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Representation of Solutions to Fuzzy Linear Conformable Differential Equations

Taoyu Yang[a], JinRong Wang[a], Donal O'Regan[b]

[a] *Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P. R. China*
[b] *School of Mathematical and Statistical Sciences, National University of Ireland, Galway, Ireland*

Abstract. In this paper, we study fuzzy linear conformable differential equations using the generalized fuzzy conformable fractional differentiability concept. We give an explicit representation of $q_{(1)}$-differentiable and $q_{(2)}$-differentiable solutions for appropriate differential equations. Finally, we give some examples to illustrate our theoretical results.

1. Introduction

Derivatives of fuzzy valued mappings were developed in [1] that generalized and extended the concept of Hukuhara differentiability (H-derivative) of set valued mappings to the class of fuzzy mappings. Using the H-derivative the author in [2] developed a theory for fuzzy differential equations. In [3] the authors introduced the notion of a fuzzy fractional derivative and the theory was developed in [4–9]. In [10] a new definition of a fuzzy fractional derivative called the fuzzy conformable fractional derivative is given, some useful results on fuzzy conformable fractional derivatives and fractional integrals are given and a class of linear fuzzy conformable fractional differential equations is considered. For first order linear fuzzy differential equations we refer the reader to [11].

The purpose of this paper is to consider the solutions of $x^{(q)}(t) = a(t)x(t) + \sigma(t)$ and $x^{(q)}(t) + a(t)x(t) = \sigma(t)$ with $x(0) = x_0$, where $x^{(q)}$ denotes the q-derivative of x, $a : I \to R$, $x_0 \in R_F$, $q \in (0, 1]$ and $\sigma \in C(I, R_F)$. In this paper, we transfer the idea of a variation of constant formula to conformable differential equations. Our results extend the representation of (1)-differentiable and (2)-differentiable solutions in [11] to the representation of $q_{(1)}$-differentiable and $q_{(2)}$-differentiable solutions. In particular, we give the explicit representation of $q_{(1)}$-differentiable and $q_{(2)}$-differentiable solutions of fuzzy conformable fractional differential equations when $a < 0$ and $a > 0$ (this is not included in [11] for the first differential equation). Also under some suitable condition, $q_{(1)}$-differentiable and $q_{(2)}$-differentiable solutions of the second differential equation exist; to achieve this we construct some H-differences.

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Email addresses: tyyangmath@126.com (Taoyu Yang), jrwang@gzu.edu.cn (JinRong Wang), donal.oregan@nuigalway.ie (Donal O'Regan)

2. Preliminaries

Denote by $R_F := \{u : R \to [0,1]\}$ the class of fuzzy subsets of the real line satisfying the following properties: u is normal (*i.e.* $\exists x_0 \in R$, $u(x_0) = 1$); u is a convex fuzzy set (*i.e.* $u(\lambda s + (1-\lambda)r) \geq \min\{u(s), u(r)\}$, $\forall \lambda \in [0,1]$, $s, r \in R$); u is upper semicontinuous on R; $[u]^0 = cl\{x \in R | u(x) > 0\}$ is compact.

Then R_F is called the space of fuzzy numbers. Now, $R \subset R_F$. Let $[u]^\alpha = \{x \in R | u(x) \geq \alpha\}$ for $0 < \alpha \leq 1$, $u \in R$. Then $[u]^\alpha$ is called the $\alpha-$level set of u, which is a nonempty compact interval for all $\alpha \in (0,1]$. The notation $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ denotes explicitly the $\alpha-$level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively. For $u \in R_F$, we use the notation $diam([u]^\alpha) = \overline{u}^\alpha - \underline{u}^\alpha$ to denote the length of u.

We define the sum $u+v$ and the produce λu as $[u+v]^\alpha = [u]^\alpha + [v]^\alpha = [\underline{u}^\alpha + \underline{v}^\alpha, \overline{u}^\alpha + \overline{v}^\alpha]$ and $[\lambda u]^\alpha = \lambda [u]^\alpha$, for $\forall \alpha \in [0,1]$, $u, v \in R_F$ and $\lambda \in R$.

The metric structure is given by the Hausdorff distance (see [12]) $D : R_F \times R_F \to R_+ \cup \{0\}$, $D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$, (R_F, D) is a complete metric space and the following properties are well known: $D(u+w, v+w) = D(u,v)$, $\forall u, v, w \in R_F$, $D(ku, kv) = |k| D(u,v)$, $\forall k \in R$, $u, v \in R_F$, $D(u+v, w+e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in R_F$.

Let $I = (0,b) \subset R$ be an interval. We denote by $C(I, R_F)$ the space of all continuous fuzzy functions from I to R_F, which is a complete metric space with respect to the metric $h(u,v) = \sup_{t \in I} D(u(t), v(t))$.

Definition 2.1 (see [11], Definition 2.1). *Let $u, v \in R_F$. If there exists $w \in R_F$ such that $u = v + w$ then w is called the H-difference of u, v and it is denoted by $u \ominus v$.*

In this paper, we use the sign $\ominus$ to represent the H-difference. Note that $u \ominus v \neq u + (-1)v$.

The generalized fuzzy conformable fractional derivative was introduced in [13].

Definition 2.2 (see [13], Definition 2). *Let $F : I \to R_F$ be a fuzzy function and $q \in (0,1]$. We say F is q-differentiable at $t \in I$, if for all $\varepsilon > 0$ sufficiently close to 0, the H-differences $F(t + \varepsilon t^{1-q}) \ominus F(t)$, $F(t) \ominus F(t - \varepsilon t^{1-q})$ exist, and the limits (in the metric D)*

$$\lim_{\varepsilon \to 0^+} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon} \tag{1}$$

exist. We write (1) *as $T_q(F)(t)$ or $F^{(q)}(t)$.*

Definition 2.3 (see [13], Definition 3). *Let $F : I \to R_F$ be a fuzzy function and $q \in (0,1]$.*

($q_{(1)}$) We say F is $q_{(1)}$-differentiable at $t \in I$, if for all $\varepsilon > 0$ sufficiently close to 0, the H-differences $F(t+\varepsilon t^{1-q}) \ominus F(t)$, $F(t) \ominus F(t - \varepsilon t^{1-q})$ exist, and the limits (in the metric D)

$$\lim_{\varepsilon \to 0^+} \frac{F(t + \varepsilon t^{1-q}) \ominus F(t)}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{F(t) \ominus F(t - \varepsilon t^{1-q})}{\varepsilon} \tag{2}$$

exist. We write (2) *as $T_{(q_{(1)})}(F)(t)$ or $F^{(q_{(1)})}(t)$.*

($q_{(2)}$) We say F is $q_{(2)}$-differentiable at $t \in I$, if for all $\varepsilon > 0$ sufficiently close to 0, the H-differences $F(t) \ominus F(t+\varepsilon t^{1-q})$, $F(t - \varepsilon t^{1-q}) \ominus F(t)$ exist and the limits (in the metric D)

$$\lim_{\varepsilon \to 0^+} \frac{F(t) \ominus F(t + \varepsilon t^{1-q})}{-\varepsilon} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{F(t - \varepsilon t^{1-q}) \ominus F(t)}{-\varepsilon} \tag{3}$$

exist. We write (3) *as $T_{(q_{(2)})}(F)(t)$ or $F^{(q_{(2)})}(t)$.*

Theorem 2.4 (see [13], Theorem 6). *Let F: $I \to R_F$ be fuzzy function, where $F_\alpha(t) = [f_1^\alpha(t), f_2^\alpha(t)]$, $\alpha \in [0,1]$: (i) If F is $q_{(1)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q-differentiable and $[F^{(q_{(1)})}(t)]^\alpha = [(f_1^\alpha)^{(q)}(t), (f_2^\alpha)^{(q)}(t)]$. (ii) If F is $q_{(2)}$-differentiable, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q-differentiable and $[F^{(q_{(2)})}(t)]^\alpha = [(f_2^\alpha)^{(q)}(t), (f_1^\alpha)^{(q)}(t)]$.*

Definition 2.5 (see [11], Definition 2.2). *Let $F : I \to R_F$ and fix $t \in I$.*

(i) *We say F is* (1)*-differentiable at t, if for all $h > 0$ sufficiently close to* 0*, the H-differences $F(t+h) \ominus F(t)$ and $F(t) \ominus F(t-h)$ exist, and the limits (in the metric D)*

$$\lim_{h\to 0^+} \frac{F(t+h)\ominus F(t)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t)\ominus F(t-h)}{h} \tag{4}$$

exist. We write (4) *as $D_1F(t)$ or $F'(t)$.*

(ii) *We say F is* (2)*-differentiable at t, if for all $h > 0$ sufficiently close to* 0*, the H-differences $F(t) \ominus F(t+h)$ and $F(t-h) \ominus F(t)$ exist, and the limits (in the metric D)*

$$\lim_{h\to 0^+} \frac{F(t)\ominus F(t+h)}{-h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t-h)\ominus F(t)}{-h} \tag{5}$$

exist. We write (5) *as $D_2F(t)$ or $F'(t)$.*

Theorem 2.6. *Let $F : I \to R_F$, $q \in (0, 1]$:*

(i) *F is* (1)*-differentiable is equivalent to F is $q_{(1)}$-differentiable, i.e.,*

$$F^{(q_{(1)})}(t) = t^{1-q}D_1F(t) = t^{1-q}F'(t).$$

(ii) *F is* (2)*-differentiable is equivalent to F is $q_{(2)}$-differentiable, i.e.,*

$$F^{(q_{(2)})}(t) = t^{1-q}D_2F(t) = t^{1-q}F'(t).$$

Proof. We present the details for case (i), since the other case is analogous.

