The \(\eta\)-Hermitian Solutions to Some Systems of Real Quaternion Matrix Equations

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\textbf{Abstract.} Let \(H^{m\times n}\) be the set of all \(m \times n\) matrices over the real quaternion algebra. We call that \(A \in H^{n\times n}\) is \(\eta\)-Hermitian if \(A = A^\eta\), where \(A^\eta = -\eta A^*\eta\), \(\eta \in \{i, j, k\}\), \(i, j, k\) are the quaternion units. In this paper, we derive some solvability conditions and the general solution to a system of real quaternion matrix equations. As an application, we present some necessary and sufficient conditions for the existence of an \(\eta\)-Hermitian solution to some systems of real quaternion matrix equations. We also give the expressions of the general \(\eta\)-Hermitian solutions to these systems when they are solvable. Some numerical examples are given to illustrate the results of this paper.

1. Introduction

Throughout, the set of all \(m \times n\) matrices over the quaternion number field \(H\)

\[H = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}\]

by \(\mathbb{H}^{m\times n}\). For a matrix \(A\), \(A^*\) stands for the conjugate transpose of \(A\). \(I\) denotes the identity matrix with appropriate sizes. The Moore-Penrose inverse \(A^+\) of \(A\) is defined to be the unique matrix \(A^+\), such that

\[(i) \quad AA^+A = A, \quad (ii) \quad A^+AA^+ = A^+, \quad (iii) \quad (AA^+)^* = AA^+, \quad (iv) \quad (A^+A)^* = A^+A.\]

Furthermore, \(L_A\) and \(R_A\) stand for the two projectors \(L_A = I - A^+A\) and \(R_A = I - AA^+\) induced by \(A\), respectively. It is known that \(L_A = L_A^*\) and \(R_A = R_A^*\). The symbol \(r(A)\) stands for the rank of a given real quaternion matrix \(A\). For a real quaternion matrix \(A\), \(r(A) = r(A^\eta)\) ([4]). A quaternion matrix \(A\) is called an \(\eta\)-Hermitian matrix if \(A = A^\eta = -\eta A^*\eta\), \(\eta \in \{i, j, k\}\) ([22]).

Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton Nowadays quaternion matrices can be used in signal and color image processing, quantum physics, computer science, and so on (e.g. [1], [19]-[21], [27]). Many problems in systems and control theory can be reduced to solving systems of quaternion matrix equations (e.g. [6]-[16], [26]).
The \( \eta \)-Hermitian matrices have some applications in widely linear modelling, convergence analysis in statistical signal processing ([21]). He and Wang ([4]) gave some solvability conditions and general solution to the real quaternion matrix equation involving \( \eta \)-Hermiticity

\[
A_1X + (A_1X)\eta + B_1YB_1^\eta + C_1ZC_1^\eta = D_1, 
\]

where \( Y \) and \( Z \) are \( \eta \)-Hermitian. Horn and Zhang ([17]) derived an analogous special singular value decomposition for \( \eta \)-Hermitian matrices. He and Wang ([2]) considered the \( \eta \)-Hermitian solution to a system of real quaternion matrix equations

\[
\begin{align*}
A_1X &= C_1, 
XB_1 &= D_1, 
A_2Y &= C_2, 
YB_2 &= D_2, 
C_3XC_3^\eta + D_3YD_3^\eta &= A_3.
\end{align*}
\]

Very recently, He, Wang and Zhang ([5]) presented a simultaneous decomposition for a set of nine real quaternion matrices involving \( \eta \)-Hermiticity: \( A_i \in \mathbb{H}^{p \times k_i}, B_i \in \mathbb{H}^{n \times l_i}, \) and \( C_i \in \mathbb{H}^{p \times p}, \) where \( C_i \) are \( \eta \)-Hermitian matrices, \( i = 1, 2, 3. \) The reference ([5]) gave some necessary and sufficient conditions for the existence of the general \( \eta \)-Hermitian solution to the system of coupled real quaternion matrix equations involving \( \eta \)-Hermiticity

\[
A_iX_iA_i^\eta + B_iX_{i+1}B_i^\eta = C_i, \quad (i = 1, 2, 3),
\]

where \( A_i \in \mathbb{H}^{p \times k_i}, B_i \in \mathbb{H}^{n \times l_i}, \) and \( C_i \in \mathbb{H}^{p \times p}, \) and \( C_i \) are \( \eta \)-Hermitian matrices.

Motivated by the work mentioned above and the recent increasing interests in \( \eta \)-Hermitian quaternion matrices and real quaternion matrix equations, we in this paper consider the \( \eta \)-Hermitian solution to the following system of real quaternion matrix equations

\[
\begin{align*}
A_1X &= C_1, 
XB_1 &= D_1, 
A_2XA_2^\eta &= C_2, 
A_3XA_3^\eta &= C_3, 
A_4XA_4^\eta &= C_4
\end{align*}
(1)
\]

where \( A_1, C_1, A_2, A_3, A_4, C_2 = C_2^\eta, C_3 = C_3^\eta, C_4 = C_4^\eta \) be known over \( \mathbb{H}, \) and \( X = X^\eta \) be unknown. We aim to give some solvability conditions and general \( \eta \)-Hermitian solution to the system of real quaternion matrix equations (1). Observe that the following system of real quaternion matrix equations

\[
\begin{align*}
A_1X &= C_1, 
XB_1 &= D_1, 
A_2XB_2 &= C_2, 
A_3XB_3 &= C_3, 
A_4XB_4 &= C_4
\end{align*}
(2)
\]

plays an important role in investigating the \( \eta \)-Hermitian solution to (1). Another goal of this paper is to give some solvability conditions and the general solution to the system (2).

The remainder of the paper is organized as follows. In Section 2, we give some lemmas which are used in this paper. In Section 3, we present some necessary and sufficient conditions for the existence of a solution to the system of real quaternion matrix equations (2) and provide the general solution to system (2). In Section 4, we derive some solvability conditions and the general \( \eta \)-Hermitian solution to the system of real quaternion matrix equations (1).

