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Interpolating Sesqui Harmonic Slant Curve in Generalized Sasakian Space Form

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Abstract. In this article, we discuss interpolating sesqui-harmonic slant curves in generalized Sasakian space form and find the necessary and sufficient conditions for slant curves to be interpolating sesqui-harmonic. Next, we study sesqui minimal slant curves in generalized Sasakian space form. In particular we give an example of interpolating sesqui-harmonic slant curve in Sasakian space form. Our paper generalizes the results of the papers [8, 16, 20].

1. Introduction

Harmonic and biharmonic maps play a vital role in geometry, analysis and physics. They are one of the most studied variational problems in geometric analysis and in theoretical physics they appear as critical points of non-linear sigma model. A smooth map $\pi : (M^m, g) \rightarrow (N^n, h)$, where M^m and N^n are smooth Riemannian manifolds, is said to be a harmonic map if it is critical point of the energy functional [7]

$$E(\pi) = \frac{1}{2}\int_M |d\pi|^2 dv_g$$

or equivalently, if the tension field

$$\tau(\pi) = tr(\nabla d\pi) \tag{1}$$

vanishes.
Biharmonic maps are a higher order generalization of harmonic maps and is defined as the critical point of the bienergy functional for a map π between two Riemannian manifolds, which is given by [7]

$$E_2(\pi) = \int_M |\tau(\pi)|^2 dv_g,$$

and characterized by the vanishing of bi-tension field

$$\tau_2(\pi) = tr(\nabla^\pi \nabla^\pi - \nabla^\pi_\nabla)\tau(\pi) - tr(R^N(d\pi, \tau(\pi))d\pi) = 0.$$

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B. Y. Chen and S. Ishikawa [6], studied biharmonic curves and surfaces in semi-Euclidean space. Moreover, Chen and Ishikawa proved the non-existence of proper biharmonic surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$. This result was further extended to surfaces in 3-dimensional space forms of non-positive curvature by R. Caddeo, S. Montaldo and C. Oniciuc [5]. Biharmonic submanifolds in the 3-sphere $\mathbb{S}^3$ were classified by Caddeo, Montaldo and Oniciuc [4]. Jiang studied the first-variational and second-variational formulas for the bi-energy functional using Euler-Lagrange equation [14]. On the other hand, E. Loubeau and S. Montaldo [18] introduced the notion of biminimal immersion. Then D. Fetcu [8] obtained biharmonic Legendre curve in Sasakian space form. Further C. Ozgur and S. Guvenc [20] extended the results in generalized Sasakian space form.
In [3], Branding introduced an action functional for maps between Riemannian manifolds that interpolated between the actions for harmonic and biharmonic maps. A map $\pi : (M^m, g) \to (N^n, h)$ is said to be interpolating sesqui-harmonic if it is a critical point of $E_{\delta_1,\delta_2}(\pi)$ [3]

$$E_{\delta_1,\delta_2}(\pi) = \delta_1 \int_M |d\pi|^2 dv_g + \delta_2 \int_M |\tau(\pi)|^2 dv_g,$$

where $\delta_1, \delta_2 \in \mathbb{R}$. The interpolating sesqui-harmonic map equation is given as

$$\tau_{\delta_1,\delta_2}(\pi) = \delta_2\tau_2(\pi) - \delta_1\tau(\pi) = 0.$$

Further, a curve φ is called *Interpolating sesqui harmonic* if the following equation satisfied [3]

$$\tau_{\delta_1,\delta_2}(\pi) \equiv \delta_2(\nabla_T\nabla_T\nabla_T T) - \delta_2 R^N(T, \nabla_T T)T - \delta_1\nabla_T T = 0, \quad (2)$$

where $\delta_1, \delta_2 \in \mathbb{R}$ and $T = \varphi'$
In [16], F. Karaca et al. studied interpolating sesqui harmonic Legendre curves in Sasakian space forms. They found a necessary and sufficient condition for Legendre curves in Sasakian space forms to be interpolating sesqui harmonic and extended the results in generalized Sasakian space forms [15].
Motivated by the above study, we consider interpolating sesqui harmonic slant curves in generalized Sasakian space forms and find a neccessary and sufficient condition for a slant curve to be interpolating sesqui harmonic. Moreover, we define interpolating sesqui minimal curve and find the condition for a slant curve to be interpolating sesqui minimal. Finally we give an example to verify our result.

2. Preliminaries

Let N^{2n+1} with the structure (ϕ, ξ, η, g) be an almost contact metric manifold such that

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y in TN, where ϕ, ξ and η are the (1-1) tensor field, characteristic vector field and one form respectively.

If $d\eta(X, Y) = g(X, \phi Y)$ for all vector fields X, Y on $N^{2n+1}(\phi, \xi, \eta, g)$, then the almost contact metric manifold $N^{2n+1}(\phi, \xi, \eta, g)$ is called a contact metric manifold. The almost contact structure of N is said to normal if

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \quad (3)$$

A normal contact metric manifold is called a Sasakian manifold [2]. An almost contact metric manifold N^{2n+1} is called a Kenmotsu manifold [17] if

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

where ∇ is the Levi-Civita connection.
An almost contact metric manifold N^{2n+1} is called a cosymplectic manifold [19] if $\nabla\phi = 0$, which implies that $\nabla\xi = 0$.

