Warped Product Pointwise Hemi-Slant Submanifolds of a Para-Kaehler Manifold

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Abstract. In this paper, we introduce pointwise hemi-slant submanifolds of para-Kaehler manifolds. Using this notion, we investigate the geometry of warped product pointwise hemi-slant submanifolds. We provide some non-trivial examples of such submanifolds.

1. Introduction

The notion of slant submanifolds was introduced by Chen in [10], and the first results on slant submanifolds were collected in his book [11]. Since then, this subject has been studied extensively by many geometers during the last two and half decades. Also, the study of slant submanifolds in a pseudo-Riemannian manifold has been initiated: Chen and Mihai classified slant surfaces in Lorentzian complex space forms in [12]. Arslan et. al defined slant submanifolds of a neutral Kaehler manifold in [6], while Alegre studied slant submanifolds of Lorentzian Sasakian and para-Sasakian manifolds in [1]. Recently, slant, bi-slant and quasi bi-slant submanifolds of (para)-Hermitian manifolds have been defined in [2, 3, 5]. As an extension of slant submanifolds, Etayo [17] defined the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds.

On the other hand, Bishop and O’Neill started the concept of warped product which is one of the most effective generalizations of semi-Riemannian manifold. The notion of warped product has recognized various significant contributions in differential geometry as well as in physics, particularly in general theory of relativity [13, 26]. Since then, the study of warped product submanifolds has been investigated by many geometers (see, e.g., [4, 9, 15, 16, 18–21, 24, 25, 27–35] among many others, and for the most up-to-date overview of this subject, see [14]).

In this paper, we introduce pointwise hemi-slant submanifolds of para-Kaehler manifolds and using this notion, we investigate the geometry of warped product pointwise hemi-slant submanifolds of the form $N_{1\varphi} \times N_{2\perp}$ in a para-Kaehler manifold $\tilde{N}$, where $N_{2\perp}$ is a totally real submanifold and $N_{1\varphi}$ is a neutral proper pointwise slant submanifold of $\tilde{N}$ with slant function $\varphi$.

In the present paper, in section 2, we give preliminaries and definitions needed for this paper. In section 3, we define and study pointwise hemi-slant submanifolds of para-Kaehler manifolds. Then, we give some non-trivial examples of pointwise hemi-slant submanifolds and investigate the geometry of the leaves.
of distributions. In section 4, we prove some preparatory results and obtain a necessary and sufficient condition for the existence of a submanifold of the form \( N_{1p} \times N_{2\perp} \) to be locally warped product and locally semi-Riemannian product. Also, we give some examples illustrating such submanifolds. In section 5 we describe the warped product submanifolds \( N \) by giving geometric inequalities in term of second fundamental form and warping function \( \varphi \) for the \( N_{1p} \times N_{2\perp} \) of a para-Kaehler manifold.

2. Preliminaries

Let \((\tilde{N}, g)\) be an almost para-Hermitian manifold with almost para-complex structure \( P \) and a semi-Riemannian metric \( g \) such that

\[
P^\sharp Y_1 = Y_1, \quad g(\nabla_{Y_1} Y_2, Y_2) + g(Y_1, Y_2) = 0, \tag{1}
\]

for all \( Y_1, Y_2 \in \Gamma(T\tilde{N}), \) where \( \tilde{\nabla} \) denotes the Levi-Civita connection on \( \tilde{N} \) of the semi-Riemannian metric \( g. \)

If the para-complex structure \( P \) satisfies

\[
(\nabla_{Y_1} P)Y_2 = 0, \tag{2}
\]

for all \( Y_1, Y_2 \in \Gamma(T\tilde{N}), \) then \( \tilde{N} \) is called a para-Kaehler manifold([23]).

Now, let \( N \) be a semi-Riemannian submanifold of \((\tilde{N}, P, g)\) and we denote by the same symbol \( g \) the semi-Riemannian metric induced on \( N. \) Let \( \Gamma(TN) \) be the Lie algebra of vector fields in \( N \) and \( \Gamma(T^\perp N) \), the set of all vector fields normal to \( N. \) If \( \tilde{\nabla} \) be the induced Levi-Civita connection on \( N, \) then the Gauss and Weingarten formulas are given by:

\[
\tilde{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \sigma(Y_1, Y_2), \tag{3}
\]

\[
\tilde{\nabla}_{Y_1} Y_3 = -\mathcal{A} Y_2 Y_1 + \nabla^\perp_{Y_1} Y_3, \tag{4}
\]

for any \( Y_1, Y_2 \in \Gamma(TN) \) and \( Y_3 \in \Gamma(T^\perp N), \) where \( \nabla^\perp \) is the normal connection in the normal bundle \( T^\perp N \) and \( \mathcal{A} \) is the shape operator of \( N \) with respect to the normal vector \( Y_3. \) Also, \( \sigma : TN \times TN \rightarrow T^\perp N \) is the second fundamental form of \( N \) in \( \tilde{N}. \) Moreover \( \mathcal{A} Y_1 \) and \( \sigma \) are related by:

\[
g(\sigma(Y_1, Y_2), Y_3) = g(\mathcal{A} Y_3 Y_1, Y_2) \tag{5}
\]

for any \( Y_1, Y_2 \in \Gamma(TN) \) and \( Y_3 \in \Gamma(T^\perp N). \)

For any \( Y_1 \) tangent to \( N \) we write

\[
P Y_1 = \alpha Y_1 + \beta Y_1, \tag{6}
\]

where \( \alpha Y_1 \) and \( \beta Y_1 \) are the tangential and normal parts of \( PY_1, \) respectively.

Also, for any \( Y_3 \in \Gamma(T^\perp N), \) we get

\[
P Y_3 = \alpha Y_3 + \beta Y_3, \tag{7}
\]

here \( \alpha Y_3 \) and \( \beta Y_3 \) are the tangential and normal parts of \( PY_3, \) respectively.

In [8], Chen and Garay introduced pointwise slant submanifold in a Kaehler manifold. Let \( N \) be a submanifold of a Kaehler manifold \((\hat{N}, P, g)\). Then the submanifold \( N \) is called pointwise slant submanifold if at each point \( p \in N, \) the slant angle \( \varphi(Y_1) \) between \( PY_1 \) and \( T_p\hat{N} \) is independent of the choice of the non-zero vector \( Y_1 \in T_pN. \) In this case, the slant angle gives rise to a real-valued function \( \varphi : TN - \{0\} \rightarrow \mathbb{R} \) which is called the slant function of the pointwise slant submanifold. If \( \alpha Y \) is the projection of \( PY_1 \) over \( N, \) they can be characterized as \( \alpha^2 = \mu l d. \)
We say that a semi-Riemannian submanifold $N$ of a para-Hermitian manifold $(\tilde{N}, P, g)$ is called a pointwise slant if for every non-lightlike $Y_1 \in \Gamma(TN)$, the quotient $g(\alpha Y_1, \alpha Y_1)/g(PY_1, PY_1)$ is non-constant. A submanifold is called invariant if it is a pointwise slant with slant function zero. It is called anti-invariant if $\alpha Y_1 = 0$ for all $Y_1 \in \Gamma(TN)$. In other cases, it is called a proper pointwise slant submanifolds.