Firstly, we establish necessity. Let F is (1)-differentiable and $[D_1F(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$. Let $h = \varepsilon t^{1-q}$. Therefore, if $h > 0$ and $\alpha \in [0, 1]$, we have $[F(t+h) \ominus F(t)]^\alpha = [f_\alpha(t+h) - f_\alpha(t), g_\alpha(t+h) - g_\alpha(t)]$. Divide both sides by h and pass to the limit, then by Definition 2.5, we have

$$\begin{aligned}
[D_1F(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)] &= \lim_{h\to 0^+}\left[\frac{f_\alpha(t+h)-f_\alpha(t)}{h}, \frac{g_\alpha(t+h)-g_\alpha(t)}{h}\right] \\
&= \lim_{\varepsilon\to 0^+}\left[\frac{f_\alpha(t+\varepsilon t^{1-q})-f_\alpha(t)}{\varepsilon t^{1-q}}, \frac{g_\alpha(t+\varepsilon t^{1-q})-g_\alpha(t)}{\varepsilon t^{1-q}}\right] \\
&= \left[t^{q-1}\lim_{\varepsilon\to 0^+}\frac{f_\alpha(t+\varepsilon t^{1-q})-f_\alpha(t)}{\varepsilon}, t^{q-1}\lim_{\varepsilon\to 0^+}\frac{g_\alpha(t+\varepsilon t^{1-q})-g_\alpha(t)}{\varepsilon}\right].
\end{aligned}$$

Thus, we have

$$\lim_{\varepsilon\to 0^+}\frac{f_\alpha(t+\varepsilon t^{1-q})-f_\alpha(t)}{\varepsilon} = t^{1-q}f'_\alpha(t) \text{ and } \lim_{\varepsilon\to 0^+}\frac{g_\alpha(t+\varepsilon t^{1-q})-g_\alpha(t)}{\varepsilon} = t^{1-q}g'_\alpha(t).$$

Similarly, we obtain

$$\lim_{\varepsilon\to 0^+}\frac{f_\alpha(t)-f_\alpha(t-\varepsilon t^{1-q})}{\varepsilon} = t^{1-q}f'_\alpha(t) \text{ and } \lim_{\varepsilon\to 0^+}\frac{g_\alpha(t)-g_\alpha(t-\varepsilon t^{1-q})}{\varepsilon} = t^{1-q}g'_\alpha(t).$$

From Definition 2.3, we get F is $q_{(1)}$-differentiable and $F^{(q_{(1)})}(t) = t^{1-q}D_1F(t) = t^{1-q}F'(t)$.

Next, we establish sufficiency. Let F is ($q_{(1)}$)-differentiable and $[F^{(q_{(1)})}(t)]^\alpha = [f^{(q)}_\alpha(t), g^{(q)}_\alpha(t)]$. Let $\varepsilon = t^{q-1}h$ and then $h = \varepsilon t^{1-q}$. Therefore, if $\varepsilon > 0$ and $\alpha \in [0, 1]$, we have

$$[F(t+\varepsilon t^{1-q}) \ominus F(t)]^\alpha = [f_\alpha(t+\varepsilon t^{1-q}) - f_\alpha(t), g_\alpha(t+\varepsilon t^{1-q}) - g_\alpha(t)].$$

Divide both sides by ε and pass to the limit, then by Definition 2.3, we have

$$[F^{(q_{(1)})}(t)]^\alpha = [f^{(q)}_\alpha(t), g^{(q)}_\alpha(t)] = \left[t^{1-q}\lim_{h\to 0^+}\frac{f_\alpha(t+h)-f_\alpha(t)}{h}, t^{1-q}\lim_{h\to 0^+}\frac{g_\alpha(t+h)-g_\alpha(t)}{h}\right].$$

Thus, we have

$$\lim_{h\to 0^+}\frac{f_\alpha(t+h)-f_\alpha(t)}{h}=t^{q-1}f_\alpha^{(q)}(t) \text{ and } \lim_{h\to 0^+}\frac{g_\alpha(t+h)-g_\alpha(t)}{h}=t^{q-1}g_\alpha^{(q)}(t).$$

Similarly, we obtain

$$\lim_{h\to 0^+}\frac{f_\alpha(t)-f_\alpha(t-h)}{h}=t^{q-1}f_\alpha^{(q)}(t) \text{ and } \lim_{h\to 0^+}\frac{g_\alpha(t)-g_\alpha(t-h)}{h}=t^{q-1}g_\alpha^{(q)}(t).$$

From Definition 2.5, we get F is (1)-differentiable and $F^{(q_{(1)})}(t)=t^{1-q}D_1F(t)=t^{1-q}F'(t)$. The proof is finished. □

Definition 2.7 (see [10], Definition 4). *Let $F\in C(I,R_F)\cap L^1(I,R_F)$, $[F(t)]^\alpha=\left[f_1^\alpha(t),f_2^\alpha(t)\right]$, $t\in I$, $\alpha\in[0,1]$. Define the fuzzy fractional integral for $c\geq 0$ and $q\in(0,1]$, $I_q^c(F)(t)=I_1^c(t^{q-1}F)(t)=\int_c^t\frac{F(s)}{s^{1-q}}\mathrm{d}x$, and $[I_q^c(F)(t)]^\alpha=[I_1^c(t^{q-1}F)(t)]^\alpha=\left[\int_c^t\frac{F}{s^{1-q}}(s)\mathrm{d}s\right]^\alpha=\left[\int_c^t\frac{f_1^\alpha}{s^{1-q}}(s)\mathrm{d}s,\int_c^t\frac{f_2^\alpha}{s^{1-q}}(s)\mathrm{d}s\right]$, where the integral $\int_c^t\frac{f_i^\alpha}{s^{1-q}}(s)\mathrm{d}s$, for $i=1,2$ is the usual Riemann improper integral.*

Lemma 2.8 (see [10], Theorem 7). *Let $q\in(0,1]$ and F, $G:I\to R_F$ be q-differentiable and $\lambda\in R$. Then (i) $(F+G)^{(q)}(t)=F^{(q)}(t)+G^{(q)}(t)$ and (ii) $(\lambda F)^{(q)}(t)=\lambda F^{(q)}(t)$.*

Lemma 2.9 (see [10], Theorem 8). *Let $F:I\to R_F$ be any continuous function on I. Then $(I_q^c(F))^{(q)}(t)=F(t)$, for $t\in I$.*

Theorem 2.10. *Let $F:I\to R_F$ be $q_{(2)}$-differentiable on I and assume that the derivative $F^{(q_{(2)})}$ is integrable over I. Then for each $t\in I$ we have $F(t)=F(0)\ominus I_q^0(-F^{(q_{(2)})})(t)$.*

Proof. Let $[F(t)]^\alpha=[f_\alpha(t),g_\alpha(t)]$, $\forall\alpha\in[0,1]$. Since F is $q_{(2)}$-differentiable, we have $[F^{(q_{(2)})}(t)]^\alpha=[g_\alpha^{(q)}(t),f_\alpha^{(q)}(t)]$. Then $[F(0)\ominus I_q^0(-F^{(q_{(2)})})(t)]^\alpha=\left[f_\alpha(0)+I_q^0(f_\alpha^{(q)})(t),g_\alpha(0)+I_q^0(g_\alpha^{(q)})(t)\right]=[f_\alpha(t),g_\alpha(t)],t\in I,\forall\alpha\in[0,1]$ and this completes the proof. □

Theorem 2.11. *Let $F:I\to R_F$ be a continuous fuzzy function on I and let $u(t)=\gamma\ominus I_q^0(-F)(t)$, $t\in I$, where $\gamma\in R_F$ is such that the previous H-difference exists for $t\in I$. Then u is $q_{(2)}$-differentiable and $u^{(q_{(2)})}(t)=F(t)$, $t\in I$.*

Proof. Let $t_0\in I$ and $\varepsilon>0$ sufficiently close to 0. We have $diam[u(t)]^\alpha=diam[\gamma]^\alpha-I_q^c(diam[F(t)]^\alpha)$.

Since $F(t)$ is continuous, $I_q^0(F)(t)$ is $q_{(1)}$-differentiable and $I_q^0(diam[F(t)]^\alpha)$ is non-decreasing in the variable t for any $\alpha\in[0,1]$, the $diam[u(t)]^\alpha$ is non-increasing in t for each $\alpha\in[0,1]$. Therefore the following H-differences exist and from Definition 2.7, we have

$$u(t_0)\ominus u(t_0+\varepsilon t^{1-q})=-\int_{t_0}^{t_0+\varepsilon t^{1-q}}s^{q-1}F(s)\mathrm{d}s,\ u(t_0-\varepsilon t^{1-q})\ominus u(t_0)=-\int_{t_0-\varepsilon t^{1-q}}^{t_0}s^{q-1}F(s)\mathrm{d}s.$$

Then, using Theorem 2.6 one has

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}D\left(\frac{u(t_0)\ominus u(t_0+\varepsilon t^{1-q})}{-\varepsilon},F(t_0)\right)\\
&=\lim_{\varepsilon\to 0^+}\sup_{\alpha\in[0,1]}\max\left\{\left|\frac{\int_{t_0}^{t_0+\varepsilon t^{1-q}}s^{q-1}\underline{F}(s)\mathrm{d}s}{\varepsilon}-\underline{F}(t_0)\right|,\left|\frac{\int_{t_0}^{t_0+\varepsilon t^{1-q}}s^{q-1}\overline{F}(s)\mathrm{d}s}{\varepsilon}-\overline{F}(t_0)\right|\right\}\\
&=\lim_{\varepsilon\to 0^+}D\left(\frac{\int_{t_0}^{t_0+\varepsilon t^{1-q}}s^{q-1}F(s)\mathrm{d}s}{\varepsilon},F(t_0)\right)=0.
\end{aligned}$$

Since F is continuous and $g(t) = I_q^0(F)(t)$ is $q_{(1)}$-differentiable, and from Lemma 2.9 we have $g^{(q)}(t) = F(t)$ for $t \in I$. Thus $\lim_{\varepsilon\to 0^+} D\left(\frac{u(t_0)\ominus u(t_0+\varepsilon t^{1-q})}{-\varepsilon}, F(t_0)\right) = 0$. Similarly, we get $\lim_{\varepsilon\to 0^+} D\left(\frac{u(t_0-\varepsilon t^{1-q})\ominus u(t_0)}{-\varepsilon}, F(t_0)\right) = 0$. Then $u^{(q_{(2)})}(t) = F(t)$, $t \in I$, and the proof is complete. □

Theorem 2.12. *Let $F, G : I \to R_F$ be two fuzzy function.*

(i) If $F(t)$ is $q_{(1)}$-differentible and $G(t)$ is $q_{(2)}$-differentible on an interval (α, β) and if the H-difference $F(t) \ominus G(t)$ exists for $t \in (\alpha, \beta)$ then $F(t) \ominus G(t)$ is $q_{(1)}$-differentible and $(F \ominus G)^{(q_{(1)})}(t) = F^{(q_{(1)})}(t) + (-1)G^{(q_{(2)})}(t)$ for all $t \in (\alpha, \beta)$.

(ii) If $F(t)$ is $q_{(2)}$-differentiable and $G(t)$ is $q_{(1)}$-differentible on an interval (α, β) and if the H-difference $F(t) \ominus G(t)$ exists for $t \in (\alpha, \beta)$ then $F(t) \ominus G(t)$ is $q_{(2)}$-differentible and $(F \ominus G)^{(q_{(2)})}(t) = F^{(q_{(2)})}(t) + (-1)G^{(q_{(1)})}(t)$.

Proof. We present the details only for case (*i*). Case (*ii*) is similar to case (*i*).