The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian (resp. Kenmotsu, cosymplectic) manifold with constant ϕ-sectional curvature c is called a Sasakian (resp. Kenmotsu, cosymplectic) space form.
The notion of a generalized Sasakian space form was introduced by Alegre et al. in [1]. An almost contact metric manifold $(N^{2n+1}, \phi, \xi, \eta, g)$ such that the curvature tensor satisfies

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned} \tag{4}$$

for certain differentiable functions f_1, f_2 and f_3 on N^{2n+1} is called a generalized Sasakian space form [1] denoted by $N^{2n+1}(f_1, f_2, f_3)$ and such space form were studied in [10], [11].
A generalized Sasakian space form is classified as:

1. If $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$, then $N^{2n+1}(f_1, f_2, f_3)$ is a Sasakian space form $N^{2n+1}(c)$.
2. If $f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}$, then $N^{2n+1}(f_1, f_2, f_3)$ is a Kenmotsu space form $N^{2n+1}(c)$.
3. If $f_1 = f_2 = f_3 = \frac{c}{4}$, then $N^{2n+1}(f_1, f_2, f_3)$ is a cosymplectic space form $N^{2n+1}(c)$.

It is to be noted that interpolating sesqui-harmonic slant curve becomes interpolating sesqui harmonic Legendre curve for $\theta = \frac{\pi}{2}$ and biharmonic Legendre curve for $\theta = \frac{\pi}{2}$, $\delta_2 = 1$ and $\delta_1 = 0$.

3. Interpolating Sesqui-Harmonic slant curves in generalized Sasakian space form

Let $\varphi : I \to (N, g)$ be an arc length curve in an n-dimensional Riemannian manifold (N, g). If $\{E_1, E_2, \cdots, E_n\}$ is orthonormal vector field then the curve φ is called Frenet curve of osculating order r, $1 \leq r \leq n$ if [21]

$$\begin{aligned} T &= E_1 = \varphi', \\ \nabla_T E_1 &= k_1 E_2, \\ \nabla_T E_i &= -k_{i-1}E_{i-1} + k_i E_{i+1}, for \quad 2 \leq i \leq n-1, \\ \nabla_T E_n &= -k_{n-1}E_{n-1}, \end{aligned} \tag{5}$$

where $\{k_1, k_2, \cdots, k_{n-1}\}$ are curvature functions.

1. A geodesic is a Frenet curve of osculating order 1.
2. A circle is a Frenet curve of osculating order 2 if k_1 is a nonzero positive constant.
3. A helix of order r is a Frenet curve of osculating order $r \geq 3$ if $k_1, \cdots, k_{r-1}$ are nonzero positive constants.

Definition 3.1. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a unit speed curve in generalized Sasakian space form. Then φ is called a slant curve if there exist a constant angle θ such that $\eta(E_1) = \cos\theta$.*

Theorem 3.2. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ is a slant curve of osculating order r, $p = min\{r, 4\}$ in generalized Sasakian space form. Then φ is interpolating sesqui harmonic if and only if there exists δ_1, δ_2 such that*

(1) *$\phi T \perp E_2$ or $\phi T \in \{E_2, \cdots, E_n\}$,*
(2) *$\xi \perp E_2$ or $\xi \in \{E_2, \cdots, E_n\}$ and*
(3) *first p of the following equations are satisfied*

$$\begin{cases} -3\delta_2 k_1 k_1' = 0, \\ \delta_2[k_1'' - k_1^3 - k_1 k_2^2] + \delta_2 f_1 k_1 - \delta_2 f_3 \cos^2\theta k_1 + 3\delta_2 f_2 k_1 g(\phi T, E_2)^2 - \delta_2 f_3 k_1 \eta(E_2)^2 - \delta_1 k_1 = 0, \\ 2\delta_2 k_1' k_2 + \delta_2 k_1 k_2' + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_3) = 0, \\ \delta_2(k_1 k_2 k_3) + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_4) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_4) = 0. \end{cases}$$

Proof. Using (1) and (5) we have

$$\begin{aligned} \nabla_T E_1 &= k_1 E_2 = \tau(\varphi), && (6) \\ \nabla_T \nabla_T T &= -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3, && (7) \\ \nabla_T \nabla_T \nabla_T T &= (-3k_1 k_1')E_1 + (k_1'' - k_1^3 - k_1 k_2^2)E_2 + (2k_1' k_2 + k_1 k_2')E_3 && (8) \\ &+ (k_1 k_2 k_3)E_4. \end{aligned}$$