**Definition 2.1.** Let $N$ be a proper pointwise slant submanifold of a para-Hermitian manifold $(\tilde{N}, P, g)$. We say that it is of

(i) type 1 if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, $\alpha Y_1$ is timelike (spacelike), and $\frac{\|\alpha Y_1\|}{\|PY_1\|} > 1$,

(ii) type 2 if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, $\alpha Y_1$ is timelike (spacelike), and $\frac{\|\alpha Y_1\|}{\|PY_1\|} < 1$.

The proof of the following result is the same as slant submanifolds (see [2] and [3]), therefore we omit its proof.

**Theorem 2.2.** Let $N$ be a semi-Riemannian submanifold of a para-Hermitian manifold $(\tilde{N}, P, g)$. Then,

(i) $N$ is a pointwise slant submanifold of type 1 if and only if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, $\alpha Y_1$ is timelike (spacelike), and there exists a function $\mu \in (1, \infty)$ such that

$$\alpha^2 Y_1 = \mu Y_1. \tag{8}$$

If $\varphi$ denotes the slant function of $N$ then $\mu = \cosh^2 \varphi$.

(ii) $N$ is a pointwise slant submanifold of type 2 if and only if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, $\alpha Y_1$ is timelike (spacelike), and there exists a function $\mu \in (0, 1)$ such that

$$\alpha^2 Y_1 = \mu Y_1. \tag{9}$$

If $\varphi$ denotes the slant function of $N$ then $\mu = \cos^2 \varphi$.

In every case, a real-valued function $\varphi$ is called the slant function of the proper pointwise slant submanifold. From the Theorem 2.2, we have:

**Corollary 2.3.** Let $D$ be a distribution on $N$. Then,

(i) $D$ is a proper pointwise slant of type 1 if and only if for any spacelike (timelike) vector field $Y_1 \in \Gamma(D)$, $\alpha Y_1$ is timelike (spacelike), and there exists a function $\mu \in (1, \infty)$ such that

$$(\alpha Q_\varphi)^2 Y_1 = \mu Y_1 \tag{10}$$

where $Q_\varphi$ denotes the orthogonal projection on $D$. Also, in this case $\mu = \cosh^2 \varphi$.

(ii) $D$ is a proper pointwise slant of type 2 if and only if for any spacelike (timelike) vector field $Y_1 \in \Gamma(D)$, $\alpha Y_1$ is timelike (spacelike), and there exists a function $\mu \in (0, 1)$ such that

$$(\alpha Q_\varphi)^2 Y_1 = \mu Y_1 \tag{11}$$

where $Q_\varphi$ denotes the orthogonal projection on $D$. Also, in this case $\mu = \cos^2 \varphi$.

In every case, a real-valued function $\varphi$ is called the slant function of the proper pointwise slant distribution.

Let us point out that for both proper pointwise slant distributions of type 1 and 2, if $Y_1$ is a spacelike tangent vector field, then $\alpha Y_1$ is a timelike tangent vector field. So, all type 1, and type 2 proper pointwise slant distributions are neutral.

Remember that a para-holomorphic distribution satisfies $PD = D$, so every para-holomorphic distribution is a pointwise slant distribution with slant function zero. It is called a totally real distribution if $PD \subseteq T^\perp N$, therefore every totally distribution is anti-invariant.

If $D$ is a para-holomorphic distribution, then $\|\alpha Y_1\| = \|PY_1\|$ for all $Y_1 \in \Gamma(D)$. If $D$ is a totally real distribution, then $\|\alpha Y_1\| = 0$, for all $Y_1 \in \Gamma(D)$.
3. Proper pointwise hemi-slant submanifolds

In this section we define and study proper pointwise hemi-slant submanifold of a para-Kaehler manifold \((\tilde{N}, P, g)\).

**Definition 3.1.** A semi-Riemannian submanifold \(N\) of a para-Hermitian \((\tilde{N}, P, g)\) is called a pointwise bi-slant submanifold if the tangent space admits a decomposition \(T_N = D_\tau \oplus D_\phi\) with both \(D_\tau\) and \(D_\phi\) pointwise slant distributions with slant functions \(\tau\) and \(\phi\).

It is called a pointwise semi-slant submanifold if the tangent space admits a decomposition \(T_N = D_\tau \oplus D_\phi\) with \(D_\tau\) a para-holomorphic distribution and \(D_\phi\) a proper pointwise slant distribution with slant function \(\phi\).

It is called a pointwise hemi-slant submanifold if the tangent space admits a decomposition \(T_N = D_\tau \oplus D_\phi\) with \(D_\tau\) a totally real distribution and \(D_\phi\) a proper pointwise slant distribution with slant function \(\phi\).

Note that given a pseudo-Euclidean space \(R_n^{2n}\) with coordinates \((x_1, ..., x_{2n})\) on \(R_n^{2n}\), we can naturally choose an almost paracomplex structure \(P\) on \(R_n^{2n}\) as follows:

\[
P(\frac{\partial}{\partial x_{2i}}) = \frac{\partial}{\partial x_{2i-1}}, \quad P(\frac{\partial}{\partial x_{2i-1}}) = -\frac{\partial}{\partial x_{2i}}
\]

where \(i = 1, ..., n\). Let \(R_n^{2n}\) be a pseudo-Euclidean space of signature \((+, r, +, ..., +, -)\) with respect to the canonical basis \((\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{2n}})\).

Now, we can present some examples of proper pointwise hemi-slant submanifolds.

