Filomat 36:10 (2022), 3331–3354
https://doi.org/10.2298/FIL2210331Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Rough Set Analysis of Graphs

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Abstract. Relational data has become increasingly important in decision analysis in recent years, and so mining knowledge which preserves relationships between objects is an important topic. Graphs can represent the knowledge which contains objects and relationships between objects. Rough set theory provides an effective tool for extracting knowledge, but it is not sufficient to extract the knowledge containing the data on relationships between objects. In order to extend the application scope and enrich the rough set theory, it is essential to develop a rough set analysis of graphs. This extension is important because graphs play a crucial role in social network analysis. In this paper, the rough set analysis of graphs based on general binary relations is investigated. We introduce three types of approximation operators of graphs: vertex graph approximation operators, edge graph approximation operators, and graph approximation operators. Relationships between approximation operators of graphs and approximation operators of sets are presented. Then we investigate the approximation operators of graphs within constructive and axiomatic approaches.

1. Introduction

Rough set theory [17, 18] as a formal tool for representing and dealing with uncertain knowledge information in database has been applied in knowledge discovery [19, 20], machine learning [13], and decision analysis [4, 8, 22], etc.

The rough set theory brings about lower and upper approximation operators, and the core idea consists in approximating an incomplete or inexact concept with a pair of complete or exact concepts—its lower and upper approximations. With the development of rough set theory, for lower and upper approximation operators, there are mainly two definition approaches—the constructive approach and the axiomatic approach. In the constructive approach, the notions of approximation operators are extended to general binary relations [33, 44], neighborhood systems [11], coverings [30, 42, 43], algebras [26], etc. In the axiomatic approach, the primitive notions are the abstract lower and upper approximation operators which are characterized by a set of axioms. In the crisp environment, the most important axiomatic studies were made by Yao and Lin [34, 36], in which various classes of crisp rough set algebras were characterized by different sets of axioms. Recently, the research of the axiomatic approach has also been extended to fuzzy

2020 *Mathematics Subject Classification*. Primary 68T30, 68T37
Keywords. Rough sets, generalized approximation space, lower and upper approximations, axioms
Received: 27 April 2020; Accepted: 21 February 2022
Communicated by Dragan S. Djordjević
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environment [12, 21, 28, 29, 31], and furthermore, in the complete residuated lattice [1, 32]. The research on axiomatic characterizations for the covering based approximation operators is also an important topic [39].

In the above, both the lower and upper approximations of a subset of universe are subsets of universe, which overlook the relationships between objects although the constructions of approximations are based on the relation. Scott [25] pointed out that "Relational data, on the other hand, are the contacts, ties and connections, the group attachments and meetings, which relate one agent to another and so cannot be reduced to the properties of the individual agents themselves". For example, in a demographic database, it is more natural to represent the parent–child relationship as a relation between individuals, instead of an attribute of the parent or the child.

Moreover, in the problem of graph pattern matching, the personalized graph pattern matching via limited simulation [5] also showed that relation between nodes (objects) cannot be reduced to the properties of the nodes (objects), which implies the necessity of the relation between nodes (objects). In order to clarify the motivation of this study more precisely, at first, we recall the definition of personalized graph pattern matching via limited simulation. In the following definition, we assume that $G = (V, E, f_V, f_E)$ is a data graph [14], where V is a finite set of nodes, $E \subseteq V \times V$ is a finite set of edges in which $(v, v') \in E$ denotes an edge from nodes v to v', f_V is a function that maps each node $v \in V$ to a node label $f_V(v)$ in Σ_V, and f_E is a function that maps each edge $e \in E$ to an edge label $f_E(e)$ in Σ_E. We say that the graph $G' = (V', E', f_{V'}, f_{E'})$ is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$, $f_{V'}(v) = f_V(v)$ for each $v \in V'$, and $f_{E'}(e) = f_E(e)$ for each edge $e \in E'$. A pattern graph is a directed node-weighted graph $P = (V_P, E_P, f_{V_P}, f_{E_P}, w)$, where $(V_P, E_P, f_{V_P}, f_{E_P})$ is a data graph and w is a weighted function for nodes, which maps each node $u \in V_P$ to an element in $\mathbb{N} \cup \{\infty\}$. Here, by $\mathbb{N}$ we denote the set of natural numbers. Let $k \in \mathbb{N}$ and $G_i = (V_i, E_i, f_{V_i}, f_{E_i})$ be a graph, where $i = 1, 2$. We say that $v_1 \in V_1$ is k-limited similar to $v_2 \in V_2$ [5], denoted by $v_1 \gtrsim_k v_2$, if the following hold:

(1) $f_{V_1}(v_1) = f_{V_2}(v_2)$ when $k = 0$;
(2) $v_1 \gtrsim_0 v_2$ and for each edge $e_2 = (v_2, v_2') \in E_2$, there exists an edge $e_1 = (v_1, v_1') \in E_1$ such that $f_{E_1}(e_1) =$, $f_{E_2}(e_2)$, $v_1 \gtrsim_{k-1} v_2$ and $v_1' \gtrsim_{k-1} v_2'$ when $k > 0$.

Then we give the definition of personalized graph pattern matching via limited simulation.

Definition 1.1. [5]. Let $P = (V_P, E_P, f_{V_P}, f_{E_P}, w)$ be a pattern graph and $G = (V, E, f_V, f_E)$ be a data graph. We say that G matches P via limited simulation, denoted by $G \rhd P$, if there exists two relations $S_V \subseteq V_P \times V$ and $S_R \subseteq E_P \times E$ such that:

(1) $V_P = \{u \,|\, (u, v) \in S_V\}$ and $E_P = \{e \,|\, (e, e') \in S_E\}$;
(2) for each pair $(u, v) \in S_V$, $v \gtrsim_{w(u)} u$;
(3) for each pair $(e, e') \in S_E$ with $e = (u, u')$ and $e' = (v, v')$, it holds that $f_{E_P}(e) = f_E(e')$ and $(u, v), (u', v') \in S_V$.

The S_V is called a node matching relation from P to G, and S_R is called an edge matching relation from P to G. Let $V' = \{v \,|\, (u, v) \in S_V\}$ and $E' = \{e' \,|\, (e, e') \in S_E\}$. The graph $G' = (V', E', f_{V|V'}, f_{E|E'})$ is a subgraph of G, called a match in G for P via limited simulation. For nodes $u \in V_P$ and $v \in V$, if $v \gtrsim_{w(u)} u$, then v is called a match in G for u. For edges $e = (u, u') \in E_P$ and $e' = (v, v') \in E$, we call e' a match in G for e if $f_{E_P}(e) = f_E(e')$, $v \gtrsim_{w(u)} u$, and $v' \gtrsim_{w(u')} u'$. It should be noted that the relation S_E can not be reflected by S_V in the problem of personalized graph pattern matching via limited simulation. For more details, including examples, we refer the reader to [5]. From the above, we know that the relation S_R on edge set cannot be neglected, which illustrates the importance of edges. The necessity of S_R also reveals that the relation on edge set is indispensable.

In rough set theory, let (U, R) be a generalized approximation space. For any $x \in U$, the set $R_p(x) = \{y \,|\, (y, x) \in R\}$ is called the predecessor neighborhood of x and $R_s(x) = \{y \,|\, (x, y) \in R\}$ is called the successor neighborhood of x. Based on the predecessor neighborhood, for any subset $V_1 \subseteq U$, the lower and upper approximations of V_1 are $\underline{R}_p(V_1) = \{x \in U \,|\, R_p(x) \subseteq V_1\}$ and $\overline{R}_p(V_1) = \{x \in U \,|\, R_p(x) \cap V_1 \neq \emptyset\}$, respectively. In terms of rough set theory, we can get that $V' = (\overline{S_V})_p(V_P)$ and $E' = (\overline{S_R})_p(E_P)$. By the property of S_V and S_R defined in Definition 1.1, we know that $E' \subseteq V' \times V'$. The above demonstrates that matching graph G' of P can be represented by the ordered pair of upper approximations of V_P and E_P based on the predecessor neighborhood. By the rough set theory, we can also derive the maximal matching graph in G for subgraph $P_1 = (V_{P_1}, E_{P_1})$ of P: $(\overline{S})_p(P_1) = ((\overline{S_U})_p(V_{P_1}), (\overline{S_R})_p(E_{P_1}))$. Sometimes, we would like to obtain the maximal matching graph $G_1'' = (V_1'', E_1'')$ of P in a target subgraph $G_1 = (V_1, E_1)$ of data graph G to narrow the pattern area: $V_1'' = \{v \in V_1 \mid (S_U)_p(v) \cap V_P \neq \emptyset\}$ and $E_1'' = \{e \in E_1 \mid (S_R)_p(e) \cap E_P \neq \emptyset\}$. The above graphs are all based on the upper approximation operators. There are also many important explanations for the graphs based on the lower approximation operators. Here, we omit the details.

Relational data between objects has become increasingly important in decision analysis in recent years. Recently, Fan [6] provided the rough set analysis of relational structures. Besides, there are many examples can be found in social network analysis, where the principal types of data are attribute data and relational data. So it is important to extract the knowledge containing the data on objects and the data on relationships between objects. We know that a graph which is an ordered pair of vertex (object) set and edge (relationship between the objects) set can contain the data on vertices and the data on edges, and hence it represents the more complex knowledge than its vertex set. In practice, graphs with irregular structures naturally occur in social networks [38, 48], knowledge networks [7, 27], and protein networks [15], etc. Many real-world applications involve the analysis of graphs, such as graph classification, node classification, node recommendation, link prediction, and node visualization.

Rough set theory provides an effective tool for extracting knowledge which does not preserve the data on relationships between objects. From the problem of personalized graph pattern matching via limited simulation, we can get the maximal matching graph of P by the predecessor neighborhood based upper approximations of V_P and E_P, which means that we can gain the knowledge represented by graphs by rough set theory. Although this is a significant application of rough set theory, the rough set analysis of graphs is rarely studied. Therefore, it is essential to extend the application scope of rough set theory to graphs, thus enriching and strengthening the rough set theory. The rough approximations of graph not only can extract the data on objects (vertices) but also on relationships (edges).