Next, making use of equation (4), we obtain

$$\begin{aligned} R(T, \nabla_T T)T &= -f_1 k_1 E_2 - 3f_2 k_1 g(\phi T, E_2)\phi T + f_3\{k_1 \cos^2\theta E_2 && (9) \\ &- k_1 \cos\theta \eta(E_2)E_1 + k_1 \eta(E_2)\xi\}. \end{aligned}$$

Further, reporting equations (6), (8) and (9) in (2), we get

$$\begin{aligned} \tau_{\delta_1,\delta_2}(\varphi) &= \delta_2[-3k_1 k_1']E_1 + [\delta_2(k_1'' - k_1^3 - k_1 k_2^2) \\ &- \delta_2 f_3 \cos^2\theta k_1 + \delta_2 f_1 k_1 - \delta_1 k_1]E_2 + \delta_2(2k_1' k_2 + k_1 k_2')E_3 + \delta_2(k_1 k_2 k_3)E_4 \\ &+ 3\delta_2 f_2 k_1 g(\phi T, E_2)\phi T - \delta_2 f_3 k_1 \eta(E_2)\xi. \end{aligned}$$

Taking inner product with E_1, E_2, E_3 and E_4 we obtain the result. □

Now we discuss the following five cases based on above theorem.

Case 1: $\phi T \perp E_2$ and $\xi \perp E_2$

Proposition 3.3. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ is a slant curve of osculating order r, $p = min\{r, 4\}$ in generalized Sasakian space form with $\phi T \perp E_2$ and $\xi \perp E_2$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{cases} k_1 = constant > 0, \\ k_1^2 + k_2^2 = f_1 - f_3 \cos^2\theta - \frac{\delta_1}{\delta_2}, \\ k_2 = constant, \\ k_2 k_3 = 0, \end{cases} \tag{10}$$

where $f_1 > f_3 \cos^2\theta + \frac{\delta_1}{\delta_2}$.

Proof. If $\phi T \perp E_2$ and $\xi \perp E_2$ then we have $g(\phi T, E_2) = 0$ and $g(E_2, \xi) = 0$. Now making use of Theorem 3.2 we obtain

$$\begin{cases} k_1 = constant > 0, \\ k_1'' - k_1^3 - k_1 k_2^2 + f_1 k_1 - f_3 \cos^2\theta k_1 - \frac{\delta_1}{\delta_2} k_1 = 0, \\ 2k_1' k_2 + k_1 k_2' = 0, \\ k_1 k_2 k_3 = 0. \end{cases} \tag{11}$$

By using $k_1 = constant > 0$ in last three equations of (11) we get the result. □

Theorem 3.4. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve of osculating order r in generalized Sasakian space form such that $\phi T \perp E_2$ and $\xi \perp E_2$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

1. *φ is of osculating order $r = 2$ and it is a circle with $k_1 = \sqrt{f_1 - f_3\cos^2(\theta) - \frac{\delta_1}{\delta_2}}$, where $f_1 - f_3\cos^2\theta > \frac{\delta_1}{\delta_2}$.*
2. *φ is of osculating order $r = 3$ and it is a helix with $k_1^2 + k_2^2 = f_1 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}$, where $f_1 > f_3\cos^2\theta + \frac{\delta_1}{\delta_2}$.*

Proof. Suppose $\phi T \perp E_2$ and $\xi \perp E_2$ then we have $g(\phi T, E_2) = 0$ and $g(\xi, E_2) = 0$.
If $f_1 > f_3\cos^2\theta + \frac{\delta_1}{\delta_2}$, then by Proposition 3.3, we get
(a) if φ is of osculating order $r = 2$ then it is a circle with

$$k_1 = \sqrt{f_1 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}},$$

where, $f_1 - f_3^2\cos^2\theta > \frac{\delta_1}{\delta_2}$.
(b) If φ is of osculating order $r = 3$ then it is helix with

$$k_1^2 + k_1^2 = f_1 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2},$$

where $f_1 > f_3\cos^2\theta + \frac{\delta_1}{\delta_2}$.
Conversely, if φ is a circle with $k_1 = \sqrt{f_1 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}}$ or a helix with $k_1^2 + k_1^2 = f_1 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}$. Then φ satisfies Theorem 3.2 and this completes the proof. □

In particular, using $f_1 = \frac{c+4}{4}$, $f_3 = \frac{c-1}{4}$ and $\theta = \frac{\pi}{2}$ in above theorem we have

Corollary 3.5. *[16] Let $\varphi : I \to N^{2n+1}(c)$ is a Legendre curve of osculating order r in Sasakian space form such that $\phi T \perp E_2$ with $c \neq 1$. Then*

1. *If $c \leq 4\frac{\delta_1}{\delta_2} - 3$, then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is geodesic.*
2. *If $c > 4\frac{\delta_1}{\delta_2} - 3$, then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if either*
   (a) *If φ is of osculating order $r = 2, n \geq 2$ and it is circle with $k_1^2 = \frac{c+3}{4} - \frac{\delta_1}{\delta_2}$, or*
   (b) *If φ is of osculating order $r = 3, n \geq 3$ and it helix with $k_1^2 + k_2^2 = \frac{(c+3)}{4} - \frac{\delta_1}{\delta_2}$.*