**Example 3.2.** Let \(N\) be a semi-Riemannian submanifold of \(R_n^{2n}\) defined by the immersion \(\phi : N \to R_n^{2n}\):

\[
\phi(u, v, t, s) = (\sin u, \cos v, \cos u, \cos v, t, k_1, k_2, s),
\]

such that \(u \neq v \neq 0\), for non-vanishing functions \(u\) and \(v\) on \(N\). Then \(N\) is a neutral pointwise hemi-slant submanifold of type 2 with neutral anti-invariant distribution \(D_\perp = \text{Span}\{Y_3 = \frac{\partial}{\partial x_1}, Y_4 = \frac{\partial}{\partial x_2}\}\) and the neutral pointwise slant distribution of type 2 \(D_\phi = \text{Span}\{Y_1 = \cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_2}, Y_2 = \cos v \frac{\partial}{\partial x_2} - \sin v \frac{\partial}{\partial x_1}\}\) with slant function \(\phi = u - v\).

**Example 3.3.** Let \(N\) be a semi-Riemannian submanifold of \(R_n^{2n}\) defined by the immersion \(\phi : N \to R_n^{2n}\):

\[
\phi(u, v, t, s) = (v, \sinh u, \cosh u, u, t, k_1, k_2, s),
\]

such that \(u > 2\) and \(v \neq 0\), for non-vanishing function \(u\) on \(N\). Then \(N\) is a neutral pointwise hemi-slant submanifold of type 1 with neutral anti-invariant distribution \(D_\perp = \text{Span}\{Y_3 = \frac{\partial}{\partial x_1}, Y_4 = \frac{\partial}{\partial x_2}\}\) and the neutral pointwise slant distribution of type 1 \(D_\phi = \text{Span}\{Y_1 = \cosh u \frac{\partial}{\partial x_1} + \sinh u \frac{\partial}{\partial x_2}, Y_2 = \frac{\partial}{\partial x_2}\}\) with slant function \(\phi = \cosh^{-1}(\sqrt{\frac{u}{\sqrt{2}}}u)\).

Let \(N\) be a proper pointwise slant hemi-slant submanifold of a para-Kaehler manifold \((\tilde{N}, P, g)\), and set the projections on the distributions \(D_\perp\) and \(D_\phi\) by \(Q_\perp\) and \(Q_\phi\), respectively. Then we can write

\[
Y_1 = Q_\perp Y_1 + Q_\phi Y_1
\]

for any spacelike (timelike) vector field \(Y_1 \in \Gamma(TN)\). Applying \(P\) to equation (12) and using (6), we get

\[
PY_1 = \beta Q_\perp Y_1 + \alpha Q_\perp Y_1 + \beta Q_\phi Y_1. \tag{13}
\]

From (13), we have

\[
\beta Q_\perp Y_1 \in \Gamma(D_\perp), \quad \alpha Q_\perp Y_1 = 0, \tag{14}
\]
for any spacelike (timelike) vector field \( Y \). Using (6) in (13), we obtain
\[
\alpha Y_1 = \alpha Q_\varphi Y_1, \quad \beta Y_1 = \beta Q_\varphi Y_1 + \beta Q_\varphi Y_1
\]
for any spacelike (timelike) vector field \( Y_1 \). Since \( \Gamma(D_\varphi) \) is a proper pointwise slant distribution, from Theorem 2.2 and Corollary 2.3, we conclude that
\[
\text{Proposition 3.5.}\quad \text{Let } N \text{ be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold.}
\]

In a similar way, we obtain:
\[
\phi \alpha Y_2 = 0 \quad \text{for any spacelike (timelike) vector field } Y_2 \in \Gamma(TN) \quad \text{orthogonal to } D.
\]

Moreover, if \( \varphi \) denotes the slant function of \( N \) then \( \mu = \cosh^2 \varphi \).

**Proof.** Let \( N \) be a proper pointwise hemi-slant submanifold of \( (\tilde{N}, P, g) \). By setting \( \mu = \cosh^2 \varphi \) and using (14) and (15), we obtain that \( D = D_\varphi \), which follows a and b. Conversely, (a) and (b) imply that \( TN = D \oplus D_\perp \).

Since \( \alpha(D) \subseteq D_\perp \), we received from (b) that \( D_\perp \) is a totally real distribution.

In a similar way, we obtain:

**Proposition 3.6.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \( (\tilde{N}, P, g) \). Then \( N \) is a proper pointwise hemi-slant submanifold if and only if \( \alpha \in (1, \infty) \) and a distribution of type 2 \( D \) on \( N \) such that
\[
\alpha Y_1 = \alpha Q_\varphi Y_1, \quad \beta Y_1 = \beta Q_\varphi Y_1 + \beta Q_\varphi Y_1
\]
for any spacelike (timelike) vector field \( Y_1 \). Since \( \alpha(D) \subseteq D_\perp \), we received from (b) that \( D_\perp \) is a totally real distribution.

From the above Propositions, we have:

**Corollary 3.7.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \( (\tilde{N}, P, g) \). Then \( D_\varphi \) is a proper pointwise slant distribution of
\[
\text{type 1 if and only if } g(\alpha Y_1, \alpha Y_2) = -\cosh^2 \varphi g(Y_1, Y_2), \quad g(\beta Y_1, \beta Y_2) = \sinh^2 \varphi g(Y_1, Y_2),
\]
\[
\text{type 2 if and only if } g(\alpha Y_1, \alpha Y_2) = -\cos^2 \varphi g(Y_1, Y_2), \quad g(\beta Y_1, \beta Y_2) = -\sin^2 \varphi g(Y_1, Y_2)
\]
for all spacelike (timelike) vector fields \( Y_1, Y_2 \in \Gamma(D_\varphi) \).

Using (1), (6) and (7), the Propositions 3.4 and 3.5, we get:

**Lemma 3.8.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \( (\tilde{N}, P, g) \). Then \( D_\varphi \) is a proper pointwise slant distribution of
\[
\text{type 1 if and only if } (a) \alpha Y_1 = (-\sinh^2 \varphi) Y_1, \quad (b) \beta Y_1 = -\beta a Y_1,
\]
\[
\text{type 2 if and only if } (a) \alpha Y_1 = (\sin^2 \varphi) Y_1, \quad (b) \beta Y_1 = -\beta a Y_1,
\]
for all spacelike (timelike) vector field \( Y_1 \in \Gamma(D_\varphi) \).

Now we examine the conditions for integrability and totally geodesic foliation of distributions associated with the definition of proper pointwise hemi-slant submanifolds of a para-Kaehler manifold.