2. Preliminaries

In this section, we review some lemmas which are used in this paper.
Lemma 2.1. ([23]) Let $A_1 \in \mathbb{H}^{m \times r}, B_1 \in \mathbb{H}^{r \times n}, C_1 \in \mathbb{H}^{n \times s},$ and $D_1 \in \mathbb{H}^{s \times k}$ be given and $X \in \mathbb{H}^{m \times r}$ be unknown. The general solution of system (4) can be expressed as
\begin{equation}
A_1X = C_1, \quad XB_1 = D_1
\end{equation}
is consistent if and only if
\[ R_{A_1}C_1 = 0, \quad D_1L_{B_1} = 0, \quad A_1D_1 = C_1B_1. \]
In this case, the general solution to (3) is
\[ X = A_1^t C_1 + L_{A_1}D_1B_1^t + L_{A_1}YR_{B_1}, \]
where $Y$ is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Lemma 2.2. ([3]) Let $A_{ii}, B_{ii},$ and $C_{ii}$ ($i = 1, 2$) be given with appropriate sizes. Set
\[ A = A_{22}L_{A_{11}}, \quad B = R_{B_{11}}, \quad C = C_{22} - A_{22}A_{11}^t C_{11}B_{11}^t B_{22}, \quad D = R_{A_{11}}A_{22}. \]
Then the system
\begin{equation}
A_{11}XB_{11} = C_{11}, \quad A_{22}XB_{22} = C_{22}
\end{equation}
is consistent if and only if
\[ R_A C_{11}L_B = 0, \quad R_A C_{ii} = 0, \quad C_{ii}L_{B_{11}} = 0, \quad i = 1, 2. \]
In this case, the general solution of system (4) can be expressed as
\[ X = A_{11}^t C_{11}B_{11}^t + L_{A_{11}}A_{11}^t CB_{11}^t - L_{A_{11}}A_{11}^t A_{22}D_{11}^t R_{A_{11}}C_{11}B_{11}^t B_{22}^t + D_{11}^t R_{A_{11}}C_{11}B_{11}^t B_{22}^t \]
\[ + L_{A_{11}}L_{A_{11}}U_{11} + U_2R_{B_{11}}R_{B_{11}} + L_{A_{11}}U_3R_{B_{11}} + L_{A_{11}}U_4R_{B_{11}}, \]
where $U_{11}, U_2, U_3,$ and $U_4$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 2.3. ([3], [25]) Let $A_1, B_1, C_3, D_3, C_4, D_4,$ and $E_1$ be given. Set
\[ A = R_A C_{31}, \quad B = D_3L_{B_1}, \quad C = R_A C_{41}, \quad D = D_4L_{B_1}, \]
\[ E = R_A E_1L_{B_1}, \quad M = R_A C_{11}, \quad N = D_4L_{B_1}, \quad S = C_{11}. \]
Then the real quaternion matrix equation
\begin{equation}
A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1
\end{equation}
is consistent if and only if
\[ R_M R_A E = 0, \quad EL_B L_N = 0, \quad R_A E L_D = 0, \quad R_C E L_B = 0. \]
In this case, the general solution can be expressed as
\[ X_1 = A_1^t (E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^t T_7 B_1 + L_{A_1}T_6, \]
\[ X_2 = R_A (E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^t + A_1A_1^t T_7 + T_8 R_{B_1}, \]
\[ X_3 = A_1^t E_1^t - A_1^t C_1^t E_1^t - A_1^t SC_1^t E_1^t DB_1^t - A_1^t ST_2 R_N DB_1^t + L_A T_4 + T_5 R_B, \]
\[ X_4 = M^t E_4^t + S^t SC_4^t E_4^t + L_M T_1 + L_M T_2 R_N + T_3 R_D, \]
where $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

The following lemma can be easily generalized to $\mathbb{H}$. 

Lemma 2.4. ([18]) Let $A \in \mathbb{H}^{m \times r}, B \in \mathbb{H}^{r \times k}, C \in \mathbb{H}^{p \times q}, D \in \mathbb{H}^{p \times s}, E \in \mathbb{H}^{s \times r}, Q \in \mathbb{H}^{m \times r},$ and $P \in \mathbb{H}^{m \times k}$ be given. Then
\begin{enumerate}
\item $r(A) + r(RAB) = r(B) + r(RBA) = r(A, B).$
\item $r(\alpha A) + r(ALA) = r(C) + r(ALC) = r\left(\frac{A}{C}\right).$
\end{enumerate}
3. Solvability conditions and general solution to the system (2)

In this section, we consider the system of real quaternion matrix equations (2). We derive solvability conditions and general solution to the system (2). Now we give the fundamental theorem of this section.

**Theorem 3.1.** Let \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, A_3, B_3, C_3, A_4, B_4, C_4 \) be known over \( \mathbb{H} \), and \( X \) be unknown. Set

\[
A_i = A_{i+1}L_{A_i}, \quad B_i = R_{B_i}B_{i+1}, \quad C_i = C_{i+1} - A_{i+1}(A_i^tC_1 + L_{A_i}D_1B_1^t)B_{i+1}, \quad (i = 1, 2, 3),
\]

(6)

\[
A = A_{22}L_{A_{11}}, \quad B = R_{B_{11}}B_{22}, \quad C = C_{22} - A_{22}A_{11}^tC_1B_{11}^tB_{22}, \quad D = R_{A_{11}}A_{22},
\]

(7)

\[
A_5 = (L_{A_{11}}L_A, L_{A_{11}}), \quad B_5 = \begin{pmatrix} R_BR_{B_{11}} \\ R_{B_{11}} \end{pmatrix},
\]

(8)

\[
C_5 = A_{33}^tC_{33}B_{33}^t - A_{11}^tC_1B_{11}^t - L_{A_{11}}A_1^tCB_{11}^t - L_{A_{11}}A_1^tA_{22}D_1^tR_4CB_{22}^t - D_1^tR_4CB_1^tR_{B_{11}},
\]

(9)

\[
A_6 = R_{A_{11}}L_{A_{11}}, \quad B_6 = R_{B_{11}}B_{11}, \quad C_6 = R_{A_{11}}L_{A_{11}}, \quad D_6 = R_{B_{11}}L_{B_{11}},
\]

(10)

\[
E = R_{A_{11}}C_{11}L_{B_{11}}, \quad M = R_{A_{11}}C_{11}, \quad N = D_6L_{B_{11}}, \quad S = C_6L_{M}.
\]

(11)

Then the following statements are equivalent:

1. The system of real quaternion matrix equations (2) is consistent.

2. \( R_{A_{11}}C_{11} = 0, D_1L_{B_{11}} = 0, A_1D_1 = C_1B_1 \)

3. \( r(A_1, C_1) = r(A_1), \quad r \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = r(B_1), A_1D_1 = C_1B_1, \)