There has been some works on analysis of graphs by rough set theory. Chen et al. [2, 3] studied the testing bipartiteness of simple undirected graphs and minimum vertex cover problem of graphs based on generalized rough sets. He and Shi [9] applied rough set theory to attributed graphs, which focuses on the roughness of multigraphs with multiple edges and essentially concentrates only on the roughness in edges. For simple graphs, Shahzamanian et al. [23] considered the roughness in Cayley graphs, they discussed lower and upper approximations edge Cayley graphs of a Cayley graph which concentrate on the roughness in edges, vertex pseudo-Cayley graphs of a pseudo-Cayley graph which focus on the roughness in vertices, and pseudo-Cayley graphs of a pseudo-Cayley graph which focus on both the roughness in vertices and in edges with respect to a normal subgroup. They showed that a Cayley graph can be approximated by two Cayley graphs — the lower and upper approximations edge Cayley graphs of the Cayley graph. In addition, Liang et al. [10] defined a specific relation over the edge set based on the notion of group, and then constructed rough graphs based on the relation. They proved that any graph can be approximated by a pair of Cayley graphs. In the above work, we know that for simple directed graphs, the roughness in graphs is based on the roughness in vertices or in edges which are based on groups. In this paper, we mainly also focus on the simple directed graphs. We further investigate the roughness in graphs based on general binary relations which has not be studied systematically. The roughness in graphs is based on both the roughness in vertices and roughness in edges which are based on the relations instead of groups. We investigate the roughness in graphs within the axiomatic approach which has not be studied. This work fills in the gap in the research of rough graphs. The edge rough graph proposed by Liang et al. [10] and the rough graph of simple directed graph introduced by He et al. [9] are special cases of this paper.

As we know, there are many different kinds of approximation operators of sets which have different meanings. Therefore, rough set theory can help us obtain different knowledges from the data graph to satisfy different requirements. In this paper, we center on four types of approximation operators of sets, for other types, we will investigate them in the future.

In this paper, we introduce vertex graph approximation operators, edge graph approximation operators, and graph approximation operators. All of them are constructed based on approximation operators of vertex sets and approximation operators of edge sets of graphs. The vertex graph approximation operators first calculate the approximation of vertex set of the graph considered, and then determined the edge set of the approximation of the graph of which the set of induced vertices is contained in the approximation of the vertex set, which implies that the property of vertices is more important than that of edges in a sense. The edge graph approximation operators first calculate the approximation of the edge set, and then determine the vertex set of the approximation of the graph of which the set of induced edges contains the approximation of the edge set, which in a way means that the property of edges is more important than that of vertices. By graph approximation operators, we can obtain approximations of the graph which do not contain isolated vertices. Based on four types of approximation operators of sets, we obtain four types of vertex graph, edge graph, graph approximation operators. We provide characterizations in the constructive approach and the axiomatic approach, and give relationships between approximation operators of graphs and approximation operators of sets.

The paper is organized as follows. In Section 2, we first review some concepts in rough set theory and graph theory, and then recall four types of approximation operators of sets and show their properties. In Section 3, we introduce four types of vertex graph, edge graph, and graph approximation operators and give characterizations in the constructive approach. The axiomatic characterizations of vertex graph approximation operators and edge graph approximation operators are given in Sections 4 and 5, respectively. In Section 6, the axiomatic characterizations of 0-type, 1-type, and 3-type graph lower approximation operators are presented. Finally, Section 7 concludes this paper.

2. Preliminaries

In this section, we first recall some concepts and notations in rough set theory and graph theory, and then review four types of approximation operators of set.

A graph is an ordered pair $G = (V, E)$ of sets V and E, where V is a set of vertices and E is a set of edges such that $E \subseteq V \times V$. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If $E(G) = \emptyset$, we say that G is an empty graph, and if $V(G) = \emptyset$ and $E(G) = \emptyset$, we denote it by $G = \boldsymbol{\emptyset}$. A graph can be referred to as a data graph with singleton vertex label and singleton edge label, which we omit for the convenience of representation. A graph G_1 is a subgraph of G if $V(G_1) \subseteq V$ and $E(G_1) \subseteq E$. A directed edge e from x to y is denoted by $e = (x, y)$, and vertices x and y are called the ends of the edge e. An isolated vertex is a vertex that is not an end of any edge. For any $V_1 \subseteq V$ and any $E_1 \subseteq E$, graph $G[V_1]$ is the vertex-induced subgraph of G with the vertex set V_1 and the set of edges of which the two ends are in V_1 and $G[E_1]$ is the edge-induced subgraph of G with the edge set E_1 and the set of vertices which are associate to the edges in E_1. Let $E'(V_1) = E(G[V_1])$ be the set of edges of which the ends are in V_1 and $V'(E_1) = V(G[E_1])$ be the set of vertices which are associate to the edges in E_1. Formally, we have that

$$E'(V_1) = \{e \in E \,|\, e = (x, y) \text{ with } x, y \in V_1\} \text{ and } V'(E_1) = \{x \in V \,|\, \exists y \in V \text{ such that } (x, y) \text{ or } (y, x) \in E_1\}.$$

We know that (U, R) is a generalized approximation space, where U is a non-empty finite universe of discourse (states or vertices) and $R \subseteq U \times U$ is a general binary relation on U. The relation R is reflexive if $(x, x) \in R$ for any $x \in U$; symmetric if $(x, y) \in R$, then $(y, x) \in R$ for any $x, y \in U$; predecessor serial if for any $x \in U$, there exists $y \in U$ such that $(y, x) \in R$. A reflexive and symmetric relation is called a tolerance relation, and we call a reflexive, symmetric, and transitive relation an equivalence relation. Actually, the concept of generalized approximation space and the concept of simple directed graph are equivalent to each other. We denote by $\mathcal{G}((U, R))$ the set of all the subgraphs of (U, R) and $\mathcal{P}(U)$ the power set of U. We know that x is an isolated vertex if and only if $R_p(x) \cup R_s(x) = \emptyset$. Let $X \subseteq U$. The inequality $(R_p(x) \cup R_s(x)) \cap X \neq \emptyset$ means that x is connected to a vertex in X, and $\cup_{y \in R_p(x)} R_s(y) \cap X \neq \emptyset$ means that x and a vertex in X are in the same successor neighborhood of some vertex in U. Based on the concepts of predecessor neighborhood and successor neighborhood, the universe U can be partitioned into ten parts $Y_i \cap X$, $Y_i \cap (U - X)$ with respect to X, where $i = 0, 1, 2, 3$, and

$$Y_0 = \{x \in U \,|\, R_p(x) \cup R_s(x) = \emptyset\},$$
$$Y_1 = \{x \in U \,|\, R_p(x) \cup R_s(x) \neq \emptyset, (R_p(x) \cup R_s(x)) \cap X \neq \emptyset \text{ and } \cup_{y \in R_p(x)} R_s(y) \cap X = \emptyset\},$$

$Y_2 = \{x \in U \mid R_p(x) \cup R_s(x) \neq \emptyset, (R_p(x) \cup R_s(x)) \cap X \neq \emptyset$ and $\cup_{y \in R_p(x)} R_s(y) \cap X \neq \emptyset\}$,
$Y_3 = \{x \in U \mid R_p(x) \cup R_s(x) \neq \emptyset, (R_p(x) \cup R_s(x)) \cap X = \emptyset$ and $\cup_{y \in R_p(x)} R_s(y) \cap X \neq \emptyset\}$,
$Y_4 = \{x \in U \mid R_p(x) \cup R_s(x) \neq \emptyset, (R_p(x) \cup R_s(x)) \cap X = \emptyset$ and $\cup_{y \in R_p(x)} R_s(y) \cap X = \emptyset\}$.

There are also other ways to partition the universe U. For example, replacing $\cup_{y \in R_p(x)} R_s(y)$ in the above with $\cup_{y \in R_s(x)} R_p(y)$, $\cup_{y \in R_p(x)} R_p(y)$, or $\cup_{y \in R_p(x)} R_s(y)$, and so we can obtain different partitions. We mainly consider the partition in this paper. The existing approximation operators in rough set theory are about sets. Now we give four types of approximation operators. For any $X \subseteq U$, the sets

$$\underline{R^0}(X) = \{x \in U \mid R_s(x) \subseteq X\} \quad \text{and} \quad \overline{R^0}(X) = \{x \in U \mid R_s(x) \cap X \neq \emptyset\}$$

are called 0-type lower and upper approximations of X [24, 34, 37], respectively; the sets

$$\underline{R^1}(X) = \cup\{R_s(x) \mid R_s(x) \subseteq X\} \quad \text{and} \quad \overline{R^1}(X) = \cup\{R_s(x) \mid R_s(x) \cap X \neq \emptyset\}$$

are called 1-type lower and upper approximations of X [39], respectively, which are based on granules; we call the sets

$$\underline{R^2}(X) = \{x \in X \mid \exists y \in X \text{ such that } (x, y) \text{ or } (y, x) \in R\},$$
$$\overline{R^2}(X) = \underline{R^2}(X) \cup \{x \in U - X \mid \exists y \in X \text{ such that}(x, y) \text{ or } (y, x) \in R\}$$

2-type lower and upper approximations of X [16], respectively. For any $X \subseteq U$, let

$$BR_L(X) = \{x \in X \mid \exists y \in X^c \text{ such that } (x, y) \text{ or } (y, x) \in R\},$$
$$BR_H(X) = \{x \in X^c \mid \exists y \in X \text{ such that } (x, y) \text{ or } (y, x) \in R\}.$$

Then let $BR(X) = BR_L(X) \cup BR_H(X) = \{x \in U \mid \exists y \in U, (x, y) \text{ or } (y, x) \in R, (x \in X \text{ and } y \in X^c) \text{ or } (x \in X^c \text{ and } y \in X)\}$ be the boundary of the set X based on R, where X^c is the complement of X in U. Based on the boundary, 3-type lower and upper approximations of X [16] are defined by

$$\underline{R^3}(X) = X - BR(X) \quad \text{and} \quad \overline{R^3}(X) = X \cup BR(X),$$

respectively. We have that $\underline{R^3}(X) \subseteq X \subseteq \overline{R^3}(X)$ for any $R \subseteq U \times U$, which remains the basic properties of the classical approximation operators. There are many other types of approximation operators of sets, in this paper, we mainly concentrate on the four types of approximation operators of sets which have different meanings. Ma et al. [16] discussed them from different aspects. In addition, by the definitions, we have that

- if we care about the part $Y_0 \cap (U - X)$, we can employ the 0-type lower approximation operator because $Y_0 \cap (U - X)$ is only contained in $\underline{R^0}(X)$;
- if we care about the part $Y_0 \cap X$, we can employ 3-type approximation operators because $Y_0 \cap X$ is only contained in $\underline{R^3}(X)$ and $\overline{R^3}(X)$;
- if we care about the part Y_1 and do not care about isolated vertices, we can employ 2-type approximation operators which are more suitable than the other three types.
- if we care about the part $Y_3 \cap (U - X)$, we can employ the 1-type approximation operators, while the other three types of approximation operators are invalid;
- The part $Y_4 \cap (U - X)$ is the set of vertices that we do not care about at all.