Since $F(t)$ is $q_{(1)}$-differentiable it follows that $F(t + \varepsilon t^{1-q}) \ominus F(t)$ exists, i.e., $\exists\, u_1(t, \varepsilon t^{1-q})$, $F(t + \varepsilon t^{1-q}) = F(t) + u_1(t, \varepsilon t^{1-q})$. Similarly since $G(t)$ is $q_{(2)}$-differentiable there exists $v_1(t, \varepsilon t^{1-q})$ such that $G(t) = G(t + \varepsilon t^{1-q}) + v_1(t, \varepsilon t^{1-q})$ and we get $F(t + \varepsilon t^{1-q}) + G(t) = F(t) + G(t + \varepsilon t^{1-q}) + u_1(t, \varepsilon t^{1-q}) + v_1(t, \varepsilon t^{1-q})$. Since the H-differences $F(t) \ominus G(t)$ and $F(t + \varepsilon t^{1-q}) \ominus G(t + \varepsilon t^{1-q})$ exist for $\varepsilon > 0$ such that $t + \varepsilon t^{1-q} \in (\alpha, \beta)$, we get $F(t + \varepsilon t^{1-q}) \ominus G(t + \varepsilon t^{1-q}) = F(t) \ominus G(t) + u_1(t, \varepsilon t^{1-q}) + v_1(t, \varepsilon t^{1-q})$, that is the H-difference $(F(t + \varepsilon t^{1-q}) \ominus G(t + \varepsilon t^{1-q})) \ominus (F(t) \ominus G(t))$ exists and

$$(F(t + \varepsilon t^{1-q}) \ominus G(t + \varepsilon t^{1-q})) \ominus (F(t) \ominus G(t)) = u_1(t, \varepsilon t^{1-q}) + v_1(t, \varepsilon t^{1-q}). \tag{6}$$

By similar reasoning we get that there exist $u_2(t, \varepsilon t^{1-q})$ and $v_2(t, \varepsilon t^{1-q})$ such that $F(t) = F(t - \varepsilon t^{1-q}) + u_2(t, \varepsilon t^{1-q})$, $G(t - \varepsilon t^{1-q}) = G(t) + v_2(t, \varepsilon t^{1-q})$ and so

$$(F(t) \ominus G(t)) \ominus (F(t - \varepsilon t^{1-q}) \ominus G(t - \varepsilon t^{1-q})) = u_2(t, \varepsilon t^{1-q}) + v_2(t, \varepsilon t^{1-q}). \tag{7}$$

Note $\lim_{\varepsilon\to 0^+} \frac{u_1(t,\varepsilon t^{1-q})}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{u_2(t,\varepsilon t^{1-q})}{\varepsilon} = F^{(q_{(1)})}(t)$ and $\lim_{\varepsilon\to 0^+} \frac{v_1(t,\varepsilon t^{1-q})}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{v_2(t,\varepsilon t^{1-q})}{\varepsilon} = (-1)G^{(q_{(2)})}(t)$. Finally, by multiplying (6) and (7) with $\frac{1}{\varepsilon}$ and passing to the limit with $\varepsilon \to 0^+$ we get that $F(t) \ominus G(t)$ is $q_{(1)}$-differentiable and $(F \ominus G)^{(q_{(1)})}(t) = F^{(q_{(2)})}(t) + (-1)G^{(q_{(1)})}(t)$. □

Theorem 2.13 (see [13], Theorem 7). *Let $F : I \to R_F$, $G : I \to R$, and G be q-differentiable.*

(i) If $G(t) \cdot G^{(q)}(t) > 0$ and F is $q_{(1)}$-differentiable, then $G \cdot F$ is $q_{(1)}$-differentiable and $(G \cdot F)^{(q_{(1)})}(t) = G(t) \cdot F^{(q_{(1)})}(t) + F(t) \cdot G^{(q)}(t)$.

(ii) If $G(t) \cdot G^{(q)}(t) < 0$ and F is $q_{(2)}$-differentiable, then $G \cdot F$ is $q_{(2)}$-differentiable and $(G \cdot F)^{(q_{(2)})}(t) = G(t) \cdot F^{(q_{(2)})}(t) + F(t) \cdot G^{(q)}(t)$.

3. The equation $x^{(q)}(t) = a(t)x(t) + \sigma(t)$

Consider the fuzzy linear conformable fractional differential equation

$$\begin{cases} x^{(q)}(t) = a(t)x(t) + \sigma(t), \ t \in I, \\ x(0) = x_0, \end{cases} \tag{8}$$

where $a : I \to R$, $x_0 \in R_F$, $q \in (0, 1]$ and $\sigma \in C(I, R_F)$.

Denote $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$, $[\sigma(t)]^\alpha = [\underline{\sigma}_\alpha(t), \overline{\sigma}_\alpha(t)]$, $[x(0)]^\alpha = [\underline{x_0}^\alpha, \overline{x_0}^\alpha]$. From Theorem 2.4, if $x(t)$ is $q_{(1)}$-differentiable then $[x^{q_{(1)}}(t)]^\alpha = [\underline{x}_\alpha^{(q)}(t), \overline{x}_\alpha^{(q)}(t)]$. If $x(t)$ is $q_{(2)}$-differentiable then $[x^{q_{(2)}}(t)]^\alpha = [\overline{x}_\alpha^{(q)}(t), \underline{x}_\alpha^{(q)}(t)]$.

Definition 3.1. *Let $x(t) : I \to R_F$ be a fuzzy function such that $x^{q_{(1)}}(t)$ or $x^{q_{(2)}}(t)$ exists. If $x(t)$ and $x^{q_{(1)}}(t)$ satisfy problem (8), we say $x(t)$ is a $q_{(1)}$-solution of problem (8). Similarly if $x(t)$ and $x^{q_{(2)}}(t)$ satisfy problem (8), we say $x(t)$ is a $q_{(2)}$-solution of problem (8).*

We study problem (8) in three cases $a(t) < 0$, $a(t) > 0$ and $a(t) = 0$ for $t \in I$.

Case 3.2. *$a(t) < 0$ for $t \in I$.*

We get a $q_{(1)}$-solution via $q_{(1)}$-differentiable and we consider the following ODEs system:

$$\begin{cases} \underline{x}^{(q)}(t) = a(t)\overline{x}(t) + \underline{\sigma}(t), \\ \overline{x}^{(q)}(t) = a(t)\underline{x}(t) + \overline{\sigma}(t), \\ \underline{x}(0) = \underline{x_0}, \\ \overline{x}(0) = \overline{x_0}. \end{cases}$$

For solving this ODEs system, we have

$$\begin{pmatrix} \underline{x}(t) \\ \overline{x}(t) \end{pmatrix}^{(q)} = \begin{pmatrix} 0 & a(t) \\ a(t) & 0 \end{pmatrix} \begin{pmatrix} \underline{x}(t) \\ \overline{x}(t) \end{pmatrix} + \begin{pmatrix} \underline{\sigma}(t) \\ \overline{\sigma}(t) \end{pmatrix}.$$

We denote

$$X(t) = \begin{pmatrix} \underline{x}(t) \\ \overline{x}(t) \end{pmatrix}, A(t) = \begin{pmatrix} 0 & a(t) \\ a(t) & 0 \end{pmatrix}, B(t) = \begin{pmatrix} \underline{\sigma}(t) \\ \overline{\sigma}(t) \end{pmatrix}$$

and the previous system is written as $X^{(q)}(t) = A(t)X(t) + B(t)$.

By the variation of constants formula for differential equation and Definition 2.7, we have

$$X(t) = e^{I_q^0(A)(t)} \left(X_0 + \int_0^t s^{q-1} \left(e^{-I_q^0(A)(s)} B(s) \right) \mathrm{d}s \right).$$

Therefore

$$\begin{aligned} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix}(t) = & \begin{pmatrix} \cosh(I_q^0(a)(t)) & \sinh(I_q^0(a)(t)) \\ \sinh(I_q^0(a)(t)) & \cosh(I_q^0(a)(t)) \end{pmatrix} \\ & \times \left(\begin{pmatrix} \underline{x_0} \\ \overline{x_0} \end{pmatrix} + \int_0^t s^{q-1} \begin{pmatrix} \cosh(I_q^0(a)(s)) & -\sinh(I_q^0(a)(s)) \\ -\sinh(I_q^0(a)(s)) & \cosh(I_q^0(a)(s)) \end{pmatrix} \begin{pmatrix} \underline{\sigma}(s) \\ \overline{\sigma}(s) \end{pmatrix} \mathrm{d}s \right). \end{aligned}$$

Then we have

$$\begin{aligned} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix}(t) = & \begin{pmatrix} \cosh(I_q^0(a)(t)) & \sinh(I_q^0(a)(t)) \\ \sinh(I_q^0(a)(t)) & \cosh(I_q^0(a)(t)) \end{pmatrix} \\ & \times \begin{pmatrix} \underline{x_0} + \int_0^t s^{q-1} \left[\underline{\sigma}(s) \cosh(I_q^0(a)(s)) - \overline{\sigma}(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \\ \overline{x_0} + \int_0^t s^{q-1} \left[-\underline{\sigma}(s) \sinh(I_q^0(a)(s)) + \overline{\sigma}(s) \cosh(I_q^0(a)(s)) \right] \mathrm{d}s \end{pmatrix}. \end{aligned}$$

Then the solution of the ODEs system is

$$\begin{aligned} \underline{x}(t) = & \cosh(I_q^0(a)(t)) \left\{ \underline{x_0} + \int_0^t s^{q-1} \left[\underline{\sigma}(s) \cosh(I_q^0(a)(s)) - \overline{\sigma}(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right\} \\ & + \sinh(I_q^0(a)(t)) \left\{ \overline{x_0} + \int_0^t s^{q-1} \left[-\underline{\sigma}(s) \sinh(I_q^0(a)(s)) + \overline{\sigma}(s) \cosh(I_q^0(a)(s)) \right] \mathrm{d}s \right\} \\ \overline{x}(t) = & \sinh(I_q^0(a)(t)) \left\{ \underline{x_0} + \int_0^t s^{q-1} \left[\underline{\sigma}(s) \cosh(I_q^0(a)(s)) - \overline{\sigma}(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right\} \\ & + \cosh(I_q^0(a)(t)) \left\{ \overline{x_0} + \int_0^t s^{q-1} \left[-\underline{\sigma}(s) \sinh(I_q^0(a)(s)) + \overline{\sigma}(s) \cosh(I_q^0(a)(s)) \right] \mathrm{d}s \right\}. \end{aligned}$$

Then for $a(t) < 0$, the $q_{(1)}$-solution of the problem (8) is

$$x(t) = \cosh(I_q^0(a)(t)) \left(x_0 + \int_0^t s^{q-1} \left[\sigma(s) \cosh(I_q^0(a)(s)) \ominus \sigma(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right) + \sinh(I_q^0(a)(t)) \left(x_0 + \int_0^t s^{q-1} \left[\sigma(s) \cosh(I_q^0(a)(s)) \ominus \sigma(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right),$$

provided the H-difference in the integral terms exists.