(15)

\[
r \begin{pmatrix} C_{i+1} \\ C_iB_{i+1} \\ A_{i+1} \\ A_1 \\ A_{i+1} \\ A_1 \end{pmatrix} = r \begin{pmatrix} A_{i+1} \\ A_1 \\ B_1 \end{pmatrix}, \quad r \begin{pmatrix} A_{i+1}D_1 \\ C_{i+1} \\ A_{i+1} \\ A_1 \end{pmatrix} = r(B_1, B_{i+1}), \quad (i = 1, 2, 3),
\]

(16)

\[
r \begin{pmatrix} -C_2 \\ A_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ A_3 \end{pmatrix} = r \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + r(B_1, B_2, B_3),
\]

(17)

\[
r \begin{pmatrix} 0 \\ 0 \\ B_2 \\ B_3 \\ B_4 \\ B_1 \\ A_2 \\ A_3 \\ 0 \\ 0 \\ C_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ A_4 \\ A_3 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_3 \\ 0 \\ 0 \\ A_4 \\ 0 \\ A_1 \\ 0 \\ 0 \\ C_4 \\ 0 \\ 0 \\ 0 \\ C_1 \end{pmatrix} + r(B_1, B_2, B_3, B_4),
\]

(18)
In this case, the general solution to system (2) can be expressed as

\[
X = A_1^* C_1 + L_{A_1} D_1 T_1 + L_{A_1} Y R_{B_1},
\]

where

\[
Y = A_{33}^* C_{33} B_{33}^* - L_{A_{33}} U_5 - U_6 R_{B_{33}},
\]

\[
\begin{pmatrix}
U_1 \\
U_2
\end{pmatrix} = A_5^* (C_5 - L_{A_5} U_5 R_{B_5} - L_{A_5} U_4 R_{B_{33}}) - A_5^* T_7 B_5 + L_{A_5} T_6,
\]

\[
(U_2, U_6) = R_{A_5}(C_5 - L_{A_5} U_5 R_{B_5} - L_{A_5} U_4 R_{B_{33}})B_5^* + A_5 A_5^* T_7 + T_5 R_{B_{33}},
\]

\[
U_3 = A_6^* E B_6^* - A_4^* C_6 M^* E B_6^* - A_6^* C_6^{\dagger} E L_B B_6^* - A_6^* S T_2 R_N D_6 B_6^* + L_{A_6} T_4 + T_5 R_{B_{33}},
\]

\[
U_4 = M^* E D_6^* + S^* C_6^{\dagger} E N^* + L_{M_1} T_1 + L_{M_2} R_N + T_3 R_{D_6},
\]

and \(T_1, \ldots, T_8\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.

**Proof.** \((1) \iff (2)\): We separate the real quaternion matrix equations in system (2) into three groups

\[
A_1 X = C_1, \quad X B_1 = D_1,
\]

\[
A_2 X B_2 = C_2, \quad A_3 X B_3 = C_3,
\]

and

\[
A_4 X B_4 = C_4.
\]
It follows from Lemma 2.1 that the system of real quaternion matrix equations (29) is consistent if and only if
\[ R_{A_1} C_1 = 0, D_1 L_{B_1} = 0, A_1 D_1 = C_1 B_1. \] (32)
In this case, the general solution to the system (29) can be expressed as
\[ X = A_1^t C_1 + L_{A_1} D_1 B_1^t + L_{A_1} Y R_{B_1}, \] (33)
where \( Y \) is an arbitrary matrix over \( \mathbb{H} \) with appropriate size. Substituting (33) into (30) and (31) gives
\[
\begin{align*}
A_2 (A_1^t C_1 + L_{A_1} D_1 B_1^t) B_2 &+ A_2 L_{A_1} Y R_{B_1} B_2 = C_2, \\
A_3 (A_1^t C_1 + L_{A_1} D_1 B_1^t) B_3 &+ A_3 L_{A_1} Y R_{B_1} B_3 = C_3
\end{align*}
\] (34)
and
\[
\begin{align*}
A_4 (A_1^t C_1 + L_{A_1} D_1 B_1^t) B_4 &+ A_4 L_{A_1} Y R_{B_1} B_4 = C_4,
\end{align*}
\] i.e.,
\[
\begin{align*}
A_{11} Y B_{11} &= C_{11}, \\
A_{22} Y B_{22} &= C_{22},
\end{align*}
\] (36)
and
\[
A_{33} Y B_{33} = C_{33},
\] (37)
where \( A_{ii}, B_{ii}, C_i \) are defined in (6). Hence, the system (2) is consistent if and only if the matrix equations (36) and (37) are consistent, respectively. By Lemma 2.2, we know that the system of real quaternion matrix equations (36) is consistent if and only if
\[ R_{A_2} C_2 = 0, R_{A_1} C_1 = 0, C_{11} L_{B_{11}} = 0, R_{A_{22}} C_{22} = 0, C_{22} L_{B_{22}} = 0. \] (38)
In this case, the general solution to the system of real quaternion matrix equations (36) can be expressed as
\[
Y = A_{11}^t C_{11} B_{11}^t + L_{A_{11}} A_{11}^t C_{11} B_{11}^t - L_{A_{11}} A_{11} A_{22} D_{11}^t R_{A_1} C_{11} B_{11}^t + D_{11}^t R_{A_1} C_{11} B_{11}^t + L_{A_{11}} U_{11} U_{11}^t R_{B_{11}},
\] (39)
where \( A, B, C, D \) are defined in (7), \( U_{11}, U_{22}, U_{33}, \) and \( U_4 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes. It follows from Lemma 2.2 that the real quaternion matrix equation (37) is consistent if and only if
\[ R_{A_3} C_{33} = 0, C_{33} L_{B_{33}} = 0. \] (40)
In this case, the general solution to the real quaternion matrix equation (37) can be expressed as
\[ Y = A_{33}^t C_{33} B_{33}^t - L_{A_{33}} U_5 - U_6 R_{B_{33}}, \] (41)
where \( U_5 \) and \( U_6 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes. Equating \( Y \) in (39) and \( Y \) in (41) gives
\[
\begin{align*}
A_{11}^t C_{11} B_{11}^t + L_{A_{11}} A_{11}^t C_{11} B_{11}^t - L_{A_{11}} A_{11} A_{22} D_{11}^t R_{A_1} C_{11} B_{11}^t + D_{11}^t R_{A_1} C_{11} B_{11}^t &+ L_{A_{11}} U_{11} U_{11}^t R_{B_{11}}, \\
A_{33}^t C_{33} B_{33}^t - L_{A_{33}} U_5 &- U_6 R_{B_{33}},
\end{align*}
\]
i.e.,
\[ A_5 \begin{pmatrix} U_1 \\ U_5 \end{pmatrix} + (U_2 \ U_6) B_5 + L_{A_{11}} U_3 R_{B_{22}} + L_{A_{11}} U_4 R_{B_{33}} = C_5, \] (42)
where \(A_5, B_5, C_5\) are defined in (8) and (9). Now we want to solve the real quaternion matrix equation (42). It follows from Lemma 2.3 that the real quaternion matrix equation (42) is consistent if and only if