As presented in [16], the 0-type and 3-type approximation operators are equivalent to each other when R is a tolerance relation and four types of approximation operators of sets are equivalent to each other when R is an equivalence relation. Four types of approximation operators of sets have the following properties.

Proposition 2.1. [33] Let (U, R) be a generalized approximation space. For any $X_1, X_2 \subseteq U$ and any $R_1, R_2 \subseteq R$, the 0-type approximation operators have the following properties.

(L1) $\underline{R}(\emptyset) \supseteq \emptyset$,
(H1) $\overline{R}(\emptyset) = \emptyset$,
(L2) $\underline{R}(U) = U$,
(H2) $\overline{R}(U) \subseteq U$,
(L3) $X_1 \subseteq X_2 \Rightarrow \underline{R}(X_1) \subseteq \underline{R}(X_2)$,
(H3) $X_1 \subseteq X_2 \Rightarrow \overline{R}(X_1) \subseteq \overline{R}(X_2)$,
(L4) $\underline{R}(X_1 \cap X_2) = \underline{R}(X_1) \cap \underline{R}(X_2)$,
(H4) $\overline{R}(X_1 \cap X_2) \subseteq \overline{R}(X_1) \cap \overline{R}(X_2)$,
(L5) $\underline{R}(X_1 \cup X_2) \supseteq \underline{R}(X_1) \cup \underline{R}(X_2)$,
(H5) $\overline{R}(X_1 \cup X_2) = \overline{R}(X_1) \cup \overline{R}(X_2)$,
(L6) $R_1 \subseteq R_2 \Rightarrow \underline{R_1}(X_1) \supseteq \underline{R_2}(X_1)$,
(H6) $R_1 \subseteq R_2 \Rightarrow \overline{R_1}(X) \subseteq \overline{R_2}(X_1)$,
(L7) $\underline{R_1 \cap R_2}(X_1) \supseteq \underline{R_1}(X_1) \cup \underline{R_2}(X_1)$,
(H7) $\overline{R_1 \cap R_2}(X_1) \subseteq \overline{R_1}(X_1) \cap \overline{R_2}(X_1)$,
(L8) $\underline{R_1 \cup R_2}(X_1) = \underline{R_1}(X_1) \cap \underline{R_2}(X_1)$,
(H8) $\overline{R_1 \cup R_2}(X_1) = \overline{R_1}(X_1) \cup \overline{R_2}(X_1)$.

Proposition 2.2. Let (U, R) be a generalized approximation space. For any $X_1, X_2 \subseteq U$ and any $R_1, R_2 \subseteq R$, we have the following.

(1) The 1-type approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 2.1, and

(L1) $\underline{R}(\emptyset) = \emptyset$,
(L2) $\underline{R}(X) \subseteq X$,
(L4) $\underline{R}(X_1 \cap X_2) \subseteq \underline{R}(X_1) \cap \underline{R}(X_2)$,
(L8) $\underline{R_1 \cup R_2}(X_1) \subseteq \underline{R_1}(X_1) \cup \underline{R_2}(X_1)$.

(2) The 2-type approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 2.1, and

(L1) $\underline{R}(\emptyset) = \emptyset$,
(L2) $\underline{R}(X) \subseteq X$,
(L4) $\underline{R}(X_1 \cap X_2) \subseteq \underline{R}(X_1) \cap \underline{R}(X_2)$,
(L6) $R_1 \subseteq R_2 \Rightarrow \underline{R_1}(X_1) \subseteq \underline{R_2}(X_1)$,
(L7) $\underline{R_1 \cap R_2}(X_1) \subseteq \underline{R_1}(X_1) \cap \underline{R_2}(X_1)$,
(L8) $\underline{R_1 \cup R_2}(X_1) = \underline{R_1}(X_1) \cup \underline{R_2}(X_1)$.

(3) [16] The 3-type approximation operators have properties (L3)–(L8), (H1), (H3)–(H8) in Proposition 2.1, and

(L1) $\underline{R}(\emptyset) = \emptyset$,
(H2) $X_1 \subseteq \overline{R}(X_1)$,
(L2) $\underline{R}(X_1) \subseteq X_1$.

Proof. The (1) and (2) can be obtained by definitions of 1-type and 2-type approximation operators, respectively. □

Remark 2.1. For the 1-type lower approximation operator, it is easy to verify that there is no relationship among $\underline{R_1 \cap R_2}(X_1)$, $\underline{R_1}(X_1)$, and $\underline{R_2}(X_1)$.

3. Vertex graph, edge graph, and graph approximation operators

The lower and upper approximations of subset X of U, both of which are empty graphs, defined in general do not contain the edges, this may result in the loss of information represented by edges. Moreover, the general approximations of X ignore the relationships between the edges in R, and we can not obtain the lower and upper approximations of non-empty graphs by approximation operators of sets defined in general. As presented in Introduction, knowledge represented by graphs naturally occur in the real world [7, 15, 27, 38, 48]. In order to fill in the gap and enrich the rough set theory, as shown in Introduction, He and Shi [9] proposed rough graph which focuses on the relationships between edges, Shahzamanian et al. [23] introduced rough graph with respect to group, and Liang et al. [10] presented rough graph based on the group induced relation. Furthermore, Zafar and Akram [45] investigated fuzzy rough graphs, and Zhan et al. [46] studied intuitionistic fuzzy rough graphs, both of which concentrate on properties in graph theory and the application in decision analysis. In this paper, we mainly focus on the numeric (crisp) environment, we will propose the concept of rough graph based on the ordered pair of relation between vertices and relation between edges which is different from the above concepts of rough graphs. In order to study the rough graph easily, we propose a concept of generalized approximation space on graph, and via it we give lower and upper approximation operators of graph.

Definition 3.1. Let U be a non-empty finite universe of discourse, $R, S_U \subseteq U \times U$ be two binary relations on set U and $S_R \subseteq R \times R$ be a binary relation on set R. We call the quadruple (U, R, S_U, S_R) a generalized approximation space on graph (U, R).

In the sequel, we call (U, R, S_U, S_R) a generalized approximation space on graph and denote (S_U, S_R) by S when there is no confusion. If both S_U and S_R are equivalence relations, then we call (U, R, S_U, S_R) an approximation space on graph. We say that $S' = (S'_U, S'_R) \subseteq S$ if $S'_U \subseteq S_U$ and $S'_R \subseteq S_R$. Based on the generalized approximation space on graph, we give vertex graph, edge graph, and graph approximation operators.

3.1. Vertex graph approximation operators

In this subsection, we give four types of vertex graph approximation operators and their properties.

Definition 3.2. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any graph $G_1 = (V_1, E_1) \in \mathcal{G}((U, R))$, the i-type vertex graph lower approximation $\underline{S^i_v}(G_1)$ and upper approximation $\overline{S^i_v}(G_1)$ of G_1 based on S, where $i = 0, 1, 2, 3$, are defined by

$$\underline{S^i_v}(G_1) = (\underline{S^i_U}(V_1), \underline{S^i_R}(E_1) \cap E'(\underline{S^i_U}(V_1))) \quad \text{and} \quad \overline{S^i_v}(G_1) = (\overline{S^i_U}(V_1), \overline{S^i_R}(E_1) \cap E'(\overline{S^i_U}(V_1))),$$

respectively.

The $\underline{S^i_v}$ and $\overline{S^i_v}$ are called the i-type vertex graph lower and upper approximation operators, respectively, where $i = 0, 1, 2, 3$. The vertex graph lower approximation operator first calculates the approximation $\underline{S^i_U}(V_1)$, and then determines the edge set $\underline{S^i_R}(E_1) \cap E'(\underline{S^i_U}(V_1))$ of $\underline{S^i_v}(G_1)$ of which the set of induced vertices is contained in $\underline{S^i_U}(V_1)$, which can be interpreted as that the property of vertices is more important than that of edges. The similar analysis is applied to vertex graph upper approximation operators. From the above definition, we know that both $\underline{S^i_v}(G_1)$ and $\overline{S^i_v}(G_1)$ are subgraphs of (U, R), the ordered pair $(\underline{S^i_v}(G_1), \overline{S^i_v}(G_1))$ is called the *i-type vertex rough graph* of G_1, where $i = 0, 1, 2, 3$. In this paper, we omit the subscript v and superscript i when there is no confusion.

Rough graphs contain the information represented by edges which are more general than rough sets. In addition, for the relationships between vertex graph approximation operators and approximation operators of set, we have the following.

Proposition 3.1. Let (U, R) be a generalized approximation space on graph. Let $V_1 \subseteq U$ and $E_1 \subseteq R$. For the relationship between i-type vertex graph approximation operators and i-type approximation operators, where $i = 0, 1, 2, 3$, we have

$$V(\underline{S^0_v}((V_1, \emptyset))) = \underline{S^0_U}(V_1), \quad E(\underline{S^0_v}((U, E_1))) = \underline{S^0_R}(E_1), \quad V(\overline{S^0_v}((V_1, \emptyset))) = \overline{S^0_U}(V_1), \quad E(\overline{S^0_v}((U, E_1))) \subseteq \overline{S^0_R}(E_1);$$

$$V(\underline{S^1_v}((V_1, \emptyset))) = \underline{S^1_U}(V_1), \quad E(\underline{S^1_v}((U, E_1))) \subseteq \underline{S^1_R}(E_1), \quad V(\overline{S^1_v}((V_1, \emptyset))) = \overline{S^1_U}(V_1), \quad E(\overline{S^1_v}((U, E_1))) \subseteq \overline{S^1_R}(E_1);$$

$$V(\underline{S^2_v}((V_1, \emptyset))) = \underline{S^2_U}(V_1), \quad E(\underline{S^2_v}((U, E_1))) \subseteq \underline{S^2_R}(E_1), \quad V(\overline{S^2_v}((V_1, \emptyset))) = \overline{S^2_U}(V_1), \quad E(\overline{S^2_v}((U, E_1))) \subseteq \overline{S^2_R}(E_1);$$

$$V(\underline{S^3_v}((V_1, \emptyset))) = \underline{S^3_U}(V_1), \quad E(\underline{S^3_v}((U, E_1))) = \underline{S^3_R}(E_1), \quad V(\overline{S^3_v}((V_1, \emptyset))) = \overline{S^3_U}(V_1), \quad E(\overline{S^3_v}((U, E_1))) = \overline{S^3_R}(E_1).$$

For the special case that $S_U = R$, we have $V(\underline{S^i_v}((V_1, \emptyset))) = \underline{R^i}(V_1)$ and $V(\overline{S^i_v}((V_1, \emptyset))) = \overline{R^i}(V_1)$, where $i = 0, 1, 2, 3$, thus the concept of the new approximation space on graph is a proper generalization of the generalized approximation space, and the vertex graph approximation operators are more general than approximation operators of sets.

