ran}(P_{n+1} - R_k) = \operatorname{codim} \overline{\operatorname{ran}(R_k) + \operatorname{ran}(Q_1) + \cdots + \operatorname{ran}(Q_m)} = \infty,$$

for $k = 1, 2$. Fix $k \in \{1, 2\}$, and set

$$A = \sum_{i=1}^{n+1} \lambda_i P_i + \sum_{j=1}^{m} \mu_j Q_j \quad \text{and} \quad B = \sum_{i=1}^{n} \lambda_i P_i + 2\lambda_{n+1} R_k + \sum_{j=1}^{m} \mu_j Q_j.$$

Since $\operatorname{ran}(R_k) \subset \operatorname{ran}(P_{n+1})$, we can write $P_{n+1} - 2R_k = I \oplus -I \oplus 0$ with respect to the orthogonal decomposition $H = \operatorname{ran}(P_{n+1} - R_k) \oplus \operatorname{ran}(R_k) \oplus \ker(P_{n+1})$. As $A - B = \lambda_{n+1}(P_{n+1} - 2R_k)$, it follows from Lemma 2.4 and induction hypothesis that

$$\begin{aligned} \Phi(A) - \Phi(B) &= \Phi(A) - \left( \sum_{i=1}^{n} h(\lambda_i) P_i + 2h(\lambda_{n+1}) R_k + \sum_{j=1}^{m} h(\mu_j) Q_j \right) \\ &= c_k (P_{n+1} - 2R_k) \end{aligned}$$

for some $c_k \in \mathbb{C}$, and so

$$\Phi(A) = \sum_{i=1}^{n} h(\lambda_i) P_i + \sum_{j=1}^{m} h(\mu_j) Q_j + c_k P_{n+1} + (2h(\lambda_{n+1}) - 2c_k) R_k.$$

We deduce that

$$c_1 P_{n+1} + (2h(\lambda_{n+1}) - 2c_1) R_1 = c_2 P_{n+1} + (2h(\lambda_{n+1}) - 2c_2) R_2,$$

and hence $2h(\lambda_{n+1}) - 2c_1 = 0$, that is $c_1 = h(\lambda_{n+1})$. Therefore

$$\Phi(A) = \sum_{i=1}^{n+1} h(\lambda_i) P_i + \sum_{j=1}^{m} h(\mu_j) Q_j,$$

the desired equality. □

From [14, Corollary 2.3], every operator on $H$ is a linear combination of sixteen orthogonal projections. It follows that every operator on $H$ is a finite linear combination of proper projections. Indeed, for a given projection $P$, $H$ can be decomposed as an orthogonal sum of $\operatorname{ran}(P)$ and $\ker(P)$. When $\operatorname{ran}(P)$ (resp. $\ker(P)$) has infinite-dimension, we decompose it as an orthogonal sum of two infinite-dimensional subspaces, and we write $P = I \oplus I \oplus 0 = I \oplus 0 \oplus 0 + 0 \oplus I \oplus 0$ (resp. $P = I \oplus 0 \oplus 0 = I \oplus I \oplus 0 - 0 \oplus I \oplus 0$).

Now, we present the proof of Theorem 1.1.

*Proof.* [Proof of Theorem 1.1] (i)⇒(ii). Let $\alpha$ and $\beta$ be the complex numbers obtained in Lemma 2.7, and let $T \in \mathcal{B}(H)$. Then, there are complex numbers $\lambda_1, \ldots, \lambda_n$ and proper projections $P_i \in \mathcal{P}_\infty(H)$, $1 \leq i \leq n$, such that $T = \sum_{i=1}^{n} \lambda_i P_i$. Taking $m = 1$ in the previous lemma, we get that

$$\begin{aligned} \Phi(T) &= \Phi\left( \sum_{i=1}^{n} \lambda_i P_i \right) = \Phi\left( \sum_{i=1}^{n-1} \lambda_i P_i + \lambda_n P_n \right) = \sum_{i=1}^{n} h(\lambda_i) P_i \\ &= \alpha \left( \sum_{i=1}^{n} \lambda_i P_i \right) + \beta \left( \sum_{i=1}^{n} \overline{\lambda_i} P_i \right) = \alpha T + \beta T^*. \end{aligned}$$

The implication (ii)⇒(i) follows immediately from the fact that for a fixed conjugation $C$ on $H$, the set of $C$-skew symmetric operators forms a $*$-closed subspace of $\mathcal{B}(H)$. □

For $T \in \mathcal{B}(\mathbb{C}^n)$, we denote by $\mathrm{tr}(T)$ the trace of $T$. We present now the proof of Theorem 1.3.

*Proof.* [Proof of Theorem 1.3] (i)⇒(ii). First, we establish the form of $\Phi$ on the set of scalar multiple of rank-one projections. Let $u \in \mathbb{C}^n$ be a unit vector, and let $\lambda \in \mathbb{C}$. We can choose an orthonormal basis $\{e_k\}_{1\le k\le n}$ of $\mathbb{C}^n$ such that $e_1 = u$. It follows from Lemma 2.6 that $\Phi(\lambda(e_1 \otimes e_1 - e_k \otimes e_k)) = h(\lambda)(e_1 \otimes e_1 - e_k \otimes e_k)$ for every $2 \le k \le n$, and hence

$$\begin{aligned}\Phi\left(n\lambda u \otimes u - \lambda I\right) &= \Phi\left(\sum_{k=2}^{n} \lambda(e_1 \otimes e_1 - e_k \otimes e_k)\right) = \sum_{k=2}^{n} \Phi\left(\lambda(e_1 \otimes e_1 - e_k \otimes e_k)\right)\\ &= \sum_{k=2}^{n} h(\lambda)\left(e_1 \otimes e_1 - e_k \otimes e_k\right) = nh(\lambda)u \otimes u - h(\lambda)I.\end{aligned}$$