**Theorem 3.9.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \( (\tilde{N}, P, g) \). Then the totally real distribution \( D_\perp \) is integrable.
Proof. It is known that is a para-Kaehler manifold, then \( dF = 0 \), where \( d \) is exterior derivative and \( F \) is the fundamental 2-form defined \( F(Y_1, Y_2) = g(Y_1, PY_2) \) for any spacelike (timelike) vector fields \( Y_1, Y_2 \in \Gamma(\mathcal{T}N) \) (see [23]). Since \( F \) is closed \( (dF = 0) \), for any spacelike (timelike) vector fields \( Y_1, Y_2, Y_3 \in \Gamma(\mathcal{D}_p) \) we have

\[
3dF(aY_1, Y_2, Y_3) = aY_1F(Y_2, Y_3) - Y_2F(aY_1, Y_3) + Y_3F(aY_1, Y_2)
= -\mathcal{F}([aY_1, Y_2], Y_3) + \mathcal{F}([aY_1, Y_3], Y_2) - \mathcal{F}([Y_2, Y_3], aY_1) = 0.
\]

Since \( \mathcal{D}_1 \) and \( \mathcal{D}_p \) are orthogonal and \( \mathcal{D}_1 \) is anti-invariant, using Proposition 3.4 and (6) we get

\[
Y_2g(\beta\alpha Y_1, Y_3) - \cosh^2 \varphi g([Y_2, Y_3], Y_1) - g([Y_2, Y_3], \beta\alpha Y_1) = 0.
\]

Since \( [Y_2, Y_3] \in \Gamma(\mathcal{T}N) \) and \( \beta\alpha Y_1 \in \Gamma(\mathcal{T}N) \), we obtain

\[
\cosh^2 \varphi g([Y_2, Y_3], Y_1) = 0.
\]

Since \( N \) is a proper pointwise hemi-slant submanifold and \( Y_1, Y_2, Y_3 \) are all non-zero, we have \([Y_2, Y_3] \in \Gamma(\mathcal{D}_p) \). \( \square \)

Note that the Theorem 3.8 holds for proper pointwise slant submanifold \( N_{sp} \) of type 2.

From the Theorem 3.8, we have:

**Corollary 3.9.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \((\tilde{N}, \mathcal{P}, g)\). Then the totally real distribution \( \mathcal{D}_1 \) is integrable if and only if for any spacelike (timelike) vector fields \( Y_1, Y_2 \in \Gamma(\mathcal{D}_1) \) the shape operator satisfies \( \mathcal{A}_{PY_2}Y_1 = \mathcal{A}_{PY_1}Y_2 \).

**Theorem 3.10.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \((\tilde{N}, \mathcal{P}, g)\). Then the totally real distribution \( \mathcal{D}_1 \) defines a totally geodesic foliation if and only if for every spacelike (timelike) vector fields \( Y_1 \in \Gamma(\mathcal{D}_1) \) and \( Y_3 \in \Gamma(\mathcal{D}_p) \), \( \mathcal{A}_{PY_1}Y_3 = \mathcal{A}_{PY_3}Y_1 \).

**Proof.** For any spacelike (timelike) vector fields \( Y_1, Y_2 \in \Gamma(\mathcal{D}_1) \) and \( Y_3 \in \Gamma(\mathcal{D}_p) \), using (1)-(7) we get

\[
\begin{aligned}
g(\nabla Y_1, Y_2, Y_3) &= -g(\nabla Y_1, PY_2, PY_3) \\
& = -g(\nabla Y_1, PY_2, \alpha Y_3) + g(\nabla Y_1, Y_2, \alpha\beta Y_3) \\
& + g(\nabla Y_1, Y_2, \beta\beta Y_3).
\end{aligned}
\]

From (4), (5) and Lemma 3.7(type 1), we obtain

\[
\begin{aligned}
g(\nabla Y_1, Y_2, Y_3) &= g(\mathcal{A}_{PY_2}Y_1, \alpha Y_3) - \sinh^2 \varphi g(\nabla Y_1, Y_2, Y_3) \\
& = -g(\mathcal{A}_{PY_3}Y_1, Y_2).
\end{aligned}
\]

Using (3), we get

\[
\cosh^2 \varphi g(\nabla Y_1, Y_2, Y_3) = g(\mathcal{A}_{PY_2}Y_3, Y_1) - g(\mathcal{A}_{PY_3}Y_2, Y_1).
\]

\( \square \)

Now, analogous to the proof of the Theorems 3.8 and 3.10 we give the following results for proper pointwise hemi-slant submanifolds.

**Theorem 3.11.** Let \( N \) be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold \((\tilde{N}, \mathcal{P}, g)\). Then the proper pointwise slant distribution \( \mathcal{D}_p \) is integrable if and only if

\[
g(\mathcal{A}_{PY_3}Y_1 - \mathcal{A}_{PY_1}Y_2, Y_3) = g(\mathcal{A}_{PY_1}Y_1 - \mathcal{A}_{PY_3}Y_2, Y_3)
\]

for every spacelike (timelike) vector fields \( Y_1 \in \Gamma(\mathcal{D}_1) \) and \( Y_2, Y_3 \in \Gamma(\mathcal{D}_p) \).
Warped products \( N_{14} \times_h N_{24} \) in para-Kaehler manifolds

Let \((N_1,g_1)\) and \((N_2,g_2)\) be two semi-Riemannian manifolds, let \(h : N_1 \rightarrow R_+\), and let \(\eta_1 : N_1 \times N_2 \rightarrow N_1\) and \(\eta_2 : N_1 \times N_2 \rightarrow N_2\) the projection maps given by \(\eta_1(r,s) = r\) and \(\eta_2(r,s) = s\) for all \((r,s) \in N_1 \times N_2\). The warped product[17] \( N = N_1 \times N_2 \) is the manifold \( N_1 \times N_2 \) equipped with the semi-Riemannian structure such that

\[
g(Y_1,Y_2) = g_1(\eta_1 Y_1, \eta_1 Y_2) + (h \circ \eta_2)^2 g_2(\eta_2 Y_1, \eta_2 Y_2)
\]

for every spacelike(timelike) vector fields \( Y_1, Y_2 \in \Gamma(TN) \), here \( \ast \) denotes the tangent map. The function \( h \) is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the manifold \( N \) is said to be trivial.

**Lemma 4.1.** (17) For spacelike(timelike) vector fields \( Y_1, Y_2 \in \Gamma(TN_1) \) and \( Y_3, Y_4 \in \Gamma(TN_2) \), we get on warped product manifold \( N = N_1 \times_h N_2 \) that

(a)\( V_1 Y_2 \in \Gamma(TN_1) \),

(b)\( V_1 Y_3 = V_2 Y_1 = (\frac{\partial}{\partial x_1}) Y_3 \),

(c)\( V_1 Y_4 = -\frac{\partial \psi(x_5 \ldots x_8)}{\partial x_4} \),

where \( V \) denotes the Levi-Civita connection on \( N \) and \( \nabla h \) is the gradient of \( h \) defined by \( g(\nabla h, Y_1) = Y_1 h \).