\[
R_M R_A E = 0, \quad E L_{B_i} L_N = 0, \quad R_{A_i} E L_{D_k} = 0, \quad R_{C_i} E L_{B_k} = 0,
\]

where \(A_6, B_6, C_6, D_6, E, M, N, S\) are defined in (10) and (11). In this case, the general solution to the real quaternion matrix equation (42) can be expressed as

\[
\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = A_5^T C_5 - L_{A_1} U_3 R_{B_2} - L_{A_2} U_4 R_{B_1}, \quad A_5^T T_7 B_5 + L_{A_5} T_6,
\]

(44)

\[
(U_2, U_6) = R_{A_1}(C_5 - L_{A_1} U_3 R_{B_2} - L_{A_2} U_4 R_{B_1} B_5^T + A_5 A_5^T T_7 + T_8 R_{B_6},
\]

(45)

\[
U_3 = A_5^T E B_6^T - A_5^T C_e M^T E B_6^T - A_5^T S C_e^T E L_{B_2} T_5 D_k B_5^T + A_5^T S T_2 R_{D_6} B_5^T + L_{A_5} T_4 + T_5 R_{B_6},
\]

(46)

\[
U_4 = M^T E D_6^T + S^T S C_e^T E N^T + L_{A_5} L_3 T_1 + L_{A_5} T_2 R_N + T_3 R_{D_6},
\]

(47)

and \(T_1, \ldots, T_8\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.

(2) \(\iff\) (3) : It follows from Lemma 2.4 that

\[
R_{A_1} C_1 = 0 \iff r(C_1, A_1) = r(A_1), \quad D_1 L_{B_1} = 0 \iff r(B_1, D_1) = r(B_1).
\]

(48)

Hence, (12) \(\iff\) (15). Then, the real quaternion matrix equations (29) has a solution, say \(X_0\). So we have

\[
A_1 X_0 = C_1, \quad X_0 B_1 = D_1.
\]

(49)

Now we want to prove (13) \(\iff\) (16) and (17). Note that

\[
R_{A_1} C_{11} = 0 \iff r(A_{11}, C_{11}) = r(A_{11}) \iff r(A_2 L_{A_1}, C_{11}) = r(A_{11})
\]

\[
\iff r\begin{pmatrix} C_{11} & A_2 \\ 0 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} \iff r\begin{pmatrix} C_{2} - A_2 X_0 B_2 & A_2 \\ 0 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}
\]

\[
\iff r\begin{pmatrix} C_2 \\ A_1 X_0 B_2 \\ A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} \iff r\begin{pmatrix} C_2 \\ C_1 B_2 \\ A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}.
\]

Similarly, we can prove

\[
R_{A_2} C_{22} = 0 \iff r\begin{pmatrix} C_3 \\ C_1 B_3 \\ A_1 \end{pmatrix} = r\begin{pmatrix} A_3 \\ A_1 \end{pmatrix},
\]

\[
R_{A_2} C_{33} = 0 \iff r\begin{pmatrix} C_4 \\ C_1 B_4 \\ A_1 \end{pmatrix} = r\begin{pmatrix} A_4 \\ A_1 \end{pmatrix},
\]

\[
C_{ii} L_{B_i} = 0 \iff r\begin{pmatrix} A_{i+1} D_{i} \\ A_{i+1} B_i \\ B_{i+1} \end{pmatrix} = r(B_i, A_{i+1}), \quad (i = 1, 2, 3).
\]
We now pay attention to $RACL_B = 0$. Note that

$$A_{11}YB_{11} = C_{11}$$

has a special solution $Y_0$

$$Y_0 = A_{11}^tC_{11}B_{11}^t.$$  

Then we have

$$A_{11}Y_0B_{11} = C_{11}.$$ 

(50)

It follows from Lemma 2.4 and (50) that

$$RACL_B = 0 \iff r\begin{pmatrix} C & A \\ B & 0 \end{pmatrix} = r(A) + r(B)$$

$$\iff r\begin{pmatrix} C & A_{22}L_{A_{11}} \\ B_{22} & A_{11} \end{pmatrix} = r(A_{22}L_{A_{11}}) + r(R_{B_{11}}B_{22})$$

$$\iff r\begin{pmatrix} C & A_{22} \\ B_{22} & 0 \\ 0 & A_{11} \end{pmatrix} = r(A_{22}) + r(B_{11}, B_{22})$$

$$\iff r\begin{pmatrix} C_{22} - A_{22}Y_0B_{22} & A_{22} \\ B_{22} & 0 \\ 0 & A_{11} \end{pmatrix} = r(A_{22}) + r(B_{11}, B_{22})$$

$$\iff r\begin{pmatrix} C_{22} & A_{22} \\ B_{22} & 0 \\ 0 & A_{11} \end{pmatrix} = r(A_{22}) + r(B_{11}, B_{22})$$

$$\iff r\begin{pmatrix} -C_2 + A_2X_0B_2 & A_2L_{A_1} \\ R_{B_2}B_2 & 0 \\ 0 & A_3L_{A_1} \end{pmatrix} = r(A_{22}L_{A_{11}}) + r(R_{B_1}B_2, R_{B_1}B_3)$$

$$\iff r\begin{pmatrix} -C_2 & A_2 \\ R_{B_2}B_2 & 0 \\ 0 & A_3 \end{pmatrix} = r(A_2L_{A_1}) + r(B_{11}, B_{22}, B_{33})$$

(17).