Now we give an example to illustrate the above vertex graph approximation operators.

Example 3.1. Let (U, R, S_U, S_R) be a generalized approximation space on graph where $U = \{x, y, z, v, w\}$, $R = \{(x, y), (x, w), (z, x), (z, v), (v, w)\}$, $S_U = R$, and $S_R = \{\{(x, y), (x, w), (z, x)\}, \{(z, v), (v, w)\}\}$ which is an equivalence relation on R. Let $G_1 = (V_1, E_1) = (\{x, y, w\}, \{(x, w)\})$. Since

$$\underline{S^0_U}(V_1) = \{x, y, v, w\}, \quad \underline{S^0_R}(E_1) = \emptyset \quad \text{and} \quad \overline{S^0_U}(V_1) = \{x, z, v\}, \quad \overline{S^0_R}(E_1) = \{(x, y), (x, w), (z, x)\},$$

we have $\underline{S^0_v}(G_1) = (\{x, y, v, w\}, \emptyset)$ and $\overline{S^0_v}(G_1) = (\{x, z, v\}, \{(z, x)\})$, respectively. Then the 0-type vertex rough graph of G_1 is $((\{x, y, v, w\}, \emptyset), (\{x, z, v\}, \{(z, x)\}))$ and the 0-type rough set of V_1 is $(\{x, y, v, w\}, \{x, z, v\})$. By

$$\underline{S^1_U}(V_1) = \{y, w\}, \quad \underline{S^1_R}(E_1) = \emptyset \quad \text{and} \quad \overline{S^1_U}(V_1) = \{x, y, v, w\}, \quad \overline{S^1_R}(E_1) = \{(x, y), (x, w), (z, x)\},$$

we obtain $\underline{S^1_v}(G_1) = (\{y, w\}, \emptyset)$ and $\overline{S^1_v}(G_1) = (\{x, y, v, w\}, \{(x, y), (x, w)\})$, respectively. Then the 1-type vertex rough graph of G_1 is $((\{y, w\}, \emptyset), (\{x, y, v, w\}, \{(x, y), (x, w)\}))$ and the 1-type rough set of V_1 is $(\{y, w\}, \{x, y, v, w\})$. Since

$$\underline{S^2_U}(V_1) = \{x, y, w\}, \quad \underline{S^2_R}(E_1) = \{(x, w)\} \quad \text{and} \quad \overline{S^2_U}(V_1) = U, \quad \overline{S^2_R}(E_1) = \{(x, y), (x, w), (z, x)\},$$

we get that $\underline{S^2_v}(G_1) = (\{x, y, w\}, \{(x, w)\})$ and $\overline{S^2_v}(G_1) = (U, \{(x, y), (x, w), (z, x)\})$. Then the 2-type vertex rough graph of G_1 is $((\{x, y, w\}, \{(x, w)\}), (U, \{(x, y), (x, w), (z, x)\}))$ and the 2-type rough set of V_1 is $(\{x, y, w\}, U)$. Finally, because

$$(BS_U)_L(V_1) = \{x, w\}, \qquad (BS_R)_L(E_1) = \{(x, w)\},$$
$$(BS_U)_U(V_1) = \{z, v\}, \qquad (BS_R)_U(E_1) = \{(x, y), (z, x)\},$$

we have

$$BS_U(V_1) = (BS_U)_L(V_1) \cup (BS_U)_U(V_1) = \{x, z, v, w\},$$
$$BS_R(E_1) = (BS_R)_L(E_1) \cup (BS_R)_U(E_1) = \{(x, y), (x, w), (z, x)\},$$

and further,

$$\underline{S^3_U}(V_1) = V_1 - (BS_U)(V_1) = \{y\}, \qquad \underline{S^3_R}(E_1) = E_1 - (BS_R)(E_1) = \emptyset,$$
$$\overline{S^3_U}(V_1) = V_1 \cup (BS_U)(V_1) = U, \qquad \overline{S^3_R}(E_1) = E_1 \cup (BS_R)(E_1) = \{(x, y), (x, w), (z, x)\},$$

we thus obtain $\underline{S^3_v}(G_1) = (\{y\}, \emptyset)$ and $\overline{S^3_v}(G_1) = (U, \{(x, y), (x, w), (z, x)\})$. Then the 3-type vertex rough graph of G_1 is $((\{y\}, \emptyset), (U, \{(x, y), (x, w), (z, x)\}))$, and the 3-type rough set of V_1 is $(\{y\}, U)$.

Now we study properties of four types of vertex graph approximation operators.

Proposition 3.2. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, the 0-type vertex graph approximation operators have the following properties.

(L1) $\underline{S}(\boldsymbol{\emptyset}) \supseteq \boldsymbol{\emptyset}$, (H1) $\overline{S}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$, (L2) $\underline{S}((U,R)) = (U,R)$,

(L3) $G_1 \subseteq G_2 \Rightarrow \underline{S}(G_1) \subseteq \underline{S}(G_2)$,

(L4) $\underline{S}(G_1 \cap G_2) = \underline{S}(G_1) \cap \underline{S}(G_2)$,

(L5) $\underline{S}(G_1 \cup G_2) \supseteq \underline{S}(G_1) \cup \underline{S}(G_2)$,

(L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \supseteq \underline{S_2}(G_1)$,

(L7) $\underline{S_1 \cap S_2}(G_1) \supseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$,

(L8) $\underline{S_1 \cup S_2}(G_1) = \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$,

(H2) $\overline{S}((U,R)) \subseteq (U,R)$,

(H3) $G_1 \subseteq G_2 \Rightarrow \overline{S}(G_1) \subseteq \overline{S}(G_2)$,

(H4) $\overline{S}(G_1 \cap G_2) \subseteq \overline{S}(G_1) \cap \overline{S}(G_2)$,

(H5) $\overline{S}(G_1 \cup G_2) \supseteq \overline{S}(G_1) \cup \overline{S}(G_2)$,

(H6) $S_1 \subseteq S_2 \Rightarrow \overline{S_1}(G_1) \subseteq \overline{S_2}(G_1)$,

(H7) $\overline{S_1 \cap S_2}(G_1) \subseteq \overline{S_1}(G_1) \cap \overline{S_2}(G_1)$,

(H8) $\overline{S_1 \cup S_2}(G_1) \supseteq \overline{S_1}(G_1) \cup \overline{S_2}(G_1)$.

Proof. We mainly prove (L4), (H4) and (H5), since the rest can be obtained directly by Proposition 2.1 and Definition 3.2.

Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. By Proposition 2.1, we have $\underline{S_U}(V_1 \cap V_2) = \underline{S_U}(V_1) \cap \underline{S_U}(V_2)$ and $\underline{S_R}(E_1 \cap E_2) = \underline{S_R}(E_1) \cap \underline{S_U}(E_2)$, and so

$$\begin{aligned}
\underline{S}(G_1 \cap G_2) &= \underline{S}(V_1 \cap V_2, E_1 \cap E_2) \\
&= (\underline{S_U}(V_1 \cap V_2), \underline{S_R}(E_1 \cap E_2) \cap E'(\underline{S_U}(V_1 \cap V_2))) \\
&= (\underline{S_U}(V_1) \cap \underline{S_U}(V_2), (\underline{S_R}(E_1) \cap \underline{S_R}(E_2)) \cap E'(\underline{S_U}(V_1) \cap \underline{S_U}(V_2))) \\
&= (\underline{S_U}(V_1) \cap \underline{S_U}(V_2), (\underline{S_R}(E_1) \cap \underline{S_R}(E_2)) \cap (E'(\underline{S_U}(V_1)) \cap E'(\underline{S_U}(V_2))) \\
&= (\underline{S_U}(V_1), \underline{S_R}(E_1) \cap E'(\underline{S_U}(V_1))) \cap (\underline{S_U}(V_2), \underline{S_R}(E_2) \cap E'(\underline{S_U}(V_2))) \\
&= \underline{S}(G_1) \cap \underline{S}(G_2).
\end{aligned}$$

Hence (L4) holds.

By Proposition 2.1 again, we obtain $\overline{S_U}(V_1 \cap V_2) \subseteq \overline{S_U}(V_1) \cap \overline{S_U}(V_2)$ and $\overline{S_R}(E_1 \cap E_2) \subseteq \overline{S_R}(E_1) \cap \overline{S_R}(E_2)$, and then

$$\begin{aligned}
\overline{S}(G_1 \cap G_2) &= \overline{S}(V_1 \cap V_2, E_1 \cap E_2) \\
&= (\overline{S_U}(V_1 \cap V_2), \overline{S_R}(E_1 \cap E_2) \cap E'(\overline{S_U}(V_1 \cap V_2))) \\
&\subseteq (\overline{S_U}(V_1) \cap \overline{S_U}(V_2), (\overline{S_R}(E_1) \cap \overline{S_R}(E_2)) \cap E'(\overline{S_U}(V_1) \cap \overline{S_U}(V_2))) \\
&= (\overline{S_U}(V_1) \cap \overline{S_U}(V_2), \overline{S_R}(E_1) \cap \overline{S_R}(E_2) \cap E'(\overline{S_U}(V_1)) \cap E'(\overline{S_U}(V_1))) \\
&= (\overline{S_U}(V_1), \overline{S_R}(E_1) \cap E'(\overline{S_U}(V_1))) \cap (\overline{S_U}(V_2), \overline{S_R}(E_2) \cap E'(\overline{S_U}(V_2))) \\
&= \overline{S}(G_1) \cap \overline{S}(G_2).
\end{aligned}$$

Thus (H4) holds.

For (H5), also by Proposition 2.1, we have $\overline{S_U}(V_1 \cup V_2) = \overline{S_U}(V_1) \cup \overline{S_U}(V_2)$ and $\overline{S_R}(E_1 \cup E_2) = \overline{S_R}(E_1) \cup \overline{S_R}(E_2)$, and so

$$\begin{aligned}
\overline{S}(G_1 \cup G_2) &= (\overline{S_U}(V_1 \cup V_2), \overline{S_R}(E_1 \cup E_2) \cap E'(\overline{S_U}(V_1 \cup V_2))) \\
&= (\overline{S_U}(V_1) \cup \overline{S_U}(V_2), (\overline{S_R}(E_1) \cup \overline{S_R}(E_2)) \cap E'(\overline{S_U}(V_1) \cup \overline{S_U}(V_1))) \\
&\supseteq (\overline{S_U}(V_1) \cup \overline{S_U}(V_2), (\overline{S_R}(E_1) \cup \overline{S_R}(E_2)) \cap (E'(\overline{S_U}(V_1)) \cup E'(\overline{S_U}(V_1)))) \\
&\supseteq (\overline{S_U}(V_1), \overline{S_R}(E_1) \cap E'(\overline{S_U}(V_1))) \cup (\overline{S_U}(V_2), \overline{S_R}(E_2) \cap E'(\overline{S_R}(V_2))) \\
&= \overline{S}(G_1) \cup \overline{S}(G_2).
\end{aligned}$$

□

The following proposition provides properties of 1-type, 2-type, and 3-type vertex graph approximation operators.