Moreover, for $\theta = \frac{\pi}{2}$, $\delta_1 = 0$ and $\delta_2 = 1$ in Theorem 3.2 we have

Corollary 3.6. *[8] Let $\varphi : I \to N^{2n+1}(c)$ be a Legendre Frenet curve in a Sasakian-space form and $\phi T \perp E_2$. Then φ is proper biharmonic if and only if either*

1. *$n \geq 2$ and φ is a circle with $k_1 = \frac{1}{2}\sqrt{c+3}$, where $c > -3$ and $\{T = E_1, E_2, \phi T, \nabla_T\phi T, \xi_1, \cdots, \xi_s\}$ is linearly independent or*
2. *$n \geq 3$ and φ is a helix with $k_1^2 + k_2^2 = c + 3$, where $c > -3$ and $\{T = E_1, E_2, \phi T, \nabla_T\phi T, \xi_1, \cdots, \xi_s\}$ is linearly independent.*

If $c \leq -3$, then φ is biharmonic if and only if it is a geodesic.

Case 2: $\phi T \| E_2$ and $\xi \perp E_2$.

Proposition 3.7. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ is a slant curve of osculating order r , $p = min\{r, 4\}$ in generalized Sasakian space form. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}, \\ k_2 &= constant, \ \ k_2k_3 = 0. \end{aligned}$$

Theorem 3.8. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve of osculating order r in generalized Sasakian space form such that $\phi T \| E_2$ and $\xi \perp E_2$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if either*

1. *φ is of osculating order $r = 2$ and it is circle with $k_1 = \sqrt{f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}}$, where $f_1 + 3f_2 - f_3\cos^2\theta > \frac{\delta_1}{\delta_2}$, or*
2. *φ is of osculating order $r = 3$ and it is a helix with $k_2 = 1$ and $k_1^2 = f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}$.*

Proof. If $\phi T || E_2$ and $\xi \perp E_2$ then we have

$$g(\phi T, E_2) = \pm 1, \quad and \quad g(\xi, E_2) = 0. \tag{12}$$

If $f_1 + 3f_2 - f_3\cos^2\theta > \frac{\delta_1}{\delta_2}$, then φ is a circle with $k_2 = 1$ and $k_1 = \sqrt{f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}}$.
Also, if φ is of osculating order $r = 3, n \geq 3$, then it is a helix with $k_2 = 1$ given by $k_1^2 = f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}$.
Conversely, if φ is helix with $k_1^2 + k_2^2 = f_1 + 3f_2 - f_3\cos^2\theta - \frac{\delta_1}{\delta_2}$ and $k_2 = 1$. Then φ satisfies Theorem 3.2. □

Case 3: $\phi T \perp E_2$ and $\xi \in span\{E_2, E_3, \cdots, E_m\}$.

Theorem 3.9. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve of osculating order $r \geq 4$ in generalized Sasakian space form such that $\phi T \perp E_2$ and $\xi \in span\{E_2, E_3, \cdots, E_m\}$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3\cos^2\theta - f_3\cos^2 u - \frac{\delta_1}{\delta_2}, \\ k_2' &= f_3 cosu\ \sin u\ \cos v, \\ k_2 k_3 &= f_3\cos u\ \sin u\ \sin v \end{aligned}$$

where u and v are real valued angle functions.

Proof. Suppose φ is an interpolating sesqui-harmonic slant curve of osculating order $r \geq 4$ in $N^{2n+1}(f_1, f_2, f_3)$. Then we have [15],

$$\xi = \cos u\, E_2 + \sin u \cos v\, E_3 + \sin u \sin v\, E_4, \tag{13}$$

where u, v are the real valued angle between ξ and E_2, E_3 and the orthogonal projection of ξ onto $span\{E_2, E_4\}$ respectively. Thus we have

$$\begin{aligned} \eta(E_2) &= \cos u, \\ \eta(E_3) &= \sin u\ \cos v, \\ \eta(E_4) &= \sin u\ \sin v. \end{aligned}$$

By using above equations and Theorem 3.2, the curve is interpolating sesqui-harmonic if

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3\cos^2\theta - f_3\cos^2 u - \frac{\delta_1}{\delta_2}, \\ k_2' &= f_3\cos u\ \sin u\ \cos v, \\ k_2 k_3 &= f_3\cos u\ \sin u\ \sin v. \end{aligned}$$

Conversely, if φ satisfies the converse statement then the above four equations in Theorem 3.2 are satisfied. Hence φ is an interpolating sesqui-harmonic. □

For $\delta_1 = 0, \delta_2 = 1$ and $\theta = \frac{\pi}{2}$, we have

Corollary 3.10. *[20] Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a Legendre Frenet curve of osculating order r in a generalized Sasakian space form. Then φ is proper biharmonic if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3 cos^2 u, \\ k_2' &= f_3 \cos u \sin u \cos v, \\ k_2 k_3 &= f_3 \cos u \sin u \sin v. \end{aligned}$$

Case 4: $\xi \perp E_2$ and $\phi T \in span\{E_2, E_3, \cdots, E_m\}$.