Similarly, we can prove

$$R_MT_AE = 0 \iff (18), \ ETL_{B_1}L_{C_1} = 0 \iff (19).$$

$$R_{AE}EL_{D_1} = 0 \iff (20), \ CR_{C_1}EL_{B_1} = 0 \iff (21).$$

Now we give an example to illustrate Theorem 3.1.
Example 3.2. Let

\[ A_1 = \begin{pmatrix} 1 + j & i - j & 1 + i + k \\ -1 - j & -i + j & -1 - i - k \end{pmatrix}, \quad B_1 = \begin{pmatrix} j - k & 1 \\ 1 + 2k & j \end{pmatrix}, \]

\[ C_1 = \begin{pmatrix} 1 + 3i + 3j - k & i + j - k & -1 + i - 3j + k \\ -2 - 4i - j + k & 1 - 2i - j + k & 2j - 2k \end{pmatrix}, \quad D_1 = \begin{pmatrix} i + k & -1 - j + k \\ 2 - 3k & 2 + i - k \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} j + k & 1 + 2i + j & 1 - i \\ i & k & 1 + j \end{pmatrix}, \quad B_2 = \begin{pmatrix} i - j + k & j \\ 1 + k & i \\ 2i & k \end{pmatrix}, \]

\[ A_3 = \begin{pmatrix} 1 + j + k & 2 + j - k & i + k \\ -1 - j - k & -2 - j + k & -1 - k \end{pmatrix}, \quad B_3 = \begin{pmatrix} 2 - 3i + k & i + k \\ i - k & -k \\ 1 + j & j \end{pmatrix}, \]

\[ A_4 = \begin{pmatrix} j - 2k & i + k & 1 \\ i & j & i \end{pmatrix}, \quad B_4 = \begin{pmatrix} i + j & k \\ 1 + 2i + k & 1 - j \\ 1 - i + k & -j \end{pmatrix}. \]

Now we consider the system of real quaternion matrix equations (2). Check that

\[ r(A_1, C_1) = r(A_1) = 2, \quad r \left( \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} \right) = r(B_1) = 2, A_1 D_1 = C_1 B_1, \]

\[ r \left( \begin{pmatrix} C_{i+1} \\ A_{i+1} \\ A_i \\ B_i \end{pmatrix} \right) = r \left( \begin{pmatrix} A_{i+1} \\ A_i \\ B_i \end{pmatrix} \right) = 3, \quad r \left( \begin{pmatrix} A_{i+1} D_1 \\ B_1 \\ C_{i+1} \end{pmatrix} \right) = r(B_1, B_{i+1}) = 3, \quad (i = 1, 2, 3), \]

\[ r \left( \begin{pmatrix} -C_2 \\ A_2 \\ 0 \\ 0 \\ B_3 \\ B_1 \\ 0 \\ A_3 \\ C_3 \\ A_2 D_4 \\ 0 \\ A_1 \\ C_1 B_3 \\ C_1 B_1 \end{pmatrix} \right) = r \left( \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ B_2 \\ B_3 \\ B_4 \\ B_1 \\ C_2 \\ 0 \\ 0 \\ C_3 \\ 0 \\ C_4 \\ 0 \\ -C_1 B_2 \\ 0 \\ -C_1 B_4 \\ 0 \\ A_1 \\ 0 \\ C_1 B_4 \\ 0 \end{pmatrix} \right) + r(B_1, B_2, B_3, B_4) = 6, \]

\[ r \left( \begin{pmatrix} A_2 \\ A_3 \\ 0 \\ A_4 \\ 0 \\ A_1 \\ 0 \\ A_1 \\ 0 \\ A_1 \end{pmatrix} \right) + r(B_1, B_2, B_3, B_4) = 9, \]

\[ r \left( \begin{pmatrix} 0 \\ B_2 \\ B_3 \\ 0 \\ B_1 \\ 0 \\ 0 \\ 0 \\ B_2 \\ 0 \\ B_4 \\ 0 \\ B_1 \\ 0 \\ A_2 \\ -C_2 \\ 0 \\ 0 \\ -A_2 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ C_4 \\ 0 \\ 0 \\ A_1 \end{pmatrix} \right) = r \left( \begin{pmatrix} B_2 \\ B_3 \\ 0 \\ B_1 \\ 0 \\ 0 \\ 0 \\ B_2 \\ 0 \\ B_4 \\ 0 \\ B_1 \\ 0 \\ A_2 \\ -C_2 \\ 0 \\ 0 \\ -A_2 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ C_4 \\ 0 \\ 0 \\ A_1 \end{pmatrix} \right) + r \left( \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \right) = 9, \]
In this case, the general solution to system (51) can be expressed as
\[
\begin{pmatrix}
0 & B_2 & B_4 & B_1 \\
A_2 & -C_2 & 0 & 0 \\
A_4 & 0 & C_4 & A_4 D_1 \\
A_1 & -C_1 B_2 & 0 & 0
\end{pmatrix}
= r \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} + r(B_1, B_2, B_4) = 6,
\]
\[
\begin{pmatrix}
0 & B_3 & B_4 & B_1 \\
A_3 & -C_3 & 0 & 0 \\
A_4 & 0 & C_4 & A_4 D_1 \\
A_1 & -C_1 B_3 & 0 & 0
\end{pmatrix}
= r \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} + r(B_1, B_3, B_4) = 6.
\]
Hence, the system of real quaternion matrix equations (2) is consistent.

Now we consider some special cases of the system (2). Let \(A_1, B_1, C_1, D_1\) vanish in Theorem 3.1. Then we can give solvability conditions and general solution to the system

\[
\begin{align*}
A_{11} X B_{11} &= C_{11}, \\
A_{22} X B_{22} &= C_{22}, \\
A_{33} X B_{33} &= C_{33}.
\end{align*}
\]

He and Wang considered the system (51) over complex field ([3]).