Proposition 3.3. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, we have the following.

(1) The 1-type vertex graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.2, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$.

(2) The 2-type vertex graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.2, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$, (L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \subseteq \underline{S_2}(G_1)$,
(L7) $\underline{S_1 \cap S_2}(G_1) \subseteq \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$, (L8) $\underline{S_1 \cup S_2}(G_1) \supseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$.

(3) The 3-type vertex graph approximation operators have properties (L3)–(L8), (H1), (H3)–(H8) in Proposition 3.2, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (H2) $G_1 \subseteq \overline{S}(G_1)$.
(L2) $\underline{S}(G_1) \subseteq G_1$,

Proof. It follows immediately from Proposition 2.2 and Definition 3.2. □

Remark 3.1. It is not difficult to obtain that for the 1-type vertex graph lower approximation operator, there is no relationship among $\underline{S_1 \cap S_2}(G_1)$, $\underline{S_1}(G_1)$, and $\underline{S_2}(G_1)$, and no relationship among $\underline{S_1 \cup S_2}(G_1)$, $\underline{S_1}(G_1)$, and $\underline{S_2}(G_1)$.

3.2. Edge graph approximation operators

The vertex graph approximation operators can be used to handle the case that the property of vertices has precedence over that of edges. For the opposite side, we give edge graph approximation operators, and further provide their properties.

Definition 3.3. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any graph $G_1 = (V_1, E_1) \in \mathcal{G}((U, R))$, the i-type edge graph lower approximation $\underline{S^i_e}(G_1)$ and upper approximation $\overline{S^i_e}(G_1)$ of G_1 based on S, where $i = 0, 1, 2, 3$, are defined by

$$\underline{S^i_e}(G_1) = (\underline{S^i_U}(V_1) \cup V'(\underline{S^i_R}(E_1)), \underline{S^i_R}(E_1)) \quad \text{and} \quad \overline{S^i_e}(G_1) = (\overline{S^i_U}(V_1) \cup V'(\overline{S^i_R}(E_1)), \overline{S^i_R}(E_1)),$$

respectively.

The $\underline{S^i_e}$ and $\overline{S^i_e}$ are called i-type edge graph lower and upper approximation operators, respectively, where $i = 0, 1, 2, 3$. The edge graph lower approximation operator first calculates the approximation $\underline{S^i_R}(E_1)$, and then determines the vertex set $\underline{S^i_U}(V_1) \cup V'(\underline{S^i_R}(E_1))$ of $\underline{S^i_e}(G_1)$ of which the set of induced edges contains the approximation $\underline{S^i_R}(E_1)$, which can be regarded as that the property of edges has priority over that of vertices. We can give the similar explanation to the edge graph upper approximation operators. From the above definition, we know that both $\underline{S^i_e}(G_1)$ and $\overline{S^i_e}(G_1)$ are subgraphs of (U, R), the ordered pair $(\underline{S^i_e}(G_1), \overline{S^i_e}(G_1))$ is called the i-type edge rough graph of G_1, where $i = 0, 1, 2, 3$. We drop the subscript e and superscript i if it is clear from the context. Besides, we have the following.

Proposition 3.4. Let (U, R) be a generalized approximation space on graph. Let $V_1 \subseteq U$ and $E_1 \subseteq R$. For the relationship between i-type edge graph approximation operators and i-type approximation operators, where $i = 0, 1, 2, 3$, we have

$$V(\underline{S_e^0}((V_1, \emptyset))) \supseteq \underline{S_U^0}(V_1), \quad E(\underline{S_e^0}((U, E_1))) = \underline{S_R^0}(E_1), \quad V(\overline{S_e^0}((V_1, \emptyset))) = \overline{S_U^0}(V_1), \quad E(\overline{S_e^0}((U, E_1))) = \overline{S_R^0}(E_1);$$

$$V(\underline{S_e^1}((V_1, \emptyset))) = \underline{S_U^1}(V_1), \quad E(\underline{S_e^1}((U, E_1))) = \underline{S_R^1}(E_1), \quad V(\overline{S_e^1}((V_1, \emptyset))) = \overline{S_U^1}(V_1), \quad E(\overline{S_e^1}((U, E_1))) = \overline{S_R^1}(E_1);$$

$$V(\underline{S_e^2}((V_1, \emptyset))) = \underline{S_U^2}(V_1), \quad E(\underline{S_e^2}((U, E_1))) = \underline{S_R^2}(E_1), \quad V(\overline{S_e^2}((V_1, \emptyset))) = \overline{S_U^2}(V_1), \quad E(\overline{S_e^2}((U, E_1))) = \overline{S_R^2}(E_1);$$

$$V(\underline{S_e^3}((V_1, \emptyset))) = \underline{S_U^3}(V_1), \quad E(\underline{S_e^3}((U, E_1))) = \underline{S_R^3}(E_1), \quad V(\overline{S_e^3}((V_1, \emptyset))) = \overline{S_U^3}(V_1), \quad E(\overline{S_e^3}((U, E_1))) = \overline{S_R^3}(E_1).$$

Try to show example that is explanation-oriented.

Example 3.2. We revisit Example 3.1. We have that

$$\underline{S_e^0}(G_1) = (\{x, y, v, w\}, \emptyset) \quad \text{and} \quad \overline{S_e^0}(G_1) = (U, \{(x, y), (x, w), (z, x)\}).$$

Then the 0-type edge rough graph of G_1 is $((\{x, y, v, w\}, \emptyset), (U, \{(x, y), (x, w), (z, x)\}))$ and the 0-type rough set of E_1 is $(\emptyset, \{(x, y), (x, w), (z, x)\})$. The 1-type edge graph lower and upper approximations of G_1 are

$$\underline{S_e^1}(G_1) = (\{y, w\}, \emptyset) \quad \text{and} \quad \overline{S_e^1}(G_1) = (U, \{(x, y), (x, w), (z, x)\}),$$

respectively. Then the 1-type edge rough graph of G_1 is $((\{y, w\}, \emptyset), (U, \{(x, y), (x, w), (z, x)\}))$ and the 1-type rough set of E_1 is $(\emptyset, \{(x, y), (x, w), (z, x)\})$. The 2-type edge graph lower and upper approximations are

$$\underline{S_e^2}(G_1) = (\{x, y, w\}, \{(x, w)\}) \quad \text{and} \quad \overline{S_e^2}(G_1) = (U, \{(x, y), (x, w), (z, x)\}),$$

respectively. Then the 2-type edge rough graph of G_1 is $((\{x, y, w\}, \{(x, w)\}), (U, \{(x, y), (x, w), (z, x)\}))$ and the 2-type rough set of E_1 is $(\{(x, w)\}, \{(x, y), (x, w), (z, x)\})$. The 3-type edge graph lower and upper approximations are

$$\underline{S_e^3}(G_1) = (\{y\}, \emptyset) \quad \text{and} \quad \overline{S_e^3}(G_1) = (U, \{(x, y), (x, w)\}),$$

respectively. Then 3-type edge rough graph of G_1 is $((\{y\}, \emptyset), (U, \{(x, y), (x, w), (z, x), (z, v)\}))$ and the 3-type rough set of E_1 is $(\emptyset, \{(x, y), (x, w), (z, x)\})$.

The following two propositions provide properties of four types of edge graph approximation operators.

Proposition 3.5. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, the 0-type edge graph approximation operators have the following properties.

(L1) $\underline{S}(\emptyset) \supseteq \emptyset$,

(L2) $\underline{S}((U, R)) = (U, R)$,

(L3) $G_1 \subseteq G_2 \Rightarrow \underline{S}(G_1) \subseteq \underline{S}(G_2)$,

(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$,

(L5) $\underline{S}(G_1 \cup G_2) \supseteq \underline{S}(G_1) \cup \underline{S}(G_2)$,

(L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \supseteq \underline{S_2}(G_1)$,

(L7) $\underline{S_1 \cap S_2}(G_1) \supseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$,

(L8) $\underline{S_1 \cup S_2}(G_1) \subseteq \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$,

(H1) $\overline{S}(\emptyset) = \emptyset$,

(H2) $\overline{S}((U, R)) \subseteq (U, R)$,

(H3) $G_1 \subseteq G_2 \Rightarrow \overline{S}(G_1) \subseteq \overline{S}(G_2)$,

(H4) $\overline{S}(G_1 \cap G_2) \subseteq \overline{S}(G_1) \cap \overline{S}(G_2)$,

(H5) $\overline{S}(G_1 \cup G_2) = \overline{S}(G_1) \cup \overline{S}(G_2)$,

(H6) $S_1 \subseteq S_2 \Rightarrow \overline{S_1}(G_1) \subseteq \overline{S_2}(G_1)$,

(H7) $\overline{S_1 \cap S_2}(G_1) \subseteq \overline{S_1}(G_1) \cap \overline{S_2}(G_1)$,

(H8) $\overline{S_1 \cup S_2}(G_1) \supseteq \overline{S_1}(G_1) \cup \overline{S_2}(G_1)$.

Proof. We omit the details since it follows directly from Proposition 2.1 and Definition 3.3. □

Proposition 3.6. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, we have the following.

(1) The 1-type edge graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.5, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$, (L8) $\underline{S_1 \cup S_2}(G_1) \subseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$.

(2) The 2-type edge graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.5, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$, (L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \subseteq \underline{S_2}(G_1)$,
(L7) $\underline{S_1 \cap S_2}(G_1) \subseteq \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$, (L8) $\underline{S_1 \cup S_2}(G_1) = \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$.

(3) The 3-type edge graph approximation operators have properties (L3)–(L8), (H1), (H3)–(H8) in Proposition 3.5, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (H2) $G_1 \subseteq \overline{S}(G_1)$.
(L2) $\underline{S}(G_1) \subseteq G_1$,

Proof. It follows immediately from Proposition 2.2 and Definition 3.3, we thus omit the proof. □

Remark 3.2. About the 1-type edge graph lower approximation operator, it is not hard to obtain that there is no relationship among $\underline{S_1 \cap S_2}(G_1)$, $\underline{S_1}(G_1)$, and $\underline{S_2}(G_1)$.