Theorem 3.11. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve of osculating order $r \geq 4$ in generalized Ssakian space form such that $\xi \perp E_2$ and $\phi T \in span\{E_2, E_3, \cdots, E_m\}$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3 \cos^2 \theta + 3 f_2 \cos^2 w - \frac{\delta_1}{\delta_2}, \\ k_2' &= -3 f_3 \cos w \ \sin w \ \cos z, \\ k_2 k_3 &= -3 f_3 \cos w \ \sin w \ \sin z \end{aligned}$$

where w and z are real valued angle function.

Proof. Suppose φ is an interpolating sesqui-harmonic slant curve of osculating order $r \geq 4$ in $N^{2n+1}(f_1, f_2, f_3)$. Then we have [15]

$$\phi T = \cos w \, E_2 + \sin w \cos z \, E_3 + \sin w \sin z \, E_4 \tag{14}$$

where w, z are the real valued angle between ξ and E_2, E_3, and the orthogonal projection of ξ onto $span\{E_2, E_4\}$ respectively. Thus we have

$$\begin{aligned} g(E_2, \phi T) &= \cos w, \\ g(E_3, \phi T) &= \sin w \ \cos z, \\ g(E_4, \phi T) &= \sin w \ \sin z. \end{aligned}$$

By using above equations and Theorem 3.2, the curve is interpolating sesqui-harmonic if

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3 \cos^2 \theta + 3 f_2 \cos^2 w - \frac{\delta_1}{\delta_2}, \\ k_2' &= -3 f_2 \cos w \ \sin w \ \cos z, \\ k_2 k_3 &= -3 f_2 \cos w \ \sin w \ \sin z. \end{aligned}$$

Conversely, if φ satisfies the converse statement then the above four equations in Theorem 3.2 are satisfied. Hence φ is an interpolating sesqui-harmonic. □

For $\delta_1 = 0, \delta_2 = 1$, and $\theta = \frac{\pi}{2}$ we have

Corollary 3.12. *[20] Let φ be a Legendre Frenet curve of osculating order r in a generalized Sasakian space form with $f_2 \neq 0, f_3 \neq 0, \phi T \in span\{E_2, ..., E_m\}$ and $\xi \perp E_2$. Then φ is proper biharmonic if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 + 3 f_2 cos^2 w, \\ k_2' &= -3 f_2 \cos w \sin w \cos z, \\ k_2 k_3 &= -3 f_2 \cos w \sin w \sin z. \end{aligned}$$

Case 5: $\xi \in span\{E_2, E_3, \cdots, E_m\}$ and $\phi T \in span\{E_2, E_3, \cdots, E_m\}$.

Making use of equations (13), (14) and Theorem 3.2 we obtain

Theorem 3.13. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve of osculating order $r \geq 4$ in generalized Sasakian space form such that $\phi T \in span\{E_2, E_3, \cdots, E_m\}$ and $\xi \in span\{E_2, E_3, \cdots, E_m\}$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 - f_3 \cos^2\theta + 3f_2 \cos^2 w + f_3 \cos^2 u - \frac{\delta_1}{\delta_2}, \\ k_2' &= f_3 \cos u \sin u \cos v - 3f_2 \cos w \sin w \cos z, \\ k_2 k_3 &= f_3 \cos u \sin u \sin v - 3f_2 \cos w \sin w \sin z. \end{aligned}$$

where u, v, w and z are real valued angle function.

Using $\delta_1 = 0$, $\delta_2 = 1$ and $\theta = \frac{\pi}{2}$ in above proposition we have

Corollary 3.14. *[20] Let φ be a Legendre Frenet curve of osculating order r in a generalized Sasakian space form such that $\phi T \in span\{E_2, \cdots, E_m\}$ and $\xi \in span\{E_2, \cdots, E_m\}$. Then φ is proper biharmonic if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= f_1 + 3f_2 cos^2 w + f_3 \cos^2 u, \\ k_2' &= -3f_2 \cos w \sin w \cos z + f_3 \cos u \sin u \cos v, \\ k_2 k_3 &= -3f_2 \cos w \sin w \sin z - f_3 \cos u \sin u \sin v. \end{aligned}$$

4. Interpolating sesqui-harmonic minimal curves

An isometric immersion $\pi : (M^m, g) \to (N^n, h)$ is said to be biminimal if it is a critical point of the bi-energy functional under all normal variations [18]. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general. In this section we obtain the minimality of interpolating sesqui-harmonic slant curves in 3-dimensional generalized Sasakian spaceform.