It is also important to note that for a warped product \( N = N_1 \times_h N_2, N_1 \) is totally geodesic and \( N_2 \) is totally umbilical in \( N \).

In this section, we investigate the existence of warped product submanifolds \( N_{14} \times_h N_{24} \) of para-Kaehler manifolds such that \( N_{14} \) is a proper pointwise slant submanifold and \( N_{24} \) is a totally real submanifold of \( N \). First, we are going to give some examples of a warped product pointwise hemi-slant submanifold of the form \( N_{14} \times_h N_{24} \).

**Example 4.2.** Consider a semi-Riemannian submanifold of \( R^8_+ \) with the cartesian coordinates \((x_1, \ldots, x_8)\) and the almost para-complex structure

\[
P(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{i-1}}, \quad P(\frac{\partial}{\partial x_{i-1}}) = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 4.
\]

Let \( R^8_+ \) be a semi-Euclidean space of signature \((+,\ldots,+,+,-\ldots,-)\) with respect to the canonical basis \((\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_8})\). Let \( N \) be defined by the immersion \( \psi \) as follows

\[
\psi(u,v,t) = (\sinh u, v, u, \cosh u, \cosh(t^3), a, \sinh(t^3), b)
\]

for any non-vanishing function \( u \) on \( N \), where \( a, b \) are constants and \( u > 1 \). Then the tangent space \( TN \) of \( N \) is spanned by the following vectors

\[
\psi_u = \cosh u \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sinh u \frac{\partial}{\partial x_4}, \quad \psi_v = \frac{\partial}{\partial x_2}.
\]
Thus, \( N \) is a warped product submanifold of the form \( N = D \phi \).

It is easy to see that \( P \chi \) and \( P \psi \) are integrable. If we denote the integral manifolds of \( D \phi \) and \( D \psi \) by \( N_{1\phi} \) and \( N_{2\phi} \), respectively, then the metric tensor of \( N \) is given by

\[
\frac{ds^2}{\sqrt{\cosh^2(2t^3)}} = 2du^2 - dv^2 + 9t^4 \cosh(2t^3) dt^2.
\]

Thus, \( N \) is a warped product submanifold of the form \( N = N_{1\phi} \times_\theta N_{2\phi} \) in \( R^8 \) with the warping function \( h = 3t^2 \sqrt{\cosh(2t^3)} \).

**Example 4.3.** Let \( N \) be an immersed semi-Riemannian submanifold of a para-Kaehler manifold \( \tilde{N} \) (as given in Example 4.2) defined by

\[
\psi(x, y, z) = (\sin x, \sin y, \cos x, \cos y, \cos \epsilon^z, a, \sin \epsilon^z, b),
\]

such that \( u \neq v \neq 0 \), for non-vanishing functions \( u \) and \( v \) on \( N \). Then the tangent space \( TN \) of \( N \) is spanned by the following vectors:

\[
\psi_x = \cos x \frac{\partial}{\partial x_1} - \sin x \frac{\partial}{\partial x_3}, \quad \psi_y = \cos y \frac{\partial}{\partial x_2} - \sin y \frac{\partial}{\partial x_4},
\]

\[
\psi_z = -\epsilon \sin(\epsilon^z) \frac{\partial}{\partial x_5} + \epsilon \cos(\epsilon^z) \frac{\partial}{\partial x_7}.
\]

Thus, we consider \( D_\phi = \text{span}\{\psi_x, \psi_y\} \) is a spacelike totally real distribution and \( D_\psi = \text{span}\{\psi_x, \psi_z\} \) is a neutral proper poitwise slant distribution of type 2 with slant function \( \varphi = u - v \). It is easy to observe that \( D_\phi \) and \( D_\psi \) are integrable. If we denote the integral manifolds of \( D_\phi \) and \( D_\psi \) by \( N_{1\phi} \) and \( N_{2\phi} \), respectively, then the metric tensor of \( N \) is given by

\[
ds^2 = dx^2 - dy^2 + \epsilon^z \, dz^2.\]

Hence, \( N \) is a 3-dimensional pointwise hemi-slant warped product submanifold of \( R^8 \) with the warping function \( h = \epsilon^z \).

**Example 4.4.** Let \( N \) be defined by the immersion \( \psi \) as follows

\[
\psi(x, y, z, t) = (\sinh x, \sinh y, \cosh x, \sinh(z + t), z, \sinh(z + t), t)
\]

for any non-vanishing functions \( x \) and \( y \) on \( N \). Then the tangent space \( TN \) of \( N \) is spanned by the following vectors

\[
\psi_x = \cosh x \frac{\partial}{\partial x_1} + \sinh x \frac{\partial}{\partial x_4}, \quad \psi_y = \cosh y \frac{\partial}{\partial x_2} + \sinh y \frac{\partial}{\partial x_3},
\]

\[
\psi_z = \cosh(z + t) \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} + \cosh(z + t) \frac{\partial}{\partial x_7}, \quad \psi_t = \cosh(z + t) \frac{\partial}{\partial x_5} + \cosh(z + t) \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}.
\]

It is easy to see that \( P \psi_x \) and \( P \psi_x \perp TN = \text{span}\{\psi_x, \psi_y, \psi_z, \psi_t\} \) and thus, we consider \( D_\phi = \text{span}\{\psi_x, \psi_y\} \) is a spacelike totally real distribution and \( D_\psi = \text{span}\{\psi_x, \psi_y\} \) is a neutral proper poitwise slant distribution of type 1 with slant function \( \alpha^2 = \cosh^2(x - y) \). It is easy to observe that \( D_\phi \) and \( D_\psi \) are integrable. If we denote the integral manifolds of \( D_\phi \) and \( D_\psi \) by \( N_{1\phi} \) and \( N_{2\phi} \), respectively, then the metric tensor of \( N \) is given by

\[
ds^2 = dx^2 - dy^2 + \cosh(2(z + t))(dz^2 + dt^2).
\]

Thus, \( N \) is a pointwise hemi-slant warped product submanifold of the form \( N = N_{1\phi} \times_\theta N_{2\phi} \) in \( R^8 \) with the warping function \( h = \sqrt{\cosh(2(z + t))} \).
Now we will consider warped product pointwise hemi-slant submanifolds \( N = N_{1p} \times h N_{2\perp} \) such that \( N_{1p} \) is a neutral proper pointwise slant submanifold and \( N_{2\perp} \) is a totally real submanifold of a para-Kaehler manifold \( \tilde{N} \).