**Corollary 3.3.** Let \(A_{ii}, B_{ij}, C_{ij}\) be known over \(\mathbb{H}\), and \(X\) be unknown, \((i, j) = (1, 2, 3)\). Set

\[
A = A_{22} L_{A_{11}}, B = R_{B_{11}} B_{22}, C = C_{22} - A_{22} A_{11}^t B_{11} L_{B_{22}}, D = R_{A_{11}} A_{22},
\]

\[
A_5 = (L_{A_{11}}, L_{A_{22}}), B_5 = \begin{pmatrix} R_B R_{B_{11}} \\ R_{B_{22}} \end{pmatrix},
\]

\[
C_5 = A_{33} A_{11}^t B_{33}^t - A_{11}^t C_{11} B_{11}^t - L_{A_{11}} A_{22}^t C_{22} B_{22}^t + L_{A_{22}} A_{11}^t C_{22} D_B^t R_A C_{22} B_{22}^t - D_A^t R_A C_{22} B_{22}^t R_B,
\]

\[
A_6 = R_{A_{11}} L_{A_{11}}, B_6 = R_{B_{22}} L_{B_{22}}, C_6 = R_{A_{11}} L_{A_{22}}, D_6 = R_{A_{11}} L_{B_{22}},
\]

\[
E = R_{A_{11}} C_5 L_{B_{11}}, M = R_{A_{11}} C_6, N = D_6 L_{B_{22}}, S = C_6 M.
\]

Then the system of real quaternion matrix equations (51) is consistent if and only if

\[
R_{A_{ii}} C_{ii} = 0, C_{ii} L_{B_{ii}} = 0, \quad (i = 1, 2, 3), \quad R_A C L_B = 0,
\]

\[
R_M R_A E = 0, EL_{B_{ii}} L_N = 0, R_{A_{ii}} E L_{D_{ii}} = 0, R_{C_{ii}} E L_{B_{ii}} = 0.
\]

In this case, the general solution to system (51) can be expressed as

\[
X = A_{11}^t C_{11} B_{11}^t + L_{A_{11}} A_{22}^t C_{22} B_{22}^t - L_{A_{22}} A_{11}^t C_{22} D_B^t R_A C_{22} B_{22}^t + D_A^t R_A C_{22} B_{22}^t R_B + L_{A_{11}} L_{A_{22}} U_1 + U_2 R_B R_{B_{11}} + L_{A_{11}} U_3 R_{B_{22}} + L_{A_{22}} U_4 R_{B_{33}},
\]

or

\[
X = A_{33}^t C_{33} B_{33}^t - L_{A_{33}} U_5 - U_6 R_{B_{33}},
\]

where

\[
\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = A_{5}^t(C_5 - L_{A_{11}} U_3 R_{B_{22}} - L_{A_{22}} U_4 R_{B_{33}}) - A_5^t T_7 B_5 + L_{A_{11}} T_6,
\]
Let $A, B, C_1, C_2$ be known over $H$, and $X$ be unknown. Set
\[
A_{ii} = A_{i+1}L_{Ai}, B_{ii} = R_{Bi}, C_{ii} = C_{i+1} - A_{i+1}(A_{i}C_{i} + L_{Ai}D_{i}B_{i+1}), (i = 1, 2),
\]

\[
A = A_{22}L_{A_{11}}, B = R_{B_{11}}B_{22}, C = C_{22} - A_{22}A_{11}B_{11}B_{22}, D = R_{A_{11}}A_{22}.
\]

Then the following statements are equivalent:
(1) The system of real quaternion matrix equations (52) is consistent.
(2) \[
R_{Ai}C_1 = 0, D_1L_{B_1} = 0, A_1D_1 = C_1B_1,
\]

\[
R_{Ai}C_2 = 0, C_{ii}L_{B_i} = 0, (i = 1, 2), R_{A_{i1}}C_{i1} = 0.
\]

(3) \[
r(A_1, C_1) = r(A_1), r\left(\begin{array}{c}
B_1 \\
D_1
\end{array}\right) = r(B_1), A_1D_1 = C_1B_1,
\]

\[
r\left(\begin{array}{c}
C_{i+1} \\
A_{i+1}
\end{array}\right)_{B_{i+1}} = r\left(\begin{array}{c}
A_{i+1} \\
B_{i+1}
\end{array}\right), r\left(\begin{array}{c}
A_{i+1} \ A_{i+1} \ C_{i+1} \ D_{i+1}
\end{array}\right)_{B_{i+1}} = r(B_1, B_{i+1}), (i = 1, 2),
\]

\[
r\left(\begin{array}{cccc}
-C_2 & A_2 & 0 & 0 \\
B_2 & 0 & B_3 & B_1 \\
0 & A_3 & C_3 & A_3D_1 \\
0 & A_1 & C_1B_3 & C_1B_1
\end{array}\right) = r\left(\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_3
\end{array}\right) + r(B_1, B_2, B_3).
\]

In this case, the general solution to system (52) can be expressed as
\[
X = A^*_{11}C_1 + L_{A_{11}}D_{1}B_{1}^* + L_{A_{11}}YR_{B_{11}},
\]

where
\[
Y = A^*_{11}C_{11}B_{11}^* + L_{A_{11}}A^*_{11}CB_{11}^* - L_{A_{11}}A^*_{11}A_{22}D^*R_{A_{11}}CB_{22}^* + D^*R_{A_{11}}CB_{11}^* R_{B_{11}},
\]

\[
+ L_{A_{11}}L_{A_{11}}U_1 + U_2R_{B_{11}}L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}},
\]

where $U_1, U_2, U_3,$ and $U_4$ are arbitrary matrices over $H$ with appropriate sizes.
4. The $\eta$-Hermitian solution to system of real quaternion matrix equations (1)

In this section, we consider the general $\eta$-Hermitian solution to system of real quaternion matrix equations (1).

**Theorem 4.1.** Let $A_1, C_1, A_2, A_3, A_4, C_2 = C_2^{op}, C_3 = C_3^{op}, C_4 = C_4^{op}$ be known over $\mathbb{H}$, and $X = X^\eta$ be unknown. Set

$$
A_i = A_i L_{A_i}, \quad C_i = C_i - A_i (A_i^\dagger C_i + L_{A_i} C_i^\dagger (A_i^\dagger)^\eta) A_i, \quad (i = 1, 2, 3),
$$

$$
A = A_{22} L_{A_{21}}, \quad C = C_{22} - A_{22} A_{11}^\dagger C_{11} (A_{11}^\dagger)^\eta A_{22}^\eta, \quad D = R_{A_{21}} A_{22},
$$

$$
A_5 = (L_{A_{41}} L_{A}, \quad L_{A_{43}}), A_6 = R_{A_4} L_{A_{41}}, B_6 = R_{A_4}^\eta L_{A_4},
$$

$$
C_5 = A_{33}^\dagger C_{33} (A_{33}^\dagger)^\eta - A_{11}^\dagger C_{11} (A_{11}^\dagger)^\eta - L_{A_{41}} A_4^\dagger C (A_{42}^\eta)^\eta + L_{A_{41}}^\eta A_4^\dagger A_{22}^\dagger D_4^\dagger A_4 (C_{42}^\eta)^\eta - D_4^\dagger R_{A_4} C (A_4)^\eta R_{A_4},
$$

$$
E = R_{A_4} C_5 L_{A_4}, M = R_{A_4} B_4^\eta, N = A_6^\eta L_{R_3}, S = B_6^\eta L_M.
$$

Then the following statements are equivalent:

(1) The system of real quaternion matrix equations (1) has an $\eta$-Hermitian solution.