Definition 4.1. *See [14] An immersion $\pi : (M, g) \to (N, h)$ is called biminimal if it is a critical point of the functional*

$$E_{2,\lambda}(\pi) := E_2(\pi) + \lambda E(\pi), \quad \lambda \in \mathbb{R}. \tag{15}$$

The Euler-Lagrange equation of biminimal immersions is

$$[\tau_2(\pi)]^{\perp} + \lambda[\tau(\pi)]^{\perp} = 0, \tag{16}$$

where $\lambda \in \mathbb{R}$ and $\perp$ stand for normal component of [.].

In the similar way f-biminmal immersion was defined by F. Gurler and C. Ozgur [9]. Motivated by these studies we define interpolating sesqui minimal curve as follows:

Definition 4.2. *An immersion π between two Riemannian manifolds M and N is called interpolating sesqui minimal if it is critical point of the energy functional $E_{\delta_1,\delta_2}(\pi)$ for variations normal to the image $\pi(M) \subset N$ with fixed energy. Equivalently, there exist a constant $\lambda \in \mathbb{R}$ such that*

$$E_{\delta_1,\delta_2,\lambda}(\pi) = E_{\delta_1,\delta_2}(\pi) + \lambda E(\pi). \tag{17}$$

The Euler Lagrange equation for *λ-interpolating sesqui minimal* immersion is

$$[\tau_{\delta_1,\delta_2,\lambda}(\pi]^{\perp} = [\tau_{\delta_1,\delta_2}(\pi)]^{\perp} - \lambda[\tau(\pi)]^{\perp} = 0, \tag{18}$$

where $\lambda \in \mathbb{R}$.
Then the tension field of π is computed as

$$\begin{aligned}\tau_{\delta_1,\delta_2}(\pi) &= [-3\delta_2 k_1 k_1']E_1 + [\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 \cos^2\theta k_1 + \delta_2 f_1 k_1 - \delta_1 k_1]E_2 \\ &+ \delta_2(2k_1' k_2 + k_1 k_2')E_3 + (k_1 k_2 k_3)E_4 + 3\delta_2 f_2 k_1 g(\phi T, E_2)\phi T - \delta_2 f_3 k_1 \eta(E_2)\xi.\end{aligned}$$

$$\begin{aligned}\tau^{\perp}_{\delta_1,\delta_2}(\pi) &= [\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 \cos^2\theta k_1 + \delta_2 f_1 k_1 - \delta_1 k_1]E_2 + \delta_2(2k_1' k_2 + k_1 k_2')E_3 \\ &+ (k_1 k_2 k_3)E_4 + 3\delta_2 f_2 g(\phi T, k_1 E_2)\phi T - \delta_2 f_3 k_1 \eta(E_2)\xi.\end{aligned}$$

Now by the interpolating sesqui minimality condition

$$\begin{aligned}\tau^{\perp}_{\delta_1,\delta_2}(\pi) - \lambda\tau^{\perp}(\pi) &= [\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 \cos^2\theta k_1 + \delta_2 f_1 k_1 - \delta_1 k_1 - \lambda k_1]E_2 \\ &+ \delta_2(2k_1' k_2 + k_1 k_2')E_3 + (k_1 k_2 k_3)E_4 + 3\delta_2 f_2 g(\phi T, k_1 E_2)\phi T \\ &- \delta_2 f_3 k_1 \eta(E_2)\xi = 0.\end{aligned}$$

Taking inner product with E_2, E_3 and E_4, respectively, we obtain

$$\begin{aligned}&[\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 cos^2(\theta)k_1 + \delta_2 f_1 k_1 - \delta_1 k_1 - \lambda k_1] \\ &\quad +3\delta_2 f_2 k_1 g(\phi T, E_2)^2 - \delta_2 f_3 k_1 (\eta(E_2))^2 = 0, \\ &\delta_2(2k_1' k_2 + k_1 k_2') + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_3) = 0, \\ &(k_1 k_2 k_3) + +3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_4) = 0.\end{aligned} \tag{19}$$

Case 1. $\phi T \perp E_2$ and $\xi \perp E_2$

$$\begin{aligned}&\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 cos^2(\theta)k_1 \\ &\quad +\delta_2 f_1 k_1 - \delta_1 k_1 + \lambda k_1 = 0, \\ &\delta_2(2k_1' k_2 + k_1 k_2') = 0, \\ &k_1 k_2 k_3 = 0.\end{aligned}$$

Case 2. $\phi T \parallel E_2$ and $\xi \perp E_2$

$$\begin{aligned}&\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 c^2 k_1 \\ &+\delta_2 f_1 k_1 - \delta_1 k_1 + \lambda k_1 + 3\delta_2 f_2 k_1 = 0, \\ &\delta_2(2k_1' k_2 + k_1 k_2') = 0, \\ &k_1 k_2 k_3 = 0.\end{aligned}$$