However, the H-difference $\sigma(s)\cosh(I_q^0(a)(s)) \ominus \sigma(s)\sinh(I_q^0(a)(s))$ always exists for $a < 0$.

Since the diameter of the α-level set of $\sigma(s)\cosh(I_q^0(a)(s))$ is $diam([\sigma(s)]^\alpha)\cosh(I_q^0(a)(s))$, which is greater than the diameter of the α-level set of $\sigma(s)\sinh(I_q^0(a)(s))$, $diam([\sigma(s)]^\alpha)(-\sinh(I_q^0(a)(s)))$. Here we have used the positive character of the function $\sinh(I_q^0(a)(s)) + \cosh(I_q^0(a)(s))$. Finally, by Theorem 2.13-(i), we see this solution is $q_{(1)}$-differentiable on I.

Thus, we have proved the following result.

Theorem 3.3. *For $a(t) < 0$, the $q_{(1)}$-solution of problem* (8) *is given by*

$$x(t) = \cosh(I_q^0(a)(t)) \left(x_0 + \int_0^t s^{q-1} \left[\sigma(s) \cosh(I_q^0(a)(s)) \ominus \sigma(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right) + \sinh(I_q^0(a)(t)) \left(x_0 + \int_0^t s^{q-1} \left[\sigma(s) \cosh(I_q^0(a)(s)) \ominus \sigma(s) \sinh(I_q^0(a)(s)) \right] \mathrm{d}s \right).$$

For finding the $q_{(2)}$-solution, change problem (8) to the corresponding ODEs system

$$\begin{cases} \underline{x}^{(q)}(t) = a(t)\underline{x}(t) + \overline{\sigma}(t), \\ \overline{x}^{(q)}(t) = a(t)\overline{x}(t) + \underline{\sigma}(t), \\ \underline{x}(0) = \underline{x_0}, \\ \overline{x}(0) = \overline{x_0}, \end{cases}$$

and we solve this system.

Theorem 3.4. *For $a(t) < 0$, the $q_{(2)}$-solution of problem* (8) *is*

$$x(t) = e^{I_q^0(a)(t)} \left(x_0 \ominus \int_0^t s^{q-1}(-\sigma(s)) e^{-I_q^0(a)(s)} \mathrm{d}s \right)$$

provided the H-difference exists.

Proof. By Theorem 2.12 and Lemma 2.9, and the H-difference $x_0 \ominus \int_0^t s^{q-1}(-\sigma(s))e^{-I_q^0(a)(s)}\mathrm{d}s$ exists, then the H-difference is $q_{(2)}$-differentiable and

$$\left(x_0 \ominus \int_0^t s^{q-1}(-\sigma(s)) e^{-I_q^0(a)(s)} \mathrm{d}s \right)^{(q_{(2)})} = \sigma(t) \cdot e^{-I_q^0(a)(t)}.$$

Denote $G(t) = e^{I_q^0(a)(t)}$ and $F(t) = x_0 \ominus \int_0^t s^{q-1}(-\sigma(s))e^{-I_q^0(a)(s)}\mathrm{d}s$, since $a(t) < 0$, so $G(t) \cdot G^{(q)}(t) < 0$ and $F(t)$ is $q_{(2)}$-differentiable, so the condition in Theorem 2.13-(ii) is satisfied. Then we get

$$x^q(t) = a(t)e^{I_q^0(a)(t)} \left(x_0 \ominus \int_0^t s^{q-1}(-\sigma(s)) e^{-I_q^0(a)(s)} \mathrm{d}s \right) + \sigma(t) e^{I_q^0(a)(t)} e^{-I_q^0(a)(t)} = a(t)x(t) + \sigma(t),$$

i.e., $x(t)$ is the $q_{(2)}$-solution of (8). $\square$

Case 3.5. *$a(t) > 0$ for $t \in I$.*

Considering $q_{(1)}$-differentiability, problem (8) is transformed into the ODEs system

$$\begin{cases} \underline{x}^{(q)}(t) = a(t)\underline{x}(t) + \underline{\sigma}(t), \\ \overline{x}^{(q)}(t) = a(t)\overline{x}(t) + \overline{\sigma}(t), \\ \underline{x}(0) = \underline{x_0}, \\ \overline{x}(0) = \overline{x_0}, \end{cases}$$

and we solve this system.

Theorem 3.6. *For $a(t) > 0$, the $q_{(1)}$-solution of problem* (8) *is*

$$x(t) = e^{I_q^0(a)(t)}\left(x_0 + \int_0^t s^{q-1}\sigma(s)e^{-I_q^0(a)(s)}\mathrm{d}s\right).$$

Proof. By Lemmas 2.8 and 2.9, since a constant function is differentiable in any case of differentiability we have $x_0 + \int_0^t s^{q-1}\sigma(s)e^{-I_q^0 a(s)}\mathrm{d}s$ is $q_{(1)}$-differentiable and $\left(x_0 + \int_0^t s^{q-1}\sigma(s)e^{-I_q^0(a)(s)}\mathrm{d}s\right)^{(q_{(1)})} = \sigma(t)e^{-I_q^0(a)(t)}$.

Denote $G(t) = e^{I_q^0(a)(t)}$ and $F(t) = x_0 + \int_0^t s^{q-1}\sigma(s)e^{-I_q^0(a)(s)}\mathrm{d}s$, since $a(t) > 0$, so $G(t)\cdot G^{(q)}(t) > 0$ and $F(t)$ is $q_{(1)}$-differentiable, so the condition in Theorem 2.13-(i) is satisfied. Then

$$x^q(t) = a(t)e^{I_q^0(a)(t)}\left(x_0 + \int_0^t s^{q-1}\sigma(s)e^{-I_q^0(a)(s)}\mathrm{d}s\right) + \sigma(t)e^{I_q^0(a)(t)}e^{-I_q^0(a)(t)} = a(t)x(t) + \sigma(t),$$

i.e., $x(t)$ is the $q_{(1)}$-solution of (8). □

For finding the $q_{(2)}$-solution, change problem (8) to the corresponding ODEs system

$$\begin{cases} \underline{x}^{(q)}(t) = a(t)\overline{x}(t) + \overline{\sigma}(t), \\ \overline{x}^{(q)}(t) = a(t)\underline{x}(t) + \underline{\sigma}(t), \\ \underline{x}(0) = \underline{x_0}, \\ \overline{x}(0) = \overline{x_0}. \end{cases}$$

The solution of this system is

$$\begin{aligned} \underline{x}(t) =& \cosh(I_q^0(a)(t))\left\{\underline{x_0} + \int_0^t s^{q-1}\left[\overline{\sigma}(s)\cosh(I_q^0(a)(s)) - \underline{\sigma}(s)\sinh(I_q^0(a)(s))\right]\mathrm{d}s\right\} \\ &+ \sinh(I_q^0(a)(t))\left\{\overline{x_0} + \int_0^t s^{q-1}\left[-\overline{\sigma}(s)\sinh(I_q^0(a)(s)) + \underline{\sigma}(s)\cosh(I_q^0(a)(s))\right]\mathrm{d}s\right\} \\ \overline{x}(t) =& \sinh(I_q^0(a)(t))\left\{\underline{x_0} + \int_0^t s^{q-1}\left[\overline{\sigma}(s)\cosh(I_q^0(a)(s)) - \underline{\sigma}(s)\sinh(I_q^0(a)(s))\right]\mathrm{d}s\right\} \\ &+ \cosh(I_q^0(a)(t))\left\{\overline{x_0} + \int_0^t s^{q-1}\left[-\overline{\sigma}(s)\sinh(I_q^0(a)(s)) + \underline{\sigma}(s)\cosh(I_q^0(a)(s))\right]\mathrm{d}s\right\}. \end{aligned}$$

Then for $a(t) > 0$, the $q_{(2)}$-differentiable solution is

$$\begin{aligned} x(t) =& \cosh(I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right]\mathrm{d}s\right) \\ &\ominus -\sinh(I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right]\mathrm{d}s\right), \end{aligned}$$

provided the H-difference

$$x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s \tag{9}$$

exists. Since for $a(t) > 0$, $\cosh(I_q^0(a)(t)) \geq \sinh(I_q^0(a)(t))$ and thus $\forall K \in R_F$, the H-difference $\cosh(I_q^0(a)(t))K \ominus (-\sinh(I_q^0(a)(t))K)$ exists.

Theorem 3.7. *For $a(t) > 0$, the $q_{(2)}$-solution of problem* (8) *is given by*

$$\begin{aligned} x(t) =& \cosh(I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s\right) \\ &\ominus -\sinh(I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\cosh(I_q^0(a)(s)) - \sigma(s)\sinh(I_q^0(a)(s))\right] \mathrm{d}s\right) \end{aligned}$$

provided that the H-difference (9) *exists.*

Proof. According to Definition 3.1, we just need to prove $x(t)$ is $q_{(2)}$-differentiable. Considering Definition 2.3-($q_{(2)}$), if the H-differences $x(t) \ominus x(t+\varepsilon t^{1-q})$ and $x(t-\varepsilon t^{1-q}) \ominus x(t)$ exist and the limits $\lim_{\varepsilon \to 0^+} \frac{x(t)\ominus x(t+\varepsilon t^{1-q})}{-\varepsilon}$ and $\lim_{\varepsilon \to 0^+} \frac{x(t-\varepsilon t^{1-q})\ominus x(t)}{-\varepsilon}$ exist, then $x(t)$ is $q_{(2)}$-differentiable. So we only need to check above the H-differences and limits exist. The details are shown as follows.