3.3. *Graph approximation operators*

We first introduce four types of graph approximation operators, and then show their properties.

Definition 3.4. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any graph $G_1 = (V_1, E_1) \in \mathcal{G}((U, R))$, the *i*-type graph lower approximation $\underline{S^i}(G_1)$ and upper approximation $\overline{S^i}(G_1)$ of G_1 based on S, where $i = 0, 1, 2, 3$, are defined by

$$\underline{S^i}(G_1) = (V'(\underline{S^i_R}(E_1) \cap E'(\underline{S^i_U}(V_1))), \underline{S^i_R}(E_1) \cap E'(\underline{S^i_U}(V_1))),$$
$$\overline{S^i}(G_1) = (V'(\overline{S^i_R}(E_1) \cap E'(\overline{S^i_U}(V_1))), \overline{S^i_R}(E_1) \cap E'(\overline{S^i_U}(V_1))),$$

respectively.

The $\underline{S^i}$ and $\overline{S^i}$ are called the *i*-type graph lower and upper approximation operators, respectively, where $i = 0, 1, 2, 3$. By graph approximation operators, we obtain that both $\underline{S^i}(G_1)$ and $\overline{S^i}(G_1)$ are subgraphs of (U, R) and do not contain isolated vertices. We call the ordered pair $(\underline{S^i}(G_1), \overline{S^i}(G_1))$ the *i*-type rough graph of G_1, where $i = 0, 1, 2, 3$. Like the above, the superscript i can be omited when there is no confusion.

It should be noted that the vertex graph, edge graph, and graph approximation operators are equivalent to each other when S_U and S_R satisfy the condition (3) in Definition 1.1, i.e., for any $(r_1, r_2) \in S_R$ with $r_1 = (u, v)$ and $r_2 = (u', v')$, it holds that $(u, v), (u', v') \in S_U$.

Graph approximation operators and approximation operators of sets have the following relationships.

Proposition 3.7. Let (U, R) be a generalized approximation space on graph. Let $V_1 \subseteq U$ and $E_1 \subseteq R$. For the relationship between i-type graph approximation operators and i-type approximation operators, where $i = 0, 1, 2, 3$, we have

$V(\underline{S^0}((V_1, E'(V_1)))) \subseteq \underline{S^0_U}(V_1)$, $E(\underline{S^0}((U, E_1))) = \underline{S^0_R}(E_1)$, $V(\overline{S^0}((V_1, \emptyset))) \subseteq \overline{S^0_U}(V_1)$, $E(\overline{S^0}((U, E_1))) \subseteq \overline{S^0_R}(E_1)$;

$V(\underline{S^1}((V_1, E'(V_1)))) \subseteq \underline{S^1_U}(V_1)$, $E(\underline{S^1}((U, E_1))) \subseteq \underline{S^1_R}(E_1)$, $V(\overline{S^1}((V_1, E'(V_1)))) \subseteq \overline{S^1_U}(V_1)$, $E(\overline{S^1}((U, E_1))) \subseteq \overline{S^1_R}(E_1)$;

$V(\underline{S^2}((V_1, E'(V_1)))) \subseteq \underline{S^2_U}(V_1)$, $E(\underline{S^2}((U, E_1))) \subseteq \underline{S^2_R}(E_1)$, $V(\overline{S^2}((V_1, E'(V_1)))) \subseteq \overline{S^2_U}(V_1)$, $E(\overline{S^2}((U, E_1))) \subseteq \overline{S^2_R}(E_1)$;

$V(\underline{S^3}((V_1, E'(V_1)))) \subseteq \underline{S^3_U}(V_1)$, $E(\underline{S^3}((U, E_1))) = \underline{S^3_R}(E_1)$, $V(\overline{S^3}((V_1, E'(V_1)))) \subseteq \overline{S^3_U}(V_1)$, $E(\overline{S^3}((U, E_1))) = \overline{S^3_R}(E_1)$.

We provide the following illustrative example.

Example 3.3. We visit Example 3.1 again. We have that

$$\underline{S^0}(G_1) = \emptyset \quad \text{and} \quad \overline{S^0}(G_1) = (\{x, z\}, \{(z, x)\}).$$

Then the 0-type rough graph of G_1 is $(\emptyset, (\{x, z\}, \{(z, x)\}))$. The 1-type graph lower and upper approximations are

$$\underline{S^1}(G_1) = \emptyset \quad \text{and} \quad \overline{S^1}(G_1) = (\{x, y, w\}, \{(x, y), (x, w)\}),$$

respectively. Then the 1-type rough graph of G_1 is $(\emptyset, (\{x, y, w\}, \{(x, y), (x, w)\}))$. The 2-type graph lower and upper approximations are

$$\underline{S^2}(G_1) = (\{x, w\}, \{(x, w)\}) \quad \text{and} \quad \overline{S^2}(G_1) = (\{x, y, z, w\}, \{(x, y), (x, w), (z, x)\}),$$

respectively. Then the 2-type rough graph of G_1 is $((\{x, w\}, \{(x, w)\}), (\{x, y, z, w\}, \{(x, y), (x, w), (z, x)\}))$. The 3-type graph lower and upper approximations are

$$\underline{S^3}(G_1) = \emptyset \quad \text{and} \quad \overline{S^3}(G_1) = (U, \{(x, y), (x, w), (z, x)\}),$$

respectively. Then the 3-type rough graph of G_1 is $(\emptyset, (U, \{(x, y), (x, w), (z, x)\}))$.

The four types of graph approximation operators have the following properties.

Proposition 3.8. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, the 0-type graph approximation operators have the following properties.

(L1) $\underline{S}(\emptyset) \supseteq \emptyset$,

(L2) $\underline{S}((U, R)) = (U, R)$,

(L3) $G_1 \subseteq G_2 \Rightarrow \underline{S}(G_1) \subseteq \underline{S}(G_2)$,

(L4) $\underline{S}(G_1 \cap G_2) = \underline{S}(G_1) \cap \underline{S}(G_2)$,

(L5) $\underline{S}(G_1 \cup G_2) \supseteq \underline{S}(G_1) \cup \underline{S}(G_2)$,

(L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \supseteq \underline{S_2}(G_1)$,

(L7) $\underline{S_1 \cap S_2}(G_1) \supseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$,

(L8) $\underline{S_1 \cup S_2}(G_1) = \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$,

(H1) $\overline{S}(\emptyset) = \emptyset$,

(H2) $\overline{S}((U, R)) \subseteq (U, R)$,

(H3) $G_1 \subseteq G_2 \Rightarrow \overline{S}(G_1) \subseteq \overline{S}(G_2)$,

(H4) $\overline{S}(G_1 \cap G_2) \subseteq \overline{S}(G_1) \cap \overline{S}(G_2)$,

(H5) $\overline{S}(G_1 \cup G_2) \supseteq \overline{S}(G_1) \cup \overline{S}(G_2)$,

(H6) $S_1 \subseteq S_2 \Rightarrow \overline{S_1}(G_1) \subseteq \overline{S_2}(G_1)$,

(H7) $\overline{S_1 \cap S_2}(G_1) \subseteq \overline{S_1}(G_1) \cap \overline{S_2}(G_1)$,

(H8) $\overline{S_1 \cup S_2}(G_1) \supseteq \overline{S_1}(G_1) \cup \overline{S_2}(G_1)$.

Proof. It follows immediately from Proposition 2.1 and Definition 3.4, and hence we omit the proof. □

Proposition 3.9. Let (U, R, S_U, S_R) be a generalized approximation space on graph. For any $G_1, G_2 \in \mathcal{G}((U, R))$ and any $S_1, S_2 \subseteq S$, we have the following.

(1) The 1-type graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.8, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$.

(2) The 2-type graph approximation operators have properties (L3), (L5), (H1)–(H8) in Proposition 3.8, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (L2) $\underline{S}(G_1) \subseteq G_1$,
(L4) $\underline{S}(G_1 \cap G_2) \subseteq \underline{S}(G_1) \cap \underline{S}(G_2)$, (L6) $S_1 \subseteq S_2 \Rightarrow \underline{S_1}(G_1) \subseteq \underline{S_2}(G_1)$,
(L7) $\underline{S_1 \cap S_2}(G_1) \subseteq \underline{S_1}(G_1) \cap \underline{S_2}(G_1)$, (L8) $\underline{S_1 \cup S_2}(G_1) \supseteq \underline{S_1}(G_1) \cup \underline{S_2}(G_1)$.

(3) The 3-type graph approximation operators have properties (L2)–(L8), (H1), (H3)–(H8) in Proposition 3.8, and

(L1) $\underline{S}(\emptyset) = \emptyset$, (H2) $G_1 \subseteq \overline{S}(G_1)$.
(L2') $\underline{S}(G_1) \subseteq G_1$,

Proof. It follows directly from Proposition 2.2 and Definition 3.4. We thus omit the details. □

Remark 3.3. It is easy to have that for the 1-type lower graph approximation operator, there is no relationship among $\underline{S_1 \cap S_2}(G_1)$, $\underline{S_1}(G_1)$, and $\underline{S_2}(G_1)$, and no relationship among $\underline{S_1 \cup S_2}(G_1)$, $\underline{S_1}(G_1)$, and $\underline{S_2}(G_1)$.

In the sequel, we always let $G_1 = (V_1, E_1) \in \mathcal{G}((U, R))$ be a graph.

4. On axiomatic characterizations of four types of vertex graph approximation operators

4.1. *On axiomatic characterizations of* 0*-type vertex graph approximation operators*

Yao [33] gave axiomatic characterizations of 0-type approximation operators. Based on the results, in this subsection, we present axiomatic characterizations of 0-type vertex graph approximation operators.

Theorem 4.1. An operator $L : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $L((U, R)) = (U, R)$,
(2) $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$,
(3) $L(G_1) = (V(L((V_1, \emptyset))), E(L((U, E_1))) \cap E'(V(L((V_1, \emptyset)))))$

if and only if there exists a unique ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $L = \underline{S_v^0}$.

Proof. "⇐" It follows immediately from Propositions 3.1 and 3.2, and Definition 3.2.