(2)

$$
R_{A_1} C_1 = 0, \quad A_1 C_1^\eta = C_1 A_1^\eta, \quad R_{A_3} C_3 = 0, \quad (i = 1, 2, 3),
$$

$$
R_{A_2} C_{22} = 0, \quad R_{A_4} R_{A_3} E = 0, \quad R_{A_4} E L_{A_4}^\eta = 0.
$$

(3)

$$
r(A_1, C_1) = r(A_1), \quad A_1 C_1^\eta = C_1 A_1^\eta,
$$

$$
r \begin{pmatrix} C_{i+1} & A_{i+1} \\ C_i & A_i \end{pmatrix} = r \begin{pmatrix} A_{i+1} \\ A_i \end{pmatrix}, \quad (i = 1, 2, 3),
$$

$$
r \begin{pmatrix} -C_2 & A_2 & 0 & 0 \\ 0 & A_3 & C_3 & A_3 C_1 \\ 0 & A_1 & C_1 A_1^\eta & C_1 A_1^\eta \end{pmatrix} = 2r \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},
$$

$$
r \begin{pmatrix} 0 & 0 & A_1^\eta & A_1^\eta & A_1^\eta \\ A_2 & A_2 & 0 & 0 & 0 \\ A_3 & 0 & 0 & C_3 & 0 \\ 0 & A_4 & 0 & C_4 & 0 \\ A_1 & 0 & -C_1 A_2^\eta & 0 & -C_1 A_4^\eta \end{pmatrix} = r \begin{pmatrix} A_2 & A_2 \\ A_3 & 0 \\ A_1 & 0 \\ A_4 & A_4 \end{pmatrix} + r \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}.
$$
In this case, the general $\eta$-Hermitian solution to system (1) can be expressed as

$$X = \frac{X + \bar{X}}{2},$$

where

\[
\bar{X} = A^*_1 C_1 + L_3 C^*_1 (A^*_1)^\eta + L_4 Y R_4^\eta,
\]

$$Y = A^*_1 C_1 (A^*_1)^\eta + L_3 A^*_1 C (A^*_1)^\eta - L_3 A^*_1 A^*_1 D^\eta R_4 C (A^*_1)^\eta + D^\eta R_4 C (A^*_1)^\eta R_4^\eta
+ L_3 A^*_1 U_1 + U_2 A^*_1 A^*_1 R_4^\eta + L_4 A^*_1 U_3 R_4^\eta + L_5 A^*_1 U_4 R_4^\eta,$$

or

$$Y = A^*_{13} C_3 (A^*_1)^\eta - L_3 A^*_1 U_5 - U_6 R_4^\eta,$$

\[
\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = A^*_1 (C_3 - L_3 A^*_1 U_3 R_4^\eta - L_5 A^*_1 U_4 R_4^\eta) - A^*_3 T_7 A^*_3 + L_5 T_8,
\]

\[
(U_2, U_6) = R_4 (C_3 - L_3 A^*_1 U_3 R_4^\eta - L_5 A^*_1 U_4 R_4^\eta)(A^*_1)^\eta + A_5 A^*_5 T_8 + T_9 R_4^\eta,
\]

\[
U_3 = A^*_1 B_1^\eta - A^*_1 B_1^\eta M^\eta B_1^\eta - A^*_1 S(B_1^\eta)^\eta Y L^\eta N^\eta A^*_1 B_1^\eta
- A^*_1 S T_2 R_5 A^*_1 B_1^\eta + L_4 T_4 + T_5 R_6,
\]

\[
U_4 = M^\eta A^*_1 B_1^\eta + S^\eta S(B_1^\eta)^\eta Y N^\eta L + L_5 T_2 R_5 + T_3 R_4^\eta
\]

and $T_1, \ldots, T_9$ are arbitrary matrices over $H$ with appropriate sizes.

**Proof.** We first prove that the system of real quaternion matrix equations (1) has an $\eta$-Hermitian solution if and only if the system of real quaternion matrix equations

\[
\begin{pmatrix} A_1 \bar{X} \\ \bar{X} A_1^\eta \\ A_2 \bar{X} A_2^\eta \\ \bar{X} A_3^\eta \\ A_4 \bar{X} A_4^\eta \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2^\eta \\ C_3 \\ C_4 \end{pmatrix}
\]

has a solution $\bar{X}$. If the system of real quaternion matrix equations (1) has an $\eta$-Hermitian solution, say $X_0$, then the system (53) clearly has a solution $\bar{X} = X_0$. Conversely, if the system (53) has a solution $\bar{X}$, then

$$X = \frac{\bar{X} + \bar{X}^\eta}{2}$$

is an $\eta$-Hermitian solution to (1). We can derive the solvability conditions to the system of real quaternion matrix equations (1) by Theorem 3.1.
Example 4.2. Let

\[
A_1 = \begin{pmatrix} i + j - k & 1 + i + j + 2k \\ i + j + 2k & 1 - j - 2k \\ k & 2 + i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & i + j & i + k \\ 1 + i & -1 + i + j + k & -1 + i + j + k \\ i & -1 + k & -1 - j \end{pmatrix},
\]

\[
A_3 = \begin{pmatrix} i + j + k & 2i + 2j - k \\ i + j & -i + k & 1 \\ i + 2j + k & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2i + j & -i + k \\ -2 & 1 & 1 \end{pmatrix},
\]

\[
C_1 = \begin{pmatrix} 2 + 2i - j - k & 2i + j - 3k & -1 + 2i - 2j + 2k \\ 0 & -2 - j + 5k & -1 + 2j + k \\ 2 & 2 + 2i - j - k & -2 + 2i + 2k & -1 + i + 3k \end{pmatrix},
\]

\[
C_2 = C_2^r = \begin{pmatrix} 1 & -i - 4i & 3 - 5i & 4 - i \\ 4 & -i & 5 + 3i & 1 + 4i \end{pmatrix},
\]

\[
C_3 = C_3^r = \begin{pmatrix} 8 - 2i - k & -1 + 3i - j - 8k & 7 + i - j - 9k \\ 4 - i + j + 9k & 3 + 7i - j - 5k & 10 + 8i - 14k \\ 7 + i - j - 9k & 3 + 7i - j - 5k & 10 + 8i - 14k \end{pmatrix},
\]

\[
C_4 = C_4^r = \begin{pmatrix} -1 - 11i & 12 - 3i - 2k & -5 - 2i + 5j - 2k \\ 12 - 3i - 2k & 6 + 11i - k & 3 - 5i + 3j + 3k \\ -5 - 2i - 5j - 2k & 3 - 5i - 3j + 3k & -2 + 2i - 6k \end{pmatrix}.
\]