**Lemma 4.5.** Let \( N = N_{1p} \times h N_{2\perp} \) be a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold \( \tilde{N} \). Then
\[
g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) = (-cosh^2 \varphi)(Y_1 \ln h)g(Y_3, Y_4) + g(\mathcal{A}_{\beta a}Y_1, Y_3, Y_4) \tag{19}
\]
for any spacelike(timelike) vector fields \( Y_1, Y_2 \in \Gamma(TN_{1p}) \) and \( Y_3, Y_4 \in \Gamma(TN_{2\perp}) \).

**Proof.** From (1)- (6) we obtain
\[
g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) = -g(\nabla_{Y_1}Y_1, Y_4) + g(\nabla_{Y_1}p\beta aY_1, Y_4). \tag{20}
\]
Using Lemma 4.1, Lemma 3.7(type 1) and (7) we get
\[
g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) = -(Y_1 \ln h)g(Y_3, Y_4) + g(\nabla_{Y_1}(-\sinh^2 \varphi)Y_1, Y_4) + g(\nabla_{Y_1}(-\beta aY_1), Y_4). \tag{21}
\]
From the fact that \( \varphi \) is slant function and using (4) we obtain
\[
g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) = -(Y_1 \ln h)g(Y_3, Y_4) - (\sinh^2 \varphi)g(\nabla_{Y_1}Y_1, Y_4) + g(\mathcal{A}_{\beta a Y_1}Y_3, Y_4),
\]
since \( g(Y_1, Y_4) = 0 \). Using (3) we have
\[
g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) = (-cosh^2)(Y_1 \ln h)g(Y_3, Y_4) + g(\mathcal{A}_{\beta a Y_1}Y_3, Y_4).
\]

\[\square\]

**Theorem 4.6.** Let \( N \) be a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold \( \tilde{N} \). Then \( N \) is locally isometric to pointwise hemi-slant warped product submanifold of the form \( N = N_{1p} \times h N_{2\perp} \) if and only if the shape operator of \( N \) satisfies
\[
\mathcal{A}_{PY_1}Y_1 - \mathcal{A}_{\beta a Y_1}Y_4 = (-cosh^2 \varphi)(Y_1 \ln h)Y_4, \tag{21}
\]
for some function \( \tau \) on \( N \) such that \( Y_3(\tau) = 0 \), where spacelike(timelike) vector fields \( Y_1, Y_2 \in \Gamma(D_{h}) \) and \( Y_3, Y_4 \in \Gamma(D_{\perp}) \).

**Proof.** Let us consider that \( N \) is a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold \( \tilde{N} \). Then, Lemma 4.5, we have (21). We know that \( h \) is a function on \( N_2 \), therefore setting \( \tau = \ln h \) implies that \( Y_3(\tau) = 0 \). Conversely we assume that \( N \) is a pointwise hemi-slant submanifold of \( \tilde{N} \) such that (21) holds. Taking the inner product of (21) with \( Y_2 \), we can say from Theorem 3.12 that the integral manifold \( N_{1p} \) of \( D_{h} \) is totally geodesic foliation in \( N \). Thus, by Corollary 3.9 the distribution \( D_{\perp} \) is integrable if and only if
\[
g(\mathcal{A}_{PY_1}a_1Y_4, a_1Y_1) = g(\mathcal{A}_{PY_1}a_1Y_1, Y_3) \tag{22}
\]
for any spacelike(timelike) vector fields \( Y_1, Y_2 \in \Gamma(D_{h}) \) and \( Y_3, Y_4 \in \Gamma(D_{\perp}) \). From (19) and (22) we obtain
\[
g(\mathcal{A}_{PY_1}a_1Y_4, a_1Y_1) = (-cosh^2 \varphi)g(\nabla_{Y_1}Y_1, Y_4) + g(\mathcal{A}_{\beta a Y_1}Y_3, Y_4) \tag{23}
\]
on the other hand, taking the inner product of (21) with \( Y_3 \) we obtain
\[
g(\mathcal{A}_{PY_1}a_1Y_1 - \mathcal{A}_{\beta a Y_1}Y_4, Y_3) = g((-cosh^2 \varphi)(Y_1 \ln h)Y_4, Y_3). \tag{24}
\]
Using type 1 (17), (3)-(5) and the fact that

\[ -g(\sigma_\perp(Y_3, Y_4), Y_1) = g(Y_1(\tau)Y_3, Y_4) = g(Y_3, Y_4)g(\nabla \tau, Y_1). \]

Thus \( \sigma_\perp(Y_3, Y_4) = g(Y_3, Y_4)(-\nabla \tau) \), here \( \sigma_\perp \) is a second fundamental form of \( D_\perp \) in \( N \) and \( \nabla \tau \) is a gradient of \( \tau = \ln h \). Hence the integrable manifold \( N_2 \) of \( D_\perp \) is totally umbilical submanifold in \( N \) and its mean curvature is non-zero and parallel and \( Y_3(\tau) = 0 \) for every spacelike (timelike) vector field \( Y_3 \in \Gamma(D_\perp) \). Therefore, from Theorem 1.2 ([22], page 211), we deduce that \( N \) is a pointwise hemi-slant warped product submanifold of \( \bar{N} \). \( \square \)

Now we maintain a necessary and sufficient condition for a warped product submanifold of the form \( N = N_{1p} \times N_{2\perp} \) to be a semi-Riemannian product.

**Theorem 4.7.** A pointwise hemi-slant warped product submanifold of the form \( N = N_{1p} \times N_{2\perp} \) of a para-Kaehler manifold \( \bar{N} \) is simply a locally semi-Riemannian product if and only if the shape operator satisfies \( \mathcal{A}_{\beta\alpha}Y_1 = 0 \) for every spacelike (timelike) vector field \( Y_3 \in \Gamma(N_{1p}) \) and \( Y_1, Y_2 \in \Gamma(N_{2\perp}) \).