Therefore, we get that

$$\Phi(\lambda u \otimes u) = h(\lambda)u \otimes u + \frac{1}{n}\left[\Phi(\lambda I) - h(\lambda)I\right].$$

Note that $h(\lambda) = \alpha\lambda + \beta\overline{\lambda}$ for every $\lambda \in \mathbb{C}$ by Lemma 2.7. Since, by the spectral theorem [4, Theorem II.7.6], every normal operator $N$ on $\mathbb{C}^n$ is a linear combination of rank-one projections $u_i \otimes u_i$, it may be concluded that

$$\begin{aligned}\Phi(N) &= \Phi\left(\sum_{i=1}^{n} \lambda_i u_i \otimes u_i\right) = \sum_{i=1}^{n} \Phi(\lambda_i u_i \otimes u_i)\\ &= \sum_{i=1}^{n}(\alpha\lambda_i + \beta\overline{\lambda_i})u_i \otimes u_i + \sum_{i=1}^{n} \frac{1}{n}\left[\Phi(\lambda_i I) - h(\lambda_i)I\right]\\ &= \alpha N + \beta N^* + \frac{1}{n}\left(\Phi(\mathrm{tr}(N)I) - h(\mathrm{tr}(N))I\right).\end{aligned}$$

This form of $\Phi$ remains true for every $T \in \mathcal{B}(\mathbb{C}^n)$ because every operator is the sum of two normal ones. If we let $\Psi(T) = \frac{1}{n}\left(\Phi(\mathrm{tr}(T)I) - h(\mathrm{tr}(T))I\right)$ for every $T \in \mathcal{B}(\mathbb{C}^n)$, then clearly $\Psi$ is an additive map on $\mathcal{B}(\mathbb{C}^n)$ that vanishes on $\mathrm{sl}_n(\mathbb{C})$, and

$$\Phi(T) = \alpha T + \beta T^* + \Psi(T).$$

(ii)⇒(i) follows from the fact that every skew-symmetric operator $T$ has trace zero. Indeed, if $M$ is the representation matrix of $T$ in some orthonormal basis that satisfy $M^{tr} = -M$, with $M^{tr}$ being the transpose of $M$, then we should have

$$\mathrm{tr}(T) = \mathrm{tr}(M) = -\mathrm{tr}(M^{tr}) = -\mathrm{tr}(M) = 0.$$

□

We close this paper by the following question:

**Question 2.14.** *It would be interesting to know if an analogue result of Theorem 1.1 can be obtained in the setting of finite-dimensional Hilbert spaces.*

## References


[1] Z. Amara, M. Oudghiri and K. Souilah, Complex symmetric operators and additive preservers problem, Advances in Operator Theory 5 (2020) 261–279.
[2] Z. Amara and M. Oudghiri, Non linear preservers problem of complex symmetric operators, Asian-European Journal of Mathematics 14 (2021) 2150162.
[3] R. An, J. Hou and L. Zhao, Adjacency preserving maps on the space of symmetric operators, Linear Algebra and its Applications 405 (2005) 311–324.
[4] J. B. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1997.
[5] G. Dolinar, Maps on $B(H)$ preserving idempotents, Linear and Multilinear Algebra 52 (2004) 335–347.
[6] H. Essannouni and A. Kaidi, Le théorème de Hua pour les algèbres artiniennes simples, Linear Algebra and its Applications 297 (1999) 9–22.
[7] S. R. Garcia and M. Putinar, Complex symmetric operators and applications, Transactions of the American Mathematical Society 358 (2006) 1285–1315.
[8] S. R. Garcia and M. Putinar, Complex symmetric operators and applications II, Transactions of the American Mathematical Society 359 (2007) 3913–3931.
[9] L. K. Hua, A theorem on matrices over a field and its applications, Acta Mathematica Sinica (English Series) 1 (1951) 109–163.
[10] L. K. Hua, Geometry of Symmetric Matrices Over Any Field with Characteristic Other Than Two, Annals of Mathematics 50 (1949) 8–31.
[11] S. Jung, E. Ko, M. Lee and J. Lee, On local spectral properties of complex symmetric operators, Journal of Mathematical Analysis and Applications 379 (2011), 325–333.
[12] C. G. Li and S. Zhu, Skew symmetric normal operators, Proceedings of the American Mathematical Society 141 (2013) 2755–2762.
[13] M. Mbekhta, A Hua type theorem and linear preserver problems, Mathematical Proceedings of the Royal Irish Academy 109 (2009) 109–121.
[14] C. Pearcy and D. Topping, Sums of small numbers of idempotents, Michigan Mathematical Journal 14 (1967) 453–465.
[15] E. Prodan, S. R. Garcia, and M. Putinar, Norm estimates of complex symmetric operators applied to quantum systems, Journal of Physics A 39 (2006) 389–400.
[16] S. M. Zagorodnyuk, On a J-polar decomposition of a bounded operator and matrices of J-symmetric and J-skew-symmetric operators, Banach Journal of Mathematical Analysis 4 (2010) 11–36.
[17] S. Zhu, On skew symmetric operators with eigenvalues, Journal of the Korean Mathematical Society 52 (2015) 1271–1286.
[18] Z. Wan, Geometry of Matrices, World Scientific Publishing Co., Singapore, 1996.