Now we consider the system (1) where \( X \) is \( j \)-Hermitian. Check that

\[
r(A_1, C_1) = r(A_1) = 2, \quad A_1 C_1^{r^*} = C_1 A_1^{r^*},
\]

\[
r\left(\begin{pmatrix} C_{i+1} & A_{i+1} \\ C_1 A_1^{r^*} & A_1 \end{pmatrix}\right) = r\left(\begin{pmatrix} A_{i+1} \\ A_1 \end{pmatrix}\right) = 3, \quad (i = 1, 2, 3),
\]

\[
r\left(\begin{pmatrix} A_2 \\ A_3^{r^*} \\ 0 \\ 0 \\ A_1 \end{pmatrix}\right) = 2r(A_2) + r(A_3) = 6,
\]

\[
r\left(\begin{pmatrix} 0 & 0 & A_2^{r^*} \\ 0 & -C_2 & 0 \\ 0 & 0 & 0 \\ A_3 & 0 & 0 \\ 0 & A_4 & 0 \\ A_1 & -C_1 A_2^{r^*} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = r\left(\begin{pmatrix} A_2 \\ A_3 \\ A_4 \\ A_1 \end{pmatrix}\right) + r\left(\begin{pmatrix} A_2 \\ A_3 \\ A_4 \\ A_1 \end{pmatrix}\right) = 9,
\]

\[
r\left(\begin{pmatrix} 0 & A_2^{r^*} \\ A_2 & -C_2 \\ 0 & 0 \\ A_3 & 0 \\ 0 & A_4 & 0 \\ A_1 & 0 \\ 0 \end{pmatrix}\right) = 2r(A_2) = 6.
\]

Hence, the system (1) has a \( j \)-Hermitian solution.
Let $A_1$ and $C_1$ vanish in Theorem 4.1. Then we obtain some necessary and sufficient conditions for the existence of an $\eta$-Hermitian solution to the following system of real quaternion matrix equations

\[
\begin{cases}
A_{11}X A_{11}^{\eta} = C_{11}, \\
A_{22}X A_{22}^{\eta} = C_{22}, \\
A_{33}X A_{33}^{\eta} = C_{33}.
\end{cases}
\]

(54)

We can also give the general $\eta$-Hermitian solution to the system (54).

**Corollary 4.3.** Let $A_{ii}, C_{ii} = C_{ii}^\eta$ be known over $\mathbb{H}$, and $X = X^\eta$ be unknown, $(i = 1, 2, 3)$. Set

\[
A = A_{22}L_{A_{11}}, C = C_{22} - A_{22}A_{11}^t C_{11}(A_{11}^t)^\eta A_{22}^\eta, D = R_{A_{11}} A_{22},
\]

\[
A_5 = (L_{A_{11}} L_A, L_{A_{11}}), A_6 = A_{11} L_{A_{11}}, B_6 = A_{22}^\eta L_{A_{11}}^\eta,
\]

\[
C_5 = A_{33}^t C_{33}(A_{33}^t)^\eta - A_{11}^t C_{11}(A_{11}^t)^\eta - L_{A_{11}} A_{22}^t D_A^t R_A C(A_{22}^t)^\eta + D_A^t R_A C(A^t)^\eta R_A^\eta A_{11}^\eta + L_{A_{11}} L_A U_1 + U_2 R_A^\eta R_A_{11}^\eta + L_{A_{11}} U_3 R_A_{11}^\eta + L_{A_{11}} U_4 R_A_{11}^\eta,
\]

\[
E = R_{A_{11}} C_{33} L_{A_{11}}^\eta, M = R_{A_{11}} B_6^\eta, N = A_6^\eta L_{R_6} + B_6^\eta L_M.
\]

Then the system of real quaternion matrix equations (54) has an $\eta$-Hermitian solution if and only if

\[
R_{A_{11}} C_{ii} = 0, (i = 1, 2, 3), R_A C_{A^\eta} = 0, R_M R_A E = 0, R_{A_{11}} E_L A_{11}^\eta = 0.
\]

In this case, the general $\eta$-Hermitian solution to system (54) can be expressed as

\[
X = \frac{\bar{X} + \bar{X}^\eta}{2},
\]

where

\[
\bar{X} = A_{33}^t C_{33}(A_{33}^t)^\eta - L_{A_{11}} U_5 - U_6 R_A_{11}^\eta,
\]

\[
\begin{pmatrix}
U_1 \\
U_6
\end{pmatrix} = A_5^2 (C_5 - L_{A_{11}} U_3 R_A_{11}^\eta - L_{A_{11}} U_4 R_A_{11}^\eta) - A_5^2 T_7 A_5^\eta + L_{A_{11}} T_6,
\]

\[
\begin{pmatrix}
U_2 \\
U_6
\end{pmatrix} = R_{A_{11}} (C_5 - L_{A_{11}} U_3 R_A_{11}^\eta - L_{A_{11}} U_4 R_A_{11}^\eta)(A_{11}^t)^\eta + A_5 A_5^t T_7 + T_8 R_A_{11}^\eta,
\]

\[
U_3 = A_6^t E B_6 - A_6^t B_6^\eta M^t E B_6^t - A_6^t S(B_6^t)^\eta E A_{11}^\eta B_6^t - A_5^t S T_2 R_N A_{11}^\eta B_6^t + L_{A_{11}} T_4 + T_5 R_N,
\]

\[
U_4 = M^t E(A_6^\eta)^\eta + S^t S(B_6^t)^\eta E T_1 + M T_2 R_N + T_3 R_A_{11}^\eta,
\]

and $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.
5. Conclusions

We have presented necessary and sufficient conditions for the existence and the general solution to the system of real quaternion matrix equations (2). As an application of the system (51), we have also given necessary and sufficient conditions for the existence and the general $\eta$-Hermitian solution to the system of real quaternion matrix equations (1). Some numerical examples are presented to illustrate the results.

6. Acknowledgement

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References