For $\alpha \in [0,1]$, $t \in I$,

$$\begin{aligned} diam[x(t)]^\alpha =& \cosh(I_q^0(a)(t)) diam\left[\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s\right)\right]^\alpha \\ &- \sinh(I_q^0(a)(t)) diam\left[\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s\right)\right]^\alpha \\ =& \left(\cosh(I_q^0(a)(t)) - \sinh(I_q^0(a)(t))\right) \\ &\times diam\left[\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s\right)\right]^\alpha. \end{aligned}$$

Since $\cosh(I_q^0(a)(s)) - \sinh(I_q^0(a)(s)) = e^{-I_q^0(a)(s)}$ is non-negative and decreasing and $x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(I_q^0(a)(s)) - \sigma(s)\cosh(I_q^0(a)(s))\right] \mathrm{d}s$ is a $q_{(2)}$-differentiable function (see Theorem 2.11), its diameter is non-increasing in the variable t for $\alpha \in [0,1]$ fixed. Then $diam[x(t)]^\alpha$ is non-increasing in t for $\alpha \in [0,1]$ fixed. Therefore the H-differences $x(t) \ominus x(t+\varepsilon t^{1-q})$ and $x(t-\varepsilon t^{1-q}) \ominus x(t)$ exist. For calculating the $q_{(2)}$-derivative, we set $f_1(t) = \cosh(I_q^0(a)(t))$ and $f_2(t) = \sinh(I_q^0(a)(t))$. We check that $\lim_{\varepsilon \to 0^+} D\left(\frac{x(t)\ominus x(t+\varepsilon t^{1-q})}{-\varepsilon}, a(t)x(t) + \sigma(t)\right) = 0$.

For this purpose, we prove that $\left|\frac{\overline{x}(t)-\overline{x}(t+\varepsilon t^{1-q})}{-\varepsilon} - (a(t)\underline{x}(t) + \underline{\sigma}(t))\right|$ and $\left|\frac{\underline{x}(t)-\underline{x}(t+\varepsilon t^{1-q})}{-\varepsilon} - (a(t)\overline{x}(t) + \overline{\sigma}(t))\right|$ tend to

0 as $\varepsilon \to 0^+$ uniformly for $\alpha \in [0,1]$. Indeed, with Definition 2.7, we have

$$
\begin{aligned}
&\frac{\overline{x}(t)-\overline{x}(t+\varepsilon t^{1-q})}{-\varepsilon}-(a(t)\underline{x}(t)+\underline{\sigma}(t))\\
=&\frac{f_2(t)}{-\varepsilon}\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}+\frac{f_1(t)}{-\varepsilon}\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}\\
&+\frac{-f_2(t+\varepsilon t^{1-q})}{-\varepsilon}\left\{\underline{x_0}+\int_0^{t+\varepsilon t^{1-q}} s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}\\
&+\frac{-f_1(t+\varepsilon t^{1-q})}{-\varepsilon}\left\{\overline{x_0}+\int_0^{t^{1-q}} s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}-\underline{\sigma}(t)\\
&-a(t)\left[f_1(t)\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}\right]\\
&-a(t)\left[f_2(t)\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}\right]\\
=&\left(\frac{f_2(t)-f_2(t+\varepsilon t^{1-q})}{-\varepsilon}-a(t)f_1(t)\right)\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}\\
&+\frac{-f_2(t+\varepsilon t^{1-q})}{-\varepsilon}\left\{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}\\
&+\left(\frac{f_1(t)-f_1(t+\varepsilon t^{1-q})}{-\varepsilon}-a(t)f_2(t)\right)\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}\\
&+\frac{-f_1(t+\varepsilon t^{1-q})}{-\varepsilon}\left\{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}-\underline{\sigma}(t)\\
=&\left(\frac{f_2(t)-f_2(t+\varepsilon t^{1-q})}{-\varepsilon}-a(t)f_1(t)\right)g_1(t)+\left(\frac{f_1(t)-f_1(t+\varepsilon t^{1-q})}{-\varepsilon}-a(t)f_2(t)\right)g_2(t)\\
&+f_2(t+\varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_1(s)\mathrm{d}s}{\varepsilon}-f_2(t+\varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon}\\
&-f_1(t+\varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon}+f_1(t+\varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_1(s)\mathrm{d}s}{\varepsilon}-\underline{\sigma}(t),
\end{aligned}
$$

where $g_1(t)=\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}$ and $g_2(t)=\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}$. Then for each t fixed, $\lim\limits_{\varepsilon\to 0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\overline{x}(t)-\overline{x}(t+\varepsilon t^{1-q})}{-\varepsilon}-(a(t)\underline{x}(t)+\underline{\sigma}(t))\right|=0$ due to the following considerations:

(1) $f_1(t)$ and $f_2(t)$ are q-differentiable at t, by Theorem 2.6, we have $f_1^{(q)}(t)=a(t)f_2(t)$ and $f_2^{(q)}(t)=a(t)f_1(t)$.

(2) $g_1(t)$ and $g_2(t)$ are bounded (in the variable α for each t fixed). Indeed, the support of x_0 is bounded, and the endpoints of the support of σ are continuous functions on the compact interval $[0,t]$ and, thus, bounded. Also, $f_1(t)$ and $f_2(t)$ are bounded on the compact interval $[0,t]$.

(3) The following limits exist for every $\alpha\in[0,1]$:

$\lim\limits_{\varepsilon\to 0^+}\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_i(s)\mathrm{d}s}{\varepsilon}=\overline{\sigma}(t)f_i(t),\ i=1,2,\ \lim\limits_{\varepsilon\to 0^+}\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_i(s)\mathrm{d}s}{\varepsilon}=\underline{\sigma}(t)f_i(t),\ i=1,2$ and they are uniformly in $\alpha\in[0,1]$. Since σ is a continuous fuzzy function, and hence, $\int_0^{\varepsilon t^{1-q}} s^{q-1}\sigma(s)f_i(s)\mathrm{d}s$ is (1)-differentiable with derivative $\sigma(t)f_i(t)$, $i=1,2$.

(4) We have $f_1^2(t)-f_2^2(t)=1$, $\forall t\in I$.

Analogously we prove $\lim\limits_{\varepsilon\to 0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\underline{x}(t)-\underline{x}(t+\varepsilon t^{1-q})}{-\varepsilon}-(a(t)\overline{x}(t)+\overline{\sigma}(t))\right|=0$.

Then $\lim\limits_{\varepsilon\to 0^+}\frac{x(t)\ominus x(t+\varepsilon t^{1-q})}{-\varepsilon}=a(t)x(t)+\sigma(t)$. Similarly $\lim\limits_{\varepsilon\to 0^+}\frac{x(t-\varepsilon t^{1-q})\ominus x(t)}{-\varepsilon}=a(t)x(t)+\sigma(t)$. Consequently, $x(t)$ is $q_{(2)}$-differentiable and $x^{(q_{(2)})}(t)=a(t)x(t)+\sigma(t)$. □

Case 3.8. *$a(t)=0$ for $t\in I$.*

Consider

$$\begin{cases}x^{(q)}(t)=\sigma(t),\ t\in I,\\ x(0)=x_0.\end{cases}$$

From the results in Case 3.2 and Case 3.5 we have

Theorem 3.9. *For $a(t)=0$ the $q_{(1)}$-solution of problem* (8) *is given by $x(t)=x_0+I_q^0(\sigma)(t)$, and the $q_{(2)}$-solution of problem* (8) *is $x(t)=x_0\ominus I_q^0(-\sigma)(t)$ provided that the H-difference exists.*

Remark 3.10. *We remark that [11, Cases 1-3] are special situations of Theorems* 3.3, 3.4, 3.6, 3.7, 3.9 *if $q=1$, i.e., [11] coincide with the first order situation.*

4. The equation $x^{(q)}(t)+a(t)x(t)=\sigma(t)$

Consider the fuzzy linear conformable fractional differential equation

$$\begin{cases}x^{(q)}(t)+a(t)x(t)=\sigma(t),\ t\in I,\\ x(0)=x_0,\end{cases}\tag{10}$$

where $a:I\to R$, $x_0\in R_F$, $q\in(0,1]$ and $\sigma\in C(I,R_F)$.

Here, we study problem (10) in three cases $a(t)>0$, $a(t)<0$ and $a(t)=0$ for $t\in I$.

Definition 4.1. *Let $x(t):I\to R_F$ be a fuzzy function such that $x^{q_{(1)}}(t)$ or $x^{q_{(2)}}(t)$ exists. If $x(t)$ and $x^{q_{(1)}}(t)$ satisfy problem* (10)*, we say $x(t)$ is a $q_{(1)}$-solution of problem* (10)*. Similarly if $x(t)$ and $x^{q_{(2)}}(t)$ satisfy problem* (10)*, we say $x(t)$ is a $q_{(2)}$-solution of problem* (10)*.*

Case 4.2. *$a(t)>0$ for $t\in I$.*

Theorem 4.3. *For $a(t)>0$, the $q_{(1)}$-solution of problem* (10) *is $x(t)=e^{-I_q^0(a)(t)}\left(x_0+\int_0^t s^{q-1}\sigma(s)e^{I_q^0(a)(s)}\mathrm{d}s\right)$ provided that $x(t+\varepsilon t^{1-q})\ominus x(t)$ and $x(t)\ominus x(t-\varepsilon t^{1-q})$ exist for ε sufficiently small.*

Proof. Problem (10) can be written as $[\underline{x}_\alpha^{(q)}(t),\overline{x}_\alpha^{(q)}(t)]+[a(t)\underline{x}_\alpha(t),a(t)\overline{x}_\alpha(t)]=[\underline{\sigma}_\alpha(t),\overline{\sigma}_\alpha(t)]$, so $\underline{x}_\alpha^{(q)}(t)+a(t)\underline{x}_\alpha(t)=\underline{\sigma}_\alpha(t)$, $\overline{x}_\alpha^{(q)}(t)+a(t)\overline{x}_\alpha(t)=\overline{\sigma}_\alpha(t)$. Thus $\left(\underline{x}_\alpha e^{I_q^0(a)(t)}\right)^{(q)}(t)=\underline{\sigma}_\alpha(t)e^{I_q^0(a)(t)}$, $\left(\overline{x}_\alpha e^{I_q^0(a)(t)}\right)^{(q)}(t)=\overline{\sigma}_\alpha(t)e^{I_q^0(a)(t)}$, therefore, by Definition 2.7, it can be deduced that

$$\underline{x}_\alpha(t)=\underline{x_0}_\alpha e^{-I_q^0(a)(t)}+e^{-I_q^0(a)(t)}\int_0^t s^{q-1}\underline{\sigma}_\alpha(s)e^{I_q^0(a)(s)}\mathrm{d}s,$$

$$\overline{x}_\alpha(t)=\overline{x_0}_\alpha e^{-I_q^0(a)(t)}+e^{-I_q^0(a)(t)}\int_0^t s^{q-1}\overline{\sigma}_\alpha(s)e^{I_q^0(a)(s)}\mathrm{d}s.$$

This proves that,

$$[x(t)]^\alpha=[x_0]^\alpha e^{-I_q^0(a)(t)}+e^{-I_q^0(a)(t)}\int_0^t s^{q-1}[\sigma(s)]^\alpha e^{I_q^0(a)(s)}\mathrm{d}s.$$

Thus $x(t) = e^{-I_q^0(a)(t)}\left(x_0 + \int_0^t s^{q-1}\sigma(s)e^{I_q^0(a)(s)}\mathrm{d}s\right)$. We get that $x(t)$ is a solution of problem (10) for $a(t) > 0$. Now we check that $x(t)$ is the $q_{(1)}$-solution of problem (10).