Theorem 4.3. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve in generalized Sasakian space form . Then φ is interpolating sesqui harmonic minimal if and only if there exists δ_1, δ_2 such that*

$$\begin{aligned}&[\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 cos^2(\theta)k_1 + \delta_2 f_1 k_1 - \delta_1 k_1 - \lambda k_1] \\ &\quad +3\delta_2 f_2 k_1 g(\phi T, E_2)^2 - \delta_2 f_3 k_1 (\eta(E_2))^2 = 0, \\ &\delta_2(2k_1' k_2 + k_1 k_2') + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_3) = 0, \\ &(k_1 k_2 k_3) + +3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta(E_2)\eta(E_4) = 0.\end{aligned} \tag{12}$$

Proposition 4.4. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve in a 3-dimensional generalized Sasakian space form with $\phi T \perp E_2$. Then φ is interpolating sesqui harmonic minimal if and only if*

$$\begin{aligned}&\delta_2(k_1'' - k_1^3 - k_1 k_2^2) - \delta_2 f_3 c^2 k_1 \\ &\quad +\delta_2 f_1 k_1 - \delta_1 k_1 + \lambda k_1 = 0,\end{aligned}$$

$$2k_1' k_2 + k_1 k_2' = 0.$$

Proposition 4.5. *Let $\varphi : I \to N^{2n+1}(f_1, f_2, f_3)$ be a slant curve in a 3-dimensional generalized Sasakian space form with $\phi T \| E_2$. Then φ is interpolating sesqui harmonic minimal if and only if*

$$\begin{aligned}&\delta_2(k_1'' - k_1^3 - k_1k_2^2) - \delta_2 f_3 c^2 k_1\\ &+\delta_2 f_1 k_1 - \delta_1 k_1 + \lambda k_1 + 3\delta_2 f_2 = 0,\end{aligned}$$

$$2k_1^{'}k_2 + k_1k_2^{'} = 0.$$

5. Example

Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian-space form with coordinate functions $\{x_1, \cdots, x_n, y_1, \cdots, y_n, z\}$. The vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, X_{n+i} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}), \xi = 2\frac{\partial}{\partial z} \tag{4}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\begin{cases}\nabla_{X_i}X_j = \nabla_{X_{n+i}}X_{n+j} = 0, \nabla_{X_i}X_{n+j} = \delta_{ij}\xi, \nabla_{X_{n+i}}X_j = -\delta_{ij}\xi,\\ \nabla_{X_i}\xi = \nabla_\xi X_i = -X_{n+i}, \nabla_{X_{n+i}}\xi = \nabla_\xi X_{n+i} = X_i.\end{cases} \tag{5}$$

Let $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t), \varphi_7(t))$ be a unit speed slant curve in $\mathbb{R}^7(-5)$. Then for a tangent vector we have

$$T = \frac{1}{2}\Big(\varphi_1'\frac{\partial}{\partial x_1} + \varphi_2'\frac{\partial}{\partial x_2} + \varphi_3'\frac{\partial}{\partial x_3} + \varphi_4'\frac{\partial}{\partial y_1} + \varphi_5'\frac{\partial}{\partial y_2} + \varphi_6'\frac{\partial}{\partial y_3} + \varphi_7'\frac{\partial}{\partial z}\Big). \tag{6}$$

From equation (4), we have

$$\begin{aligned}&X_1 = 2\frac{\partial}{\partial y_1}, \quad X_2 = 2\frac{\partial}{\partial y_2}, \quad X_3 = 2\frac{\partial}{\partial y_3},\\ &X_4 = \phi X_1 = 2\Big(\frac{\partial}{\partial x_1} + y_1(\frac{\partial}{\partial z})\Big), X_5 = \phi X_2 = 2\Big(\frac{\partial}{\partial x_2} + y_2(\frac{\partial}{\partial z})\Big),\\ &X_6 = \phi X_3 = 2\Big(\frac{\partial}{\partial x_3} + y_3(\frac{\partial}{\partial z})\Big), \quad \xi_1 = 2\frac{\partial}{\partial z}.\end{aligned}$$

By using these values we have

$$\begin{aligned}T &= \frac{1}{2}\Big(\varphi_4'X_1 + \varphi_5'X_2 + \varphi_6'X_3 + \varphi_1'X_4 + \varphi_2'X_5 + \varphi_3'X_6\\ &+ (\varphi_7' - \varphi_1'\varphi_4 - \varphi_2'\varphi_5 - \varphi_3'\varphi_6)\xi\Big)\end{aligned} \tag{4}$$

and

$$\phi T = \frac{1}{2}\Big(-\varphi_1'X_1 - \varphi_2'X_2 - \varphi_3'X_3 + \varphi_4'X_4 + \varphi_5'X_5 + \varphi_6'X_6\Big). \tag{4}$$

For slant curve $\eta(T) = \cos(\theta)$, we have

$$\varphi_7' = \varphi_1'\varphi_4 + \varphi_2'\varphi_5 + \varphi_3'\varphi_6 + 2\cos(\theta). \tag{5}$$