**Proof.** For all spacelike (timelike) vector fields \( Y_3 \in \Gamma(N_{1p}) \) and \( Y_1, Y_2 \in \Gamma(N_{2\perp}) \), using (1)-(3) we have

\[ g(\nabla Y_1 Y_3, Y_2) = -g(\nabla Y_1 PY_3, PY_2). \]

From (1),(2) and (6) we get

\[ g(Y_1, Y_3, Y_2) = g(\nabla Y_1, a^2 Y_3, Y_2) + g(\nabla Y_1, \beta a Y_3, Y_2) - g(\nabla Y_1, \beta Y_3, PY_2). \]

Using type 1 (17), (3)-(5) and the fact that \( g(Y_2, Y_3) = 0 \), we obtain

\[ g(\nabla Y_1, Y_3, Y_2) = (\cosh^2 \varphi)g(\nabla Y_1, Y_3, Y_2) - g(\sigma(Y_1 Y_2), \beta a Y_3) - g(\nabla Y_1^\perp, \beta Y_3, PY_2). \]  (25)

Hence, from Lemma 4.1, we get

\[ (\sinh^2 \varphi)(Y_3 \ln h)g(Y_1, Y_2) = g(\sigma(Y_1 Y_2), \beta a Y_3) + g(\nabla Y_1^\perp, \beta Y_3, PY_2). \]  (26)

Interchanging \( Y_1 \) and \( Y_2 \) in (26) and then, subtracting from (26), we have:

\[ g(\nabla Y_1^\perp, \beta Y_3, PY_2) = g(\nabla Y_1^\perp, \beta Y_3, PY_1). \]  (27)

Furthermore from (1),(3),(4) and (6) we obtain

\[ g(\nabla Y_1^\perp, \beta Y_3, PY_2) = -(Y_3 \ln h)g(Y_1, Y_2) - g(\nabla Y_1, a Y_3, PY_2). \]  (28)

Again by interchanging \( Y_1 \) and \( Y_2 \) in (28) we conclude that (27) holds if and only if

\[ g(\nabla Y_1, a Y_3, PY_2) = -g(\nabla Y_1, PY_2, a Y_3) = 0. \]  (29)

Using type 1,(17) and (1)-(6) we have

\[ (-\cosh^2 \varphi)(Y_3 \ln h)g(Y_1, Y_2) + g(\sigma(Y_1 Y_2), \beta a Y_3) = 0. \]  (30)

Hence, from (30) we can say that \( h \) is constant if and only if \( g(\sigma(Y_1 Y_2), \beta a Y_3) = 0 \), since \( N_{1p} \) is proper pointwise slant submanifold and \( Y_3 \) is non-zero spacelike (timelike) vector field. \( \square \)

We say that a hemi-slant submanifold is mixed geodesic if

\[ \sigma(Y_1, Y_3) = 0 \]  (31)

for all spacelike (timelike) vector fields \( Y_1 \in \Gamma(D\varphi) \) and \( Y_3 \in \Gamma(D\perp) \).
Lemma 4.8. For a mixed geodesic pointwise hemi-slant warped product submanifold $N = N_{1p} \times_b N_{2\perp}$ of a para-Kaehler manifold $\tilde{N}$, we obtain
\[
\begin{align*}
g(\alpha(Y_1, Y_2), PY_3) &= 0 \\
(\alpha_1 \ln h) g(\sigma(Y_3, Y_4) = g(\sigma(Y_3, Y_4), \beta Y_1) \\
\end{align*}
\]
for spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(N_{1p})$ and $Y_3, Y_4 \in \Gamma(N_{2\perp})$.

Proof. From (1) and (2) we obtain $g(\sigma(Y_1, Y_2), PY_3) = -g(\tilde{\nabla}_Y Y_2, Y_3)$. From here, $g(\sigma(Y_1, Y_2), Y_3) = g(\tilde{\nabla}_Y Y_2, Y_3) = 0$. Using (6), we have $g(\sigma(Y_1, Y_2), Y_3) = g(\tilde{\nabla}_Y Y_2, Y_3) + g(\sigma(Y_1, Y_3), \beta Y_2)$. Thus from Lemma 4.1 we get (32).

In a similar way, we have (33). □

Note that the Lemma 4.8 holds for proper pointwise slant submanifold $N_{1p}$ of type 2.

5. An optimal inequality

We establish general sharp geometric inequality for proper pointwise hemi-slant warped product submanifolds of the form $N_{1p} \times_b N_{2\perp}$ of a para-Kaehler manifold $(\tilde{N}, P, g)$.

Let $x \in N$ and $\{E_1, ..., E_m, \tilde{E}_1, ..., \tilde{E}_n, PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ be an orthonormal basis of the tangent space $T_x N$ such that $\{E_1, ..., E_m, \tilde{E}_1, ..., \tilde{E}_n\}$ are tangent to $N$ at $x$ and $\{PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ are normal to $\tilde{N}$, and thus $T_x \tilde{N} = T_x N \oplus T_x^\perp N$. Now, we can take $\{E_1, ..., E_m, \tilde{E}_1, ..., \tilde{E}_n\}$ in such a way that $\{E_1, ..., E_m\}$ form an orthonormal basis of $\mathcal{D}_\perp$ and $\{\tilde{E}_1, ..., \tilde{E}_n\}$ form an orthonormal basis of $\mathcal{D}_\parallel$, where $\dim \mathcal{D}_\perp = m$ and $\dim \mathcal{D}_\parallel = n$. We can take $\{PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ in such a way that $\{PE_1, ..., PE_m\}$ form an orthonormal frame of $P(\mathcal{D}_\perp)$ and $\{\tilde{E}_1, ..., \tilde{E}_n\}$ form an orthonormal frame of $\beta(\mathcal{D}_\parallel)$. Since the metric on $N_{1p}$ of a warped product $N_{1p} \times_b N_{2\perp}$ is neutral, it is even-dimensional([9]). Thus $n = 2p$. Then, we can choose a orthonormal frames $\{\tilde{E}_1, ..., \tilde{E}_{2p}\}$ of $\mathcal{D}_\parallel$ and $\{E_1, ..., E_{2p}\}$ of $\beta(\mathcal{D}_\parallel)$ in such a way that
\[
\begin{align*}
\tilde{E}_1 &= \sech \varphi \tilde{E}_1, ..., \tilde{E}_{2p} = \sech \varphi \tilde{E}_{2p-1}, \text{ (type1)} \\
\tilde{E}_1 &= \csch \varphi \tilde{E}_1, ..., \tilde{E}_{2p} = \csch \varphi \tilde{E}_{2p}, \text{ (type1)}
\end{align*}
\]
where $\varphi$ is the slant function. We note that such an orthonormal frame is called an adapted frame ([2]).