"⇒" We first construct $S = (S_U, S_R)$. For S_U, $(x, y) \in S_U$ if and only if there exists a subset $V \subseteq U$ such that $y \in V$, $x \in V(L((V, \emptyset)))$ and $x \notin V(L((V - \{y\}, \emptyset)))$. For S_R, $(r_1, r_2) \in S_R$ if and only if there exists a subset $E \subseteq U$ such that $r_2 \in E$, $r_1 \in E(L((U, E)))$ and $r_1 \notin E(L((U, E - \{r_2\})))$. From the construction we have that for any $x \in U$, $x \in V(L(((S_U)_s(x), \emptyset)))$ and $x \notin V(L(((S_U)_s(x) - \{y\}, \emptyset)))$ for any $y \in (S_U)_s(x)$, and for any $r_1 \in R$, $r_1 \in E(L((U, (S_R)_s(r_1))))$ and $r_1 \notin E(L((U, (S_R)_s(r_1) - \{r_2\})))$ for any $r_2 \in (S_R)_s(r_1)$. Obviously, $\underline{S_v^0}((U, R)) = (U, R) = L((U, R))$. Now we prove $\underline{S_v^0}(G_1) = L(G_1)$. By the definition of 0-type lower approximation operator and (2), we obtain that for any $x \in U$, $x \in \underline{S^0_U}(V_1)$ if and only if $(S_U)_s(x) \subseteq V_1$ if and only if $x \in V(L((V_1, \emptyset)))$, and hence $\underline{S^0_U}(V_1) = V(L((V_1, \emptyset)))$. Similarly, $\underline{S^0_R}(E_1) = E(L((U, E_1)))$. Therefore, by (3) and Definition 3.2, we have $\underline{S^0_v}(G_1) = (\underline{S^0_U}(V_1), \underline{S^0_R}(E_1) \cap E'(\underline{S^0_U}(V_1))) = (V(L((V_1, \emptyset))), E(L((U, E_1))) \cap E'(V(L((V_1, \emptyset))))) = L(G_1)$. By Definition 3.2 and Theorem 6 of [39], we can obtain the uniqueness of S. □

For the axiomatic characterization of the 0-type vertex graph upper approximation operator, we have the following.

Theorem 4.2. An operator $H : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $H(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
(2) $V(H((V_1 \cup V_2, \emptyset))) = V(H((V_1, \emptyset))) \cup V(H((V_2, \emptyset)))$,
(3) $E(H((U, E_1 \cup E_2))) = E(H((U, E_1))) \cup E(H((U, E_2)))$,
(4) $H(G_1) = (V(H((V_1, \emptyset))), E(H((U, E_1))) \cap E'(V(H((V_1, \emptyset)))))$

if and only if there exists a unique ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $H = \overline{S^0_v}$.

Proof. "⇐" It is easy to get by Propositions 2.1, 3.1 and 3.2, and Definition 3.2.

"⇒" We construct $S = (S_U, S_R)$ by the following way: for any $x, y \in U$, $(x, y) \in S_U$ if and only if $x \in V(H((\{y\}, \emptyset)))$, and for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if $r_1 \in E((H(U, \{r_2\})))$. By the construction and (2), we have that for any $x \in U$ and any subset $V \subseteq U$, $(S_U)_s(x) \cap V \neq \emptyset$ if and only if $x \in V(H((V, \emptyset)))$, and by the construction and (3), we obtain that for any $r \in R$ and any subset $E \subseteq R$, $(S_R)_s(r) \cap E \neq \emptyset$ if and only if $r \in E(H((U, E)))$. By Proposition 3.2, we have $\overline{S^0_v}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = H(\boldsymbol{\emptyset})$. Then we prove that $\overline{S^0_v}(G_1) = H(G_1)$, for any $x \in U$, from the above results, we have that $x \in \overline{S^0_U}(V_1)$ if and only if $(S_U)_s(x) \cap V_1 \neq \emptyset$ if and only if $x \in V(H((V_1, \emptyset)))$, which implies that $\overline{S^0_U}(V_1) = V(H((V_1, \emptyset)))$. Analogously, $\overline{S^0_R}(E_1) = E(H((U, E_1)))$. Hence $(\overline{S^0_U}(V_1), \overline{S^0_R}(E_1) \cap E'(\overline{S^0_U}(V_1))) = (V(H((V_1, \emptyset))), E(H((U, E_1))) \cap E'(V(H((V_1, \emptyset)))))$, i.e., $\overline{S^0_v}(G_1) = H(G_1)$. The uniqueness of S can be obtained immediately from Definition 3.2 and Theorem 5 of [39]. This completes the proof. □

4.2. *On axiomatic characterizations of* 1*-type vertex graph approximation operators*

Zhu [40, 41] first gave the axiomatic characterization of the 1-type lower approximation operator, and then Zhang et al. [47] established axiomatic systems and examined that the obtained axioms are all independent. The approximation operator is also studied by Yao [35], and Zhu et al. [39] from different aspects. In this subsection, we give the axiomatic characterization of the 1-type vertex graph lower approximation operator, for the 1-type vertex graph upper approximation operator, we will investigate it in the future.

Theorem 4.3. An operator $L : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $L(G_1) \subseteq G_1$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $V(L((U, \emptyset))) = U$,
(4) if $V(L((V_1, \emptyset))) = V'_1$, then $V(L((V'_1, \emptyset))) = V'_1$,
(5) if $E(L((U, E_1))) = E'_1$, then $E(L((U, E'_1))) = E'_1$,
(6) $L(G_1) = (V(L((V_1, \emptyset))), E(L((U, E_1))) \cap E'(V(L((V_1, \emptyset)))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of predecessor serial binary relation S_U on U and binary relation S_R on R such that $L = \underline{S^1_v}$.

Proof. "⇐" It follows immediately from Propositions 3.1 and 3.3, and Definition 3.2.

"⇒" We first construct $S = (S_U, S_R)$. Let

$$O_1 = \{V \subseteq U \mid V(L((V, \emptyset))) = V\},$$
$$O_2 = \{V \in O_1 \mid \text{ for any } V' \in O_1, \text{ if } V' \subseteq V, \text{ then } V' = V\},$$
$$O_3 = \{V \in O_1 - O_2 \mid V \text{ is not the union of some elements in } O_2\}.$$

Then $O_1, O_2, O_3 \subseteq \mathcal{P}(U)$. Assume that $|O_2| = m$ and $|O_3| = l$. Let $O_2 = \{V'_1, V'_2, \cdots, V'_m\}$ and $O_3 = \{V'_{m+1}, V'_{m+2}, \cdots, V'_{m+l}\}$. Let us take $m + l$ different elements $y_1, y_2, \cdots, y_{m+l}$ from U. Set $S_U = \{(y_i, x) \in U \times U \mid x \in V_i, i = 1, 2, \cdots, m + l\}$. Then we have $(S_U)_s(y_i) = V_i$, where $i = 1, 2, \cdots, m + l$. By the construction, we know that S_U is a predecessor serial relation. For the S_R, let

$$T_1 = \{E \subseteq R \mid E(L((U, E))) = E\},$$
$$T_2 = \{E \in T_1 \mid \text{ for any } E' \in T_1, \text{ if } E' \subseteq E, \text{ then } E' = E\},$$
$$T_3 = \{E \in T_1 - T_2 \mid E \text{ is not the union of some elements in } T_2\}.$$

Then $T_1, T_2, T_3 \subseteq \mathcal{P}(R)$. Assume that $|T_2| = n$ and $|T_3| = q$. Let $T_2 = \{E'_1, E'_2, \cdots, E'_n\}$ and $T_3 = \{E'_{n+1}, E'_{n+2}, \cdots, E'_{n+q}\}$. Taking $n + q$ different elements $r_1, r_2, \cdots, r_{n+q}$ from R, and set $S_R = \{(r_i, r) \in R \times R \mid r \in E_i, i = 1, 2, \cdots, n + q\}$. Then we have $(S_R)_s(r_i) = E_i$, where $i = 1, 2, \cdots, n + q$.

It is obvious that $\underline{S^1_v}(G_1) \subseteq G_1$. Now we prove that $\underline{S^1_v}(G_1) = L(G_1)$, for any $x \in \underline{S^1_U}(V_1)$, we have that there exists $y \in \{y_1, y_2, \cdots, y_{m+l}\}$, assume that $y = y_k$, such that $x \in (S_U)_s(y_k) = \overline{V'_k} \subseteq V_1$, and so $x \in V(L(((V'_k, \emptyset))) \subseteq V(L(((V_1, \emptyset)))$ by the construction and (2). Hence $\underline{S^1_U}(V_1) \subseteq V(L(((V_1, \emptyset)))$. Conversely, for any $x \in V(L((V_1, \emptyset)))$, assume that $V(L((V_1, \emptyset))) = V_3$, then by (4), we have $V(L((V_3, \emptyset))) = V_3$, and so $x \in V(L((V_3, \emptyset)))$. By the construction of S_U, we know that there exists $y \in \{y_1, y_2, \cdots, y_{m+l}\}$, assume that $y = y_k$, such that $x \in (S_U)_s(y_k) = V'_k \subseteq V_3$. Hence $x \in \underline{S^1_U}(V_3) \subseteq \underline{S^1_U}(V_1)$ by the definition of 1-type lower approximation operator. Thus $V(L((V_1, \emptyset))) \subseteq \underline{S^1_U}(V_1)$. Therefore, we have $\underline{S^1_U}(V_1) = V(L((V_1, \emptyset)))$. Analogously, we obtain $\underline{S^1_R}(E_1) = E(L((U, E_1)))$. Thus by (6) and Definition 3.2, we have $\underline{S^1_v}(G_1) = L(G_1)$. This completes the proof. □

4.3. *On axiomatic characterizations of* 2*-type vertex graph approximation operators*

Ma and Mi [16] introduced 2-type approximation operators. Based on them, we introduced 2-type vertex graph approximation operators. Now we present the axiomatic characterizations.

Theorem 4.4. An operator $L : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $L(G_1) \subseteq G_1$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $V(L((U, \emptyset))) = U$,
(4) for any $x \in U$ and any $V \subseteq U$, $x \in V(L((V, \emptyset)))$ if and only of $\exists y \in V$ such that $V(L((\{x, y\}, \emptyset))) = \{x, y\}$,
(5) for any $r_1 \in R$ and any $E \subseteq R$, $r_1 \in E(L((U, E)))$ if and only of $\exists r_2 \in E$ such that $E(L((U, \{r_1, r_2\}))) = \{r_1, r_2\}$,
(6) $L(G_1) = (V(L((V_1, \emptyset))), E(L((U, E_1))) \cap E'(V(L((V_1, \emptyset)))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of serial and symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $L = \underline{S^2_v}$.

Proof. "⇐" It follows from Propositions 3.1 and 3.3, and Definition 3.2.