Differentiating equation (4) and making use of (5)

$$\nabla_T T = \frac{1}{2}[\varphi_4''X_1 + \varphi_5''X_2 + \varphi_6''X_3 + \varphi_1''X_4 + \varphi_2''X_5 + \varphi_3''X_6]. \tag{6}$$

Since $\phi T \perp E_2$ if and only if

$$\varphi_1'\varphi_1'' + \varphi_2'\varphi_5'' + \varphi_3'\varphi_6'' = \varphi_4'\varphi_1'' + \varphi_5'\varphi_2'' + \varphi_6'\varphi_3''.$$

Then using $\theta = \frac{\pi}{3}$ and $\varphi_4 = \varphi_5 = \varphi_6 = 0$ in above equations, we get $\varphi_1 = \sqrt{3}sint, \varphi_2 = 0$ and $\varphi_3 = \sqrt{3}cost$. Therefore we have $\varphi(t) = (\sqrt{3}sint, 0, \sqrt{3}cost, 0, 0, 0, \frac{t}{2})$. Now making use of equation (6) we have

$$\nabla_T T = \frac{1}{2}[\sqrt{3}costX_4 - \sqrt{3}sintX_6].$$

Taking inner product of above equation with itself we have $k_1 = \sqrt{3}$, which satisfy Theorem 3.2 for the case of osculating order 2, $\phi T \perp E_2, \delta_1 = -25$ and $\delta_2 = 8$.

6. Applications

For particular values of f_1, f_2 and f_3, we have the following results for Sasakian, cosymplectic and Kenmotsu space forms.

Corollary 6.1. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in Sasakian space form such that $\varphi T \perp E_2$, $p = min\{r, 4\}$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= \frac{c+3}{4} - \frac{c-1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}, \\ k_2k_3 &= 0. \end{aligned}$$

Corollary 6.2. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in cosymplectic space form such that $\phi T \perp E_2$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= \frac{c}{4} - \frac{c}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}, \\ k_2k_3 &= 0. \end{aligned}$$

Corollary 6.3. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in Kenmotsu space form such that $\phi T \perp E_2$, $p = min\{r, 4\}$. Then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$\begin{aligned} k_1 &= constant > 0, \\ k_1^2 + k_2^2 &= \frac{c-3}{4} - \frac{c+1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}, \\ k_2k_3 &= 0. \end{aligned}$$

Theorem 6.4. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in Sasakian space form such that $\phi T \perp E_2$. Then*

1. *If $\frac{c+3}{4} \leq \frac{c-1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is geodesic.*

2. *If $\frac{c+3}{4} > \frac{c-1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if either one of the following holds:*
   (a) *φ is of osculating order $r = 2, n \geq 2$ and it is a circle with*
   $k_1 = \sqrt{\frac{c+3}{4} - \frac{c-1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}}$, where, $\frac{c+3}{4} - \frac{c-1}{4}\cos^2\theta > \frac{\delta_1}{\delta_2}$.

   (b) *φ is of osculating order $r = 3, n \geq 3$ and it is a helix with*
   $k_1^2 + k_2^2 = \frac{c+3}{4} - \frac{c-1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}$, where $\frac{c+3}{4} > \frac{c-1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$.

Corollary 6.5. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in cosymplectic space form such that $\phi T \perp E_2$. Then*

1. *If $\frac{c}{4} \leq \frac{c}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is geodesic.*

2. *If $\frac{c}{4} > \frac{c}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if either one of the following holds:*
   (a) *φ is of osculating order $r = 2, n \geq 2$ and it is circle with*
   $k_1 = \sqrt{\frac{c}{4} - \frac{c}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}}$*, where* $\frac{c}{4} - \frac{c}{4}\cos^2\theta > \frac{\delta_1}{\delta_2}$.

   (b) *φ is of osculating order $r = 3, n \geq 3$ and it is a helix with*
   $k_1^2 + k_2^2 = \frac{c}{4} - \frac{c}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}$*, where* $\frac{c}{4} > \frac{c}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$.

Corollary 6.6. *Let $\varphi : I \to N^{2n+1}(c)$ be a slant curve of osculating order r in Kenmotsu space form such that $\phi T \perp E_2$ and . Then*

1. *If $\frac{c-3}{4} \leq \frac{c+1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is geodesic.*

2. *If $\frac{c-3}{4} > \frac{c+1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$ then φ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if either one of the following holds:*
   (a) *φ is of osculating order $r = 2, n \geq 2$ and it is circle with*
   $k_1 = \sqrt{\frac{c-3}{4} - \frac{c+1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}}$*, where* $\frac{c-3}{4} - \frac{c+1}{4}\cos^2\theta > \frac{\delta_1}{\delta_2}$.
   (b) *φ is of osculating order $r = 3, n \geq 3$ and it is a helix with*
   $k_1^2 + k_2^2 = \frac{c-3}{4} - \frac{c+1}{4}\cos^2\theta - \frac{\delta_1}{\delta_2}$*, where* $\frac{c-3}{4} > \frac{c+1}{4}\cos^2\theta + \frac{\delta_1}{\delta_2}$.

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