The H-differences $x(t+\varepsilon t^{1-q}) \ominus x(t)$ and $x(t) \ominus x(t-\varepsilon t^{1-q})$ exist. For calculating the $q_{(2)}$-derivative, we set $f_1(t) = e^{-I_q^0(a)(t)}$ and $f_2(t) = e^{I_q^0(a)(t)}$. We check that $\lim\limits_{\varepsilon\to 0^+} D\left(\frac{x(t+\varepsilon t^{1-q})\ominus x(t)}{\varepsilon}, \sigma(t)\ominus a(t)x(t)\right) = 0$. For this purpose, we prove that $\left|\frac{\underline{x}(t+\varepsilon t^{1-q})-\underline{x}(t)}{\varepsilon} - (\underline{\sigma}(t) - a(t)\underline{x}(t))\right|$ and $\left|\frac{\overline{x}(t+\varepsilon t^{1-q})-\overline{x}(t)}{\varepsilon} - (\overline{\sigma}(t) - a(t)\overline{x}(t))\right|$ tend to 0 as $\varepsilon \to 0^+$ uniformly for $\alpha \in [0,1]$. Indeed, with Definition 2.7, we have

$$
\begin{aligned}
&\frac{\underline{x}(t+\varepsilon t^{1-q}) - \underline{x}(t)}{\varepsilon} - (\underline{\sigma}(t) - a(t)\underline{x}(t))\\
=&\frac{f_1(t+\varepsilon t^{1-q})}{\varepsilon}\left[\underline{x_0} + \int_0^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s\right] - \frac{f_1(t)}{\varepsilon}\left[\underline{x_0} + \int_0^t s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s\right]\\
&+ a(t)f_1(t)\left[\underline{x_0} + \int_0^t s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s\right] - \underline{\sigma}(t)\\
=&\left(\frac{f_1(t+\varepsilon t^{1-q}) - f_1(t)}{\varepsilon} + a(t)f_1(t)\right)\left[\underline{x_0} + \int_0^t s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s\right]\\
&+ \frac{f_1(t+\varepsilon t^{1-q})}{\varepsilon}\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s - \underline{\sigma}(t)\\
=&\left(\frac{f_1(t+\varepsilon t^{1-q}) - f_1(t)}{\varepsilon} + a(t)f_1(t)\right)g(t) + \frac{f_1(t+\varepsilon t^{1-q})}{\varepsilon}\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s - \underline{\sigma}(t),
\end{aligned}
$$

where $g(t) = \underline{x_0} + \int_0^t s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s$.

Then for each t fixed, $\lim\limits_{\varepsilon\to 0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\underline{x}(t+\varepsilon t^{1-q})-\underline{x}(t)}{\varepsilon} - (\underline{\sigma}(t) - a(t)\underline{x}(t))\right| = 0$ due to the following considerations:

(1) $f_1(t)$ is q-differentiable at t, by Theorem 2.6, we have $f_1^{(q)}(t) = -a(t)f_1(t)$.

(2) $g(t)$ are bounded (in the variable α for each t fixed). Indeed, the support of x_0 is bounded, and the endpoints of the support of σ are continuous functions on the compact interval $[0,t]$ and, thus, bounded. Also, $f_1(t)$ and $f_2(t)$ are bounded on the compact interval $[0,t]$.

(3) The following limits exist for every $\alpha \in [0,1]$:
$\lim\limits_{\varepsilon\to 0^+}\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon} = \underline{\sigma}(t)f_2(t)$, $\lim\limits_{\varepsilon\to 0^+}\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon} = \overline{\sigma}(t)f_2(t)$, and they are uniformly in $\alpha \in [0,1]$. Since σ is a continuous fuzzy function, and hence, $\int_0^{\varepsilon t^{1-q}} s^{q-1}\sigma(s)f_2(s)\mathrm{d}s$ is (1)-differentiable with derivative $\sigma(t)f_2(t)$.

(4) We have $f_1(t)\cdot f_2(t) = 1$, $\forall t \in I$.

Analogously we prove $\lim\limits_{\varepsilon\to 0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\overline{x}(t+\varepsilon t^{1-q})-\overline{x}(t)}{\varepsilon} - (\overline{\sigma}(t) - a(t)\overline{x}(t))\right| = 0$.

Then $\lim\limits_{\varepsilon\to 0^+}\frac{x(t+\varepsilon t^{1-q})\ominus x(t)}{\varepsilon} = \sigma(t)\ominus a(t)x(t)$. Similarly $\lim\limits_{\varepsilon\to 0^+}\frac{x(t)\ominus x(t-\varepsilon t^{1-q})}{\varepsilon} = \sigma(t)\ominus a(t)x(t)$. Consequently, $x(t)$ is $q_{(1)}$-differentiable and $x^{(q_{(1)})}(t) = \sigma(t)\ominus a(t)x(t)$, i.e., $x^{(q_{(1)})}(t) + a(t)x(t) = \sigma(t)$. □

Remark 4.4. *We remark that ([10], Theorem 12) is special case of Theorem* 4.3 *when $a(t) = 1$ for $t \in I$.*

For finding the $q_{(2)}$-solution, change problem (10) to the corresponding ODEs system

$$
\begin{cases}
\underline{x}^{(q)}(t) + a(t)\overline{x}(t) = \overline{\sigma}(t),\\
\overline{x}^{(q)}(t) + a(t)\underline{x}(t) = \underline{\sigma}(t),\\
\underline{x}(0) = \underline{x_0},\\
\overline{x}(0) = \overline{x_0}.
\end{cases}
$$

The solution of this system is

$$
\begin{aligned}
\underline{x}(t) =& \cosh(-I_q^0(a)(t))\left\{\underline{x_0} + \int_0^t s^{q-1}\left[\overline{\sigma}(s)\cosh(-I_q^0(a)(s)) - \underline{\sigma}(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\
&+ \sinh(-I_q^0(a)(t))\left\{\overline{x_0} + \int_0^t s^{q-1}\left[-\overline{\sigma}(s)\sinh(-I_q^0(a)(s)) + \underline{\sigma}(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\
\overline{x}(t) =& \sinh(-I_q^0(a)(t))\left\{\underline{x_0} + \int_0^t s^{q-1}\left[\overline{\sigma}(s)\cosh(-I_q^0(a)(s)) - \underline{\sigma}(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\
&+ \cosh(-I_q^0(a)(t))\left\{\overline{x_0} + \int_0^t s^{q-1}\left[-\overline{\sigma}(s)\sinh(-I_q^0(a)(s)) + \underline{\sigma}(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}.
\end{aligned}
$$

Then for $a(t) > 0$, the $q_{(2)}$-differentiable solution of problem (10) is

$$
\begin{aligned}
x(t) =& \cosh(-I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(-I_q^0(a)(s)) - \sigma(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right)\\
&\ominus - \sinh(-I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(-I_q^0(a)(s)) - \sigma(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right),
\end{aligned}
$$

provided the H-difference

$$
x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(-I_q^0(a)(s)) - \sigma(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s \tag{11}
$$

exists. Since for $a(t) > 0$, $\cosh(-I_q^0(a)(t)) > \sinh(-I_q^0(a)(t))$ and thus $\forall K \in R_F$, the H-difference $\cosh(I_q^0(a)(t))K \ominus (-\sinh(I_q^0(a)(t))K)$ exists.

Theorem 4.5. *Let $a(t) > 0$ and then the fuzzy function $x(t)$ given by*

$$
\begin{aligned}
x(t) =& \cosh(-I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(-I_q^0(a)(s)) - \sigma(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right)\\
&\ominus - \sinh(-I_q^0(a)(t))\left(x_0 \ominus \int_0^t s^{q-1}\left[\sigma(s)\sinh(-I_q^0(a)(s)) - \sigma(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right),
\end{aligned}
$$

is the $q_{(2)}$-solution to problem (10)*, provided that H-difference* (11) *exists and $x(t) \ominus x(t + \varepsilon t^{1-q})$ and $x(t - \varepsilon t^{1-q}) \ominus x(t)$ exist for ε sufficiently small.*

Proof. The H-differences $x(t) \ominus x(t + \varepsilon t^{1-q})$ and $x(t - \varepsilon t^{1-q}) \ominus x(t)$ exist. For calculating the $q_{(2)}$-derivative, we set $f_1(t) = \cosh(-I_q^0(a)(t))$ and $f_2(t) = \sinh(-I_q^0(a)(t))$.

We check that $\lim_{\varepsilon\to 0^+} D\left(\frac{x(t)\ominus x(t+\varepsilon t^{1-q})}{-\varepsilon}, \sigma(t) \ominus a(t)x(t)\right) = 0$.

For this purpose, we prove that $\left|\frac{\overline{x}(t)-\overline{x}(t+\varepsilon t^{1-q})}{-\varepsilon} - (\underline{\sigma}(t) - a(t)\underline{x}(t))\right|$, $\left|\frac{\underline{x}(t)-\underline{x}(t+\varepsilon t^{1-q})}{-\varepsilon} - (\overline{\sigma}(t) - a(t)\overline{x}(t))\right|$ tend to 0 as $\varepsilon \to 0^+$ uniformly for $\alpha \in [0, 1]$. Indeed, with Definition 2.7, we have

$$
\begin{aligned}
&\frac{\overline{x}(t) - \overline{x}(t + \varepsilon t^{1-q})}{-\varepsilon} - (\underline{\sigma}(t) - a(t)\underline{x}(t))\\
=&\left(\frac{f_2(t) - f_2(t + \varepsilon t^{1-q})}{-\varepsilon} + a(t)f_1(t)\right)g_1(t) + \left(\frac{f_1(t) - f_1(t + \varepsilon t^{1-q})}{-\varepsilon} + a(t)f_2(t)\right)g_2(t)\\
&+ f_2(t + \varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_1(s)\mathrm{d}s}{\varepsilon} - f_2(t + \varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon}\\
&- f_1(t + \varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\overline{\sigma}(s)f_2(s)\mathrm{d}s}{\varepsilon} + f_1(t + \varepsilon t^{1-q})\frac{\int_t^{t+\varepsilon t^{1-q}} s^{q-1}\underline{\sigma}(s)f_1(s)\mathrm{d}s}{\varepsilon} - \underline{\sigma}(t),
\end{aligned}
$$

where $g_1(t)=\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\overline{\sigma}(s)f_1(s)-\underline{\sigma}(s)f_2(s)\right]\mathrm{d}s\right\}$ and $g_2(t)=\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\overline{\sigma}(s)f_2(s)+\underline{\sigma}(s)f_1(s)\right]\mathrm{d}s\right\}$. Then for each t fixed, $\lim\limits_{\varepsilon\to0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\overline{x}(t)-\overline{x}(t+\varepsilon t^{1-q})}{-\varepsilon}-(\underline{\sigma}(t)-a(t)\underline{x}(t))\right|=0$ due to the following considerations:

(1) $f_1(t)$ and $f_2(t)$ are q-differentiable at t, by Theorem 2.6, we get $f_1^{(q)}(t)=-a(t)f_2(t)$ and $f_2^{(q)}(t)=-a(t)f_1(t)$.