Let us consider
\begin{itemize}
  \item on $\mathcal{D}_\perp$ : an orthonormal basis $\{E_i\}_{i=1, ..., m}$ where $m = \text{boy} \mathcal{D}_\perp$; moreover, one can suppose that $e_i = g(E_i, E_i) = 1$.
  \item on $P(\mathcal{D}_\perp)$ : an orthonormal basis $\{PE_i\}_{i=1, ..., m}$ where $m = \text{boy} P(\mathcal{D}_\perp)$ and $e_i' = g(P E_i, E_i) = -1$.
  \item on $\mathcal{D}_\parallel$ : an orthonormal basis $\{\tilde{E}_i\}_{i=1, ..., n}$ where $n = \text{boy} \mathcal{D}_\parallel$ and $\tilde{e}_i = g(\tilde{E}_i, \tilde{E}_i) = \mp 1$.
  \item on $\beta(\mathcal{D}_\parallel)$ : an orthonormal basis $\{\tilde{E}_i\}_{i=1, ..., n}$, where $n = \text{boy} \beta(\mathcal{D}_\parallel)$ and $\tilde{e}_i = g(\tilde{E}_i, \tilde{E}_i) = \mp 1$.
\end{itemize}

Theorem 5.1. Let $N = N_{1p} \times_b N_{2\perp}$ be a mixed geodesic warped product submanifold of a para-Kaehler manifold $\tilde{N}$ such that $N_{1p}$ is a $n$–dimensional neutral proper pointwise slant submanifold and $N_{2\perp}$ is a $m$–dimensional totally real submanifold of $N$. Suppose that $N_{2\perp}$ is spacelike. Then, the squared norm of the second fundamental form $\|\sigma\|^2$ of $N$ satisfies
\[
\|\sigma\|^2 \leq m \coth^2 \varphi \|\nabla \ln h\|^2,
\]
where $\nabla \ln h$ is the gradient of $\ln h$.

Proof. Since $\|\sigma\|^2 = \|\sigma(\mathcal{D}_\parallel, \mathcal{D}_\parallel)\|^2 + 2\|\sigma(\mathcal{D}_\parallel, \mathcal{D}_\perp)\|^2 + \|\sigma(\mathcal{D}_\perp, \mathcal{D}_\perp)\|^2$, if $N$ is mixed geodesic we obtain
\[
\|\sigma\|^2 = \|\sigma(\mathcal{D}_\parallel, \mathcal{D}_\parallel)\|^2 + \|\sigma(\mathcal{D}_\perp, \mathcal{D}_\perp)\|^2.
\]
The first factor of the right hand side of (35) can be written as
\[ ||\sigma(D_{\psi}, D_{\phi})||^2 = \sum_{r=1}^{2^{p+m}} \sum_{c,d=1}^{2^p} g(\sigma(\mathcal{E}_r, \mathcal{E}_d), E_r)^2. \]

Using the adapted frame, we have
\[ ||\sigma(D_{\psi}, D_{\phi})||^2 = \sum_{i=1}^{m} \sum_{c,d=1}^{2^p} g(\sigma(\mathcal{E}_c, \mathcal{E}_d), PE_i)^2 + \sum_{s=1}^{2^p} \sum_{c,d=1}^{2^p} g(\sigma(\mathcal{E}_c, \mathcal{E}_d), csch\phi\beta\mathcal{E}_s)^2. \] (36)

From (32), we get
\[ ||\sigma(D_{\psi}, D_{\phi})||^2 = \sum_{s=1}^{2^p} \sum_{c,d=1}^{2^p} g(\sigma(\mathcal{E}_c, \mathcal{E}_d), csch\phi\beta\mathcal{E}_s)^2. \] (37)

On the other hand we can write the second factor of the right side of (35) as
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), E_i)^2. \]

Using the adapted frame we arrive at
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + \sum_{c=1}^{2^p} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), csch\phi\beta\mathcal{E}_c)^2. \] (38)

From (33), we get
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + m \sum_{c=1}^{2^p} csch^2\varphi(a\mathcal{E}_c \ln h)^2. \] (39)

Further we can write (39) as
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + m(csch^2\varphi(a\mathcal{E}_1 \ln h)^2
\hspace{1cm} + csch^2\varphi(a\mathcal{E}_2 \ln h)^2 + \ldots + csch^2\varphi(a\mathcal{E}_{2^p} \ln h)^2). \] (40)

From (40) and using the adapted frame, we have
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + m(coth^2\varphi(sech\phi\alpha\mathcal{E}_1 \ln h)^2
\hspace{1cm} + csch^2\varphi(sech\phi\alpha\mathcal{E}_1 \ln h)^2 + coth^2\varphi(sech\phi\alpha\mathcal{E}_2 \ln h)^2
\hspace{1cm} + csch^2\varphi(sech\phi\alpha\mathcal{E}_2 \ln h)^2 + \ldots + coth^2\varphi(sech\phi\alpha\mathcal{E}_{2^p-1} \ln h)^2
\hspace{1cm} + csch^2\varphi(sech\phi\alpha\mathcal{E}_{2^p-1} \ln h)^2). \]

Using the Proposition 3.4, we obtain
\[ ||\sigma(D_{\perp}, D_{\perp})||^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + m \sum_{c=1}^{2^p} (coth^2(\mathcal{E}_{2^p-1} \ln h)^2 + \mathcal{E}_{2^p-1} \ln h)^2
\hspace{1cm} \sum_{k=1}^{m} \sum_{i,j=1}^{m} g(\sigma(E_i, E_j), PE_k)^2 + m \coth^2 ||\mathcal{V}(\ln h)||^2. \] (41)
From (35), (37) and (41) we obtain (34). If the equality sign of (34) holds identically, then \( N_{1\varphi} \) is totally geodesic and \( N_{2\perp} \) a totally umbilical submanifold in \( N \).

**Remark 5.2.** If the manifold \( N_{2\perp} \) of Theorem 5.1 is timelike, then (34) shall be replaced by

\[
\|\sigma\|^2 \geq m \coth^2 \varphi \|\nabla (\ln h)\|^2. 
\]

(42)

In a similar way, for proper pointwise slant submanifold \( N_{1\varphi} \) of type 2, we obtain the following result:

**Theorem 5.3.** Let \( N = N_{1\varphi} \times_h N_{2\perp} \) be a mixed geodesic warped product submanifold of a para-Kaehler manifold \( \tilde{N} \) such that \( N_{1\varphi} \) is an \( n \)-dimensional neutral proper pointwise slant submanifold and \( N_{2\perp} \) is a \( m \)-dimensional totally real submanifold of \( N \). Suppose that \( N_{2\perp} \) is spacelike (respectively, timelike). Then, the squared norm of the second fundamental form \( \|\sigma\|^2 \) of \( N \) satisfies

\[
\|\sigma\|^2 \leq m \cot^2 \varphi \|\nabla (\ln h)\|^2 \quad \text{(respectively, } \|\sigma\|^2 \geq m \cot^2 \varphi \|\nabla (\ln h)\|^2) 
\]

(43)

where \( \nabla (\ln h) \) is the gradient of \( \ln h \).

**References**