(2) $g_1(t)$ and $g_2(t)$ are bounded (in the variable α for each t fixed). Indeed, the support of x_0 is bounded, and the endpoints of the support of σ are continuous functions on the compact interval $[0,t]$ and, thus, bounded. Also, $f_1(t)$ and $f_2(t)$ are bounded on the compact interval $[0,t]$.

(3) The following limits exist for every $\alpha\in[0,1]$: $\lim\limits_{\varepsilon\to0^+}\frac{\int_t^{t+\varepsilon t^{1-q}}s^{q-1}\overline{\sigma}(s)f_i(s)\mathrm{d}s}{\varepsilon}=\overline{\sigma}(t)f_i(t),\ i=1,\ 2,$ $\lim\limits_{\varepsilon\to0^+}\frac{\int_t^{t+\varepsilon t^{1-q}}s^{q-1}\underline{\sigma}(s)f_i(s)\mathrm{d}s}{\varepsilon}=\underline{\sigma}(t)f_i(t),\ i=1,2$ and they are uniformly in $\alpha\in[0,1]$. Since σ is a continuous fuzzy function, and hence, $\int_0^{\varepsilon t^{1-q}}s^{q-1}\sigma(s)f_i(s)\mathrm{d}s$ is (1)-differentiable with derivative $\sigma(t)f_i(t)$, $i=1,2$.

(4) We have $f_1^2(t)-f_2^2(t)=1,\ \forall t\in I$.

Analogously we prove $\lim\limits_{\varepsilon\to0^+}\sup\limits_{\alpha\in[0,1]}\left|\frac{\underline{x}(t)-\underline{x}(t+\varepsilon t^{1-q})}{-\varepsilon}-(\overline{\sigma}(t)-a(t)\overline{x}(t))\right|=0$.

Then $\lim\limits_{\varepsilon\to0^+}\frac{x(t)\ominus x(t+\varepsilon t^{1-q})}{-\varepsilon}=\sigma(t)\ominus a(t)x(t)$. Similarly $\lim\limits_{\varepsilon\to0^+}\frac{x(t-\varepsilon t^{1-q})\ominus x(t)}{-\varepsilon}=\sigma(t)\ominus a(t)x(t)$. Consequently, $x(t)$ is $q_{(2)}$-differentiable and $x^{(q_{(2)})}(t)=\sigma(t)\ominus a(t)x(t)$ i.e. $x^{(q_{(2)})}(t)+a(t)x(t)=\sigma(t)$. □

Case 4.6. *$a(t)<0$ for $t\in I$.*

Considering $q_{(1)}$-differentiability, problem (10) is transformed into the ODEs system

$$\begin{cases}\underline{x}^{(q)}(t)+a(t)\underline{x}(t)=\underline{\sigma}(t),\\ \overline{x}^{(q)}(t)+a(t)\overline{x}(t)=\overline{\sigma}(t),\\ \underline{x}(0)=\underline{x_0},\\ \overline{x}(0)=\overline{x_0}.\end{cases}$$

The solution of this system is

$$\begin{aligned}\underline{x}(t)=&\cosh(-I_q^0(a)(t))\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\underline{\sigma}(s)\cosh(-I_q^0(a)(s))-\overline{\sigma}(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\&+\sinh(-I_q^0(a)(t))\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\underline{\sigma}(s)\sinh(-I_q^0(a)(s))+\overline{\sigma}(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\ \overline{x}(t)=&\sinh(-I_q^0(a)(t))\left\{\underline{x_0}+\int_0^t s^{q-1}\left[\underline{\sigma}(s)\cosh(-I_q^0(a)(s))-\overline{\sigma}(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}\\&+\cosh(-I_q^0(a)(t))\left\{\overline{x_0}+\int_0^t s^{q-1}\left[-\underline{\sigma}(s)\sinh(-I_q^0(a)(s))+\overline{\sigma}(s)\cosh(-I_q^0(a)(s))\right]\mathrm{d}s\right\}.\end{aligned}$$

Then for $a(t)<0$, the $q_{(1)}$-solution of the problem (10) is

$$\begin{aligned}x(t)=&\cosh(-I_q^0(a)(t))\left(x_0+\int_0^t s^{q-1}\left[\sigma(s)\cosh(-I_q^0(a)(s))\ominus\sigma(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right)\\&+\sinh(-I_q^0(a)(t))\left(x_0+\int_0^t s^{q-1}\left[\sigma(s)\cosh(-I_q^0(a)(s))\ominus\sigma(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right),\end{aligned}$$

provided the H-difference in the integral terms exists. Since for $a(t)<0$, we have $(-I_q^0(a)(t))>0$, so

$$diam([\sigma(s)]^\alpha)\cosh(-I_q^0(a)(s))\geq diam([\sigma(s)]^\alpha)\sinh(-I_q^0(a)(s)),$$

then H-difference $\sigma(s)\cosh(-I_q^0(a)(s))\ominus\sigma(s)\sinh(-I_q^0(a)(s))$ always exists. By Theorem 2.13-(i), and $\cosh(-I_q^0(a)(s))+\sinh(-I_q^0(a)(s))>0$, we get this solution is $q_{(1)}$-differentiable on I.

Theorem 4.7. *For $a(t) < 0$, the $q_{(1)}$-solution of problem* (10) *is given by*

$$x(t) = \cosh(-I_q^0(a)(t))\left(x_0 + \int_0^t s^{q-1}\left[\sigma(s)\cosh(-I_q^0(a)(s)) \ominus \sigma(s) sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right)$$
$$+ \sinh(-I_q^0(a)(t))\left(x_0 + \int_0^t s^{q-1}\left[\sigma(s)\cosh(-I_q^0(a)(s)) \ominus \sigma(s)\sinh(-I_q^0(a)(s))\right]\mathrm{d}s\right).$$

Theorem 4.8. *For $a(t) < 0$, the $q_{(2)}$-solution of problem* (10) *is given by*

$$x(t) = x_0 e^{-I_q^0(a)(t)} \ominus e^{-I_q^0(a)(t)} \int_0^t s^{q-1}(-\sigma(s)) e^{I_q^0(a)(s)} \mathrm{d}s.$$

provided the H-difference exists and $x(t) \ominus x(t + \varepsilon t^{1-q})$ and $x(t - \varepsilon t^{1-q}) \ominus x(t)$ exist for ε sufficiently small.

Proof. For finding the $q_{(2)}$-solution, problem (10) can be written as $[\overline{x}_\alpha^{(q)}(t), \underline{x}_\alpha^{(q)}(t)] + [a(t)\overline{x}_\alpha(t), a(t)\underline{x}_\alpha(t)] = [\underline{\sigma}_\alpha(t), \overline{\sigma}_\alpha(t)]$, so $\underline{x}_\alpha^{(q)}(t) + a(t)\underline{x}_\alpha(t) = \overline{\sigma}_\alpha(t)$, $\overline{x}_\alpha^{(q)}(t) + a(t)\overline{x}_\alpha(t) = \underline{\sigma}_\alpha(t)$. Thus $\left(\underline{x}_\alpha e^{I_q^0(a)(t)}\right)^{(q)}(t) = \overline{\sigma}_\alpha(t) e^{I_q^0(a)(t)}$, $\left(\overline{x}_\alpha e^{I_q^0(a)(t)}\right)^{(q)}(t) = \underline{\sigma}_\alpha(t) e^{I_q^0(a)(t)}$, and, therefore, it can be deduced that

$$\underline{x}_\alpha(t) = \underline{x_0}_\alpha e^{-I_q^0(a)(t)} - e^{-I_q^0(a)(t)} \int_0^t s^{q-1}(-\overline{\sigma}_\alpha(s)) e^{I_q^0(a)(s)} \mathrm{d}s,$$

$$\overline{x}_\alpha(t) = \overline{x_0}_\alpha e^{-I_q^0(a)(t)} - e^{-I_q^0(a)(t)} \int_0^t s^{q-1}(-\underline{\sigma}_\alpha(s)) e^{I_q^0(a)(s)} \mathrm{d}s.$$

This proves that, $[x(t)]^\alpha = [x_0]^\alpha e^{-I_q^0(a)(t)} - e^{-I_q^0(a)(t)} \int_0^t s^{q-1}[-\overline{\sigma}_\alpha(s), -\underline{\sigma}_\alpha(s)] e^{I_q^0(a)(s)} \mathrm{d}s$. So

$$x(t) = x_0 e^{-I_q^0(a)(t)} \ominus e^{-I_q^0(a)(t)} \int_0^t s^{q-1}(-\sigma(s)) e^{I_q^0(a)(s)} \mathrm{d}s,$$

provided the H-difference exists. The H-differences $x(t) \ominus x(t + \varepsilon t^{1-q})$ and $x(t - \varepsilon t^{1-q}) \ominus x(t)$ exist. Similar to the proof of Theorem 4.3, we get that $x(t)$ is the $q_{(2)}$-differentiable, i.e., $x(t)$ is the $q_{(2)}$-solution of problem (10). □

Case 4.9. $a(t) = 0$ *for $t \in I$.*

When $a(t) = 0$ case 4.9 of problem (10) is equivalent with case 3.8 of problem (8). In other words, they have the same solutions; so we omit the details here.

5. Examples

Example 5.1. *Consider the fuzzy linear conformable fractional differential equation*

$$\begin{cases} x^{(0.5)}(t) = tx(t) + 2t\gamma, \; t \geq 0, \\ x(0) = \gamma, \end{cases}$$

where $[\gamma]^\alpha = [\alpha - 1, 1 - \alpha]$.

From Theorem 3.6, we have the $q_{(1)}$-solution is $x(t) = \gamma\left(3e^{\frac{2}{3}t^{1.5}} - 2\right)$, that is shown in Figure 1.

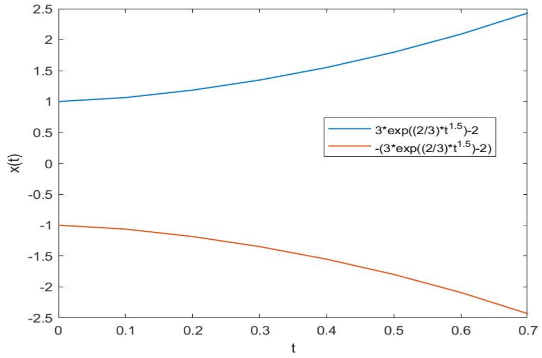

Figure 1: $q_{(1)}$-Solution of Examples 5.1 and 5.2 using the $q_{(1)}$-differentiability.

For finding the $q_{(2)}$-solution, the solution of the corresponding ODEs system is

$$\begin{aligned}\underline{x}(t) =&(\alpha-1)\cosh\left(\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}\\&+(1-\alpha)\sinh\left(\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\},\\ \overline{x}(t) =&(\alpha-1)\sinh\left(\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}\\&+(1-\alpha)\cosh\left(\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}.\end{aligned}$$

For $t\in\left[0,\ \sqrt[1.5]{1.5\ln 1.5}\right]$, $\gamma\ominus\int_0^t s^{-0.5}\left[2s\gamma\sinh\left(\frac{2}{3}s^{1.5}\right)-2s\gamma\cosh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s$ exists, since $diam([\gamma]^\alpha)\geq diam\left(\int_0^t s^{-0.5}\left[2s[\gamma]^\alpha\sinh\left(\frac{2}{3}s^{1.5}\right)-2s[\gamma]^\alpha\cosh\left(\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right)$. So we have the $q_{(2)}$-solution is $x(t)=\gamma(3e^{-\frac{2}{3}t^{1.5}}-2)$. This solution is shown in Figure 2.

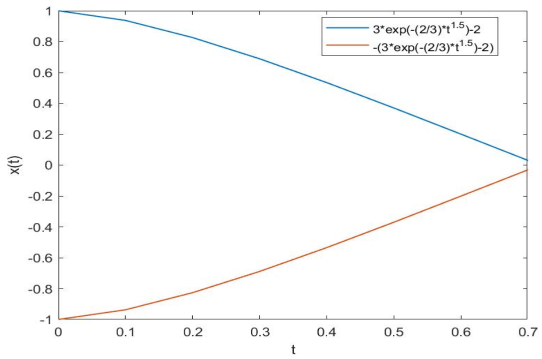

Figure 2: $q_{(2)}$-Solution of Examples 5.1 and 5.2 using the $q_{(2)}$-differentiability.

Example 5.2. *Consider the fuzzy linear conformable fractional differential equation*

$$\begin{cases}x^{(0.5)}(t)=-tx(t)+2t\gamma,\ t\geq 0,\\ x(0)=\gamma,\end{cases}$$

where $[\gamma]^\alpha = [\alpha - 1, 1 - \alpha]$.

For finding the $q_{(1)}$-solution, the solution of the corresponding ODEs system is

$$\begin{aligned}
\underline{x}(t) =&(\alpha-1)\cosh\left(-\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}\\
&+(1-\alpha)\sinh\left(-\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\},\\
\overline{x}(t) =&(\alpha-1)\sinh\left(-\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}\\
&+(1-\alpha)\cosh\left(-\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]\mathrm{d}s\right\}.
\end{aligned}$$

We note that the H-difference $2s\gamma\cosh\left(-\frac{2}{3}s^{1.5}\right)\ominus 2s\gamma\sinh\left(-\frac{2}{3}s^{1.5}\right)$ exists and we have the $q_{(1)}$-solution is $x(t) = \gamma\left(3e^{\frac{2}{3}t^{1.5}} - 2\right)$, that is shown in Figure 1.

For $t \in [0, 0.7]$, $\gamma \ominus \int_0^t -2s^{0.5}\gamma e^{\frac{2}{3}s^{1.5}}\mathrm{d}s$ exists, since $diam([\gamma]^\alpha) \geq diam\left(\int_0^t -2s^{0.5}[\gamma]^\alpha e^{\frac{2}{3}s^{1.5}}\mathrm{d}s\right)$. From Thoerem 3.4 we have the $q_{(2)}$-solution is $x(t) = \gamma\left(3e^{-\frac{2}{3}t^{1.5}} - 2\right)$. This solution is shown in Figure 2.

Example 5.3. *Consider the fuzzy linear conformable fractional differential equation*

$$\begin{cases} x^{(0.5)}(t) + tx(t) = 2t\gamma,\ t \geq 0,\\ x(0) = \gamma, \end{cases}$$

where $[\gamma]^\alpha = [\alpha - 1, 1 - \alpha]$.

From Theorem 4.3, we have the $q_{(1)}$-solution is $x(t) = \gamma\left(2 - e^{-\frac{2}{3}t^{1.5}}\right)$ that is shown in Figure 3. (It is easy to check that $x(t + \varepsilon t^{1-q}) \ominus x(t)$ and $x(t) \ominus x(t - \varepsilon t^{1-q})$ exist over $[0, 0.7]$.)

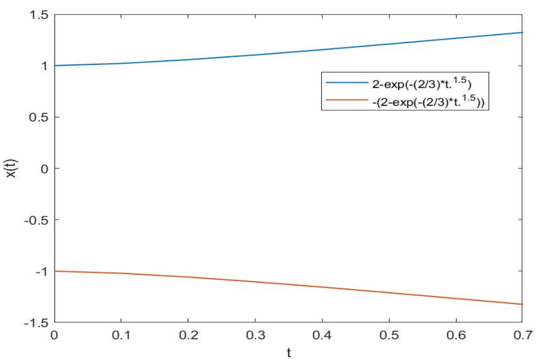

Figure 3: $q_{(1)}$-Solution of Examples 5.3 and 5.4 using the $q_{(1)}$-differentiability.

For finding the $q_{(2)}$-solution, the solution of the corresponding ODEs system is

$$\begin{aligned}\underline{x}(t) =&(\alpha-1)\cosh\left(-\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]ds\right\}\\&+(1-\alpha)\sinh\left(-\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]ds\right\},\\ \overline{x}(t) =&(\alpha-1)\sinh\left(-\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]ds\right\}\\&+(1-\alpha)\cosh\left(-\frac{2}{3}t^{1.5}\right)\left\{1-2\int_0^t s^{0.5}\left[\cosh\left(-\frac{2}{3}s^{1.5}\right)+\sinh\left(-\frac{2}{3}s^{1.5}\right)\right]ds\right\}.\end{aligned}$$

For $t\in[0,0.7]$, $\gamma\ominus\int_0^t s^{-0.5}\left[2s\gamma\sinh\left(-\frac{2}{3}s^{1.5}\right)-2s\gamma\cosh\left(-\frac{2}{3}s^{1.5}\right)\right]ds$ exists, since

diam$([\gamma]^\alpha)\geq$ *diam*$\left(\int_0^t s^{-0.5}\left[2s[\gamma]^\alpha\sinh\left(-\frac{2}{3}s^{1.5}\right)-2s[\gamma]^\alpha\cosh\left(-\frac{2}{3}s^{1.5}\right)\right]ds\right)$. So we have the $q_{(2)}$-solution $x(t)=\gamma\left(2-e^{\frac{2}{3}t^{1.5}}\right)$. This solution is shown in Figure 4. (It is easy to check that $x(t)\ominus x(t+\varepsilon t^{1-q})$ and $x(t-\varepsilon t^{1-q})\ominus x(t)$ exist over $[0,0.7]$.)

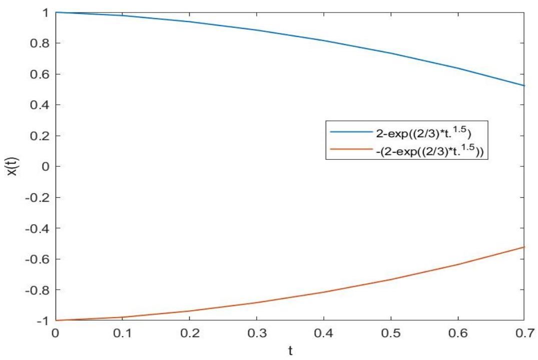

Figure 4: $q_{(2)}$-Solution of Examples 5.3 and 5.4 using the $q_{(2)}$-differentiability.

Example 5.4. *Consider the fuzzy linear conformable fractional differential equation*

$$\begin{cases}x^{(0.5)}(t)-tx(t)=2t\gamma,\ t\geq 0,\\ x(0)=\gamma,\end{cases}$$

where $[\gamma]^\alpha=[\alpha-1,1-\alpha]$.

For finding the $q_{(1)}$-solution, the solution of the corresponding ODEs system is

$$\begin{aligned}\underline{x}(t) =&(\alpha-1)\cosh\left(\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]ds\right\}\\&+(1-\alpha)\sinh\left(\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]ds\right\},\\ \overline{x}(t) =&(\alpha-1)\sinh\left(\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]ds\right\}\\&+(1-\alpha)\cosh\left(\frac{2}{3}t^{1.5}\right)\left\{1+2\int_0^t s^{0.5}\left[\cosh\left(\frac{2}{3}s^{1.5}\right)+\sinh\left(\frac{2}{3}s^{1.5}\right)\right]ds\right\}.\end{aligned}$$

We note that the H-difference $2s\gamma\cosh\left(\frac{2}{3}s^{1.5}\right)\ominus 2s\gamma\sinh\left(\frac{2}{3}s^{1.5}\right)$ exists and we have the $q_{(1)}$-solution is $x(t) = \gamma\left(2-e^{-\frac{2}{3}t^{1.5}}\right)$, that is shown in Figure 3.

For $t\in[0,0.7]$, $\gamma\ominus\int_0^t -2s^{0.5}\gamma e^{-\frac{2}{3}s^{1.5}}\mathrm{d}s$ exists, since $diam([\gamma]^\alpha)\geq diam\left(\int_0^t -2s^{0.5}[\gamma]^\alpha e^{-\frac{2}{3}s^{1.5}}\mathrm{d}s\right)$. From Theorem 4.8 we have the $q_{(2)}$-solution is $x(t) = \gamma\left(2-e^{\frac{2}{3}t^{1.5}}\right)$. That is shown in Figure 4. (It is easy to check that $x(t)\ominus x(t+\varepsilon t^{1-q})$ and $x(t-\varepsilon t^{1-q})\ominus x(t)$ exist over $[0,0.7]$.)

6. Conclusion

A basic procedure for finding explicit $q_{(1)}$-differentiable and $q_{(2)}$-differentiable solutions of linear fuzzy conformable fractional differential equations with constant function coefficients is established. From the upper and lower branches, the linear fuzzy conformable fractional differential equations are transformed into linear ordinary differential equations, and the variation of constant formula is used to derive $q_{(1)}$-differentiable solutions and $q_{(2)}$-differentiable solutions.

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