"⇒" We first construct $S = (S_U, S_R)$ as follows: for any $x, y \in U$, $(x, y) \in S_U$ if and only if $V(L((\{x, y\}, \emptyset))) = \{x, y\}$; for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if $E(L((E, \{r_1, r_2\}))) = \{r_1, r_2\}$. By the construction, (3) and (4), we know that S_U is a serial and symmetric binary relation and S_R is a symmetric binary relation. It is easy to show that $\underline{S^2_v}(G_1) \subseteq G_1$. Then we prove that $\underline{S^2_v}(G_1) = L(G_1)$, for any $x \in \underline{S^2_U}(V_1)$, by the definition of 2-type lower approximation operator, there exists $y \in V_1$ such that (x, y) or $(y, x) \in S_U$, and then by the construction and (2), we have $\{x, y\} = V(L((\{x, y\}, \emptyset))) \subseteq V(L((V_1, \emptyset)))$. Hence $\underline{S^2_U}(V_1) \subseteq V(L((V_1, \emptyset)))$. Conversely, for any $x \in V(L((V_1, \emptyset)))$, by (4), there exists $y \in V_1$ such that $\{x, y\} = \overline{V}(L((\{x, y\}, \emptyset)))$, and so $(x, y) \in S_U$ by the construction. In addition, by (1), we have $x \in V_1$, and hence $x \in \underline{S^2_U}(V_1)$ by the definition of 2-type lower approximation operator. Thus $V(L((V_1, \emptyset))) \subseteq \underline{S^2_U}(V_1)$. Therefore, we have $\underline{S^2_U}(V_1) = V(L((V_1, \emptyset)))$. By the same way, we can obtain that $\underline{S^2_R}(E_1) = E(L((U, E_1)))$. Hence $\underline{S^2_v}(G_1) = L(G_1)$. This completes the proof. □

Theorem 4.5. An operator $H : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $H(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
(2) $H(G_1 \cup G_2) \supseteq H(G_1) \cup H(G_2)$,
(3) for any $x, y \in U$, $x \in V(H((\{y\}, \emptyset)))$ if and only if $y \in V(H((\{x\}, \emptyset)))$, for any $r_1, r_2 \in R$, $r_1 \in E(H((U, \{r_2\})))$ if and only if $r_2 \in E(H((U, \{r_1\})))$,
(4) for any $x \in U$ and any $V \subseteq U$, $x \in V(H((V, \emptyset)))$ if and only if $\exists y \in V$ such that $x \in V(H((\{y\}, \emptyset)))$, for any $r_1 \in R$ and $E \subseteq R$, $r_1 \in V(H((U, E)))$ if and only if $\exists r_2 \in R$ such that $r_1 \in V(H((U, \{r_2\})))$,
(5) $H(G_1) = (V(H((V_1, \emptyset))), E(H((U, E_1))) \cap E'(V(H((V_1, \emptyset)))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $H = \overline{S^2_v}$.

Proof. "⇐" It can be obtained directly by Propositions 3.1 and 3.3, and Definition 3.2.

"⇒" We construct $S = (S_U, S_R)$ as the way: for any $x, y \in U$, $(x, y) \in S_U$ if and only if $x \in V(H((\{y\}, \emptyset)))$; for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if $r_1 \in E(H((U, \{r_2\})))$. We know that both S_U and S_R are symmetric binary relations. It is obvious that $\overline{S^2_v}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = H(\boldsymbol{\emptyset})$. Now we show $\overline{S^2_v}(G_1) = H(G_1)$, for any $x \in \overline{S^2_U}(V_1)$, there exists $y \in V_1$ such that (x, y) or $(y, x) \in S_U$, and then $x \in V(H((\{y\}, \emptyset))) \subseteq V(H((V_1, \emptyset)))$ by the construction and (2). Hence $\overline{S^2_U}(V_1) \subseteq V(H((V_1, \emptyset)))$. Conversely, for any $x \in V(H((V_1, \emptyset)))$, by (4), there exists $y \in V_1$ such that $x \in V(H((\{y\}, \emptyset)))$, and so $(x, y) \in S_U$ by the construction. Then $x \in \overline{S^2_U}(\{y\}) \subseteq \overline{S^2_U}(V_1)$. Thus $V(H((V_1, \emptyset))) \subseteq \overline{S^2_U}(V_1)$. Therefore, we have $\overline{S^2_U}(V_1) = V(H((V_1, \emptyset)))$. Similarly, we obtain $\overline{S^2_R}(E_1) = E(H((U, E_1)))$. By (5) and Definition 3.2, we have $\overline{S^2_v}(G_1) = H(G_1)$. This completes the proof. □

4.4. *On axiomatic characterizations of* 3*-type vertex graph approximation operators*

The axiomatic characterizations of 3-type approximation operators are studied by Ma and Mi [16], as a generalization, we investigate axiomatic characterizations of 3-type vertex graph approximation operators in this subsection.

Theorem 4.6. An operator $L : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $L((U, R)) = (U, R)$,
(2) $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$,
(3) $L(G_1) \subseteq G_1$,
(4) $L(G_1) = (V(L((V_1, \emptyset))), E(L((U, E_1))) \cap E'(V(L((V_1, \emptyset)))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $L = \underline{S^3_v}$.

Proof. "⇐" It follows immediately from Propositions 3.1 and 3.3, and Definition 3.2.

"⇒" We can construct $S = (S_U, S_R)$ as follows: for any $x, y \in U$, $(x, y), (y, x) \in S_U$ if and only if there exists a subset $V \subseteq U$ such that $x \in V(L((V, \emptyset)))$ and $x \notin V(L((V - \{y\}, \emptyset)))$, and similarly, we can construct S_R. We know that both S_U and S_R are symmetric relations. The $\underline{S^3_v}((U, R)) = (U, R) = L((U, R))$ is obvious. Then we prove $\underline{S_v^3}(G_1) = L(G_1)$. From the construction and (3), we obtain that $(S_U)_s(x) \cup (S_U)_p(x) \cup \{x\} \subseteq V_1$ if and only if $x \in V(L((V_1, \emptyset)))$, for any $x \in U$. Besides, by the definition of 3-type lower approximation operator, we have $x \in \underline{S_U^3}(V_1)$ if and only if $(S_U)_s(x) \cup (S_U)_p(x) \cup \{x\} \subseteq V_1$. Hence $\underline{S_U^3}(V_1) = V(L((V_1, \emptyset)))$. In the same way, we can obtain that $\underline{S_R^3}(E_1) = E(L((U, E_1)))$. By (4) and Definition 3.2, we have $\underline{S_v^3}(G_1) = L(G_1)$. This completes the proof. □

Theorem 4.7. An operator $H : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $H(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
(2) $H(G_1 \cup G_2) \supseteq H(G_1) \cup H(G_2)$,
(3) $G_1 \subseteq H(G_1)$,
(4) for any $x, y \in U$, $x \in V(H((\{y\}, \emptyset)))$ if and only if $y \in V(H((\{x\}, \emptyset)))$, for any $r_1, r_2 \in R$, $r_1 \in E(H((U, \{r_2\})))$ if and only if $r_2 \in E(H((U, \{r_1\})))$,
(5) for any $x \in U$ and any $V \subseteq U$, $x \in V(H((V, \emptyset)))$ if and only if $\exists y \in V$ such that $x \in V(H((\{y\}, \emptyset)))$, for any $r_1 \in R$ and any $E \subseteq R$, $r_1 \in V(H((U, E)))$ if and only if $\exists r_2 \in R$ such that $r_1 \in V(H((U, \{r_2\})))$,
(6) $H(G_1) = (V(H((V_1, \emptyset))), E(H((U, E_1))) \cap E'(V(H((V_1, \emptyset)))))$

if and only if there exists a unique ordered pair $S = (S_U, S_R)$ of tolerance binary relation S_U on U and tolerance binary relation S_R on R such that $H = \overline{S_v^3}$.

Proof. "⇐" It follows from Propositions 3.1 and 3.3, and Definition 3.2.

"⇒" We construct $S = (S_U, S_R)$ by the following way: for any $x, y \in U$, $(x, y) \in S_U$ if and only if $x \in V(H((\{y\}, \emptyset)))$; for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if $r_1 \in E(H((U, \{r_2\})))$. From the construction and (3), we know that both S_U and S_R are tolerance relations. It is easy to show that $\overline{S_v^3}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = H(\boldsymbol{\emptyset})$. Now we prove that $\overline{S_v^3}(G_1) = H(G_1)$, for any $x \in \overline{S_U^3}(V_1)$, by the definition of 3-type upper approximation operator and S_U is reflexive, there exists $y \in V_1$ such that (x, y) or $(y, x) \in S_U$, hence $x \in V(H((\{y\}, \emptyset))) \subseteq V(H((V_1, \emptyset)))$ by the construction and (2). Thus $\overline{S_U^3}(V_1) \subseteq V(H((V_1, \emptyset)))$. Conversely, for any $x \in V(H((V_1, \emptyset)))$, by (4), there exists $y \in V_1$ such that $x \in V(H((\{y\}, \emptyset)))$, and so by the construction, we have $(x, y) \in S_U$, which implies that $x \in \overline{S_U^3}(\{y\}) \subseteq \overline{S_U^3}(V_1)$ by the definition of 3-type upper approximation operator. Hence $V(H((V_1, \emptyset))) \subseteq \overline{S_U^3}(V_1)$. Therefore, we have that $\overline{S_U^3}(V_1) = V(H((V_1, \emptyset)))$. Analogously, we obtain that $\overline{S_R^3}(E_1) = E(H((U, E_1)))$. By (6) and Definition 3.2, we obtain $\overline{S_v^3}(G_1) = H(G_1)$. This completes the proof. □

Remark 4.1. The uniqueness of ordered pairs of equivalence relations which have the same 1-type vertex graph lower approximation operator as the abstract vertex graph lower approximation operator, the same 2-type vertex graph lower approximation operator as the abstract vertex graph lower approximation operator, or the same 2-type vertex graph upper approximation operator as the abstract vertex graph upper approximation operator can be obtained since 0-type, 1-type, and 2-type approximation operators are equivalent to each other with respect to equivalence relations [16] and Theorems 4.1 and 4.2. By Theorem 3 in [16] and Theorem 4.1, we have that the uniqueness of ordered pair of tolerance binary relations which has the same 3-type vertex graph lower approximation operator as the abstract vertex graph lower approximation operator.

5. On axiomatic characterizations of four types of edge graph approximation operators

In this section, like four types of vertex graph approximation operators, we provide axiomatic characterizations of four types of edge graph approximation operators.

5.1. On axiomatic characterizations of 0*-type edge graph approximation operators*

We first present the axiomatic characterization of the 0-type edge graph lower approximation operator which can be obtained by the same way as that of the 0-type vertex graph lower approximation operator.

Theorem 5.1. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L((U,R)) = (U,R)$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $L(G_1) = (V(L((V_1,\emptyset))) \cup V'(E(L((U,E_1)))), E(L((U,E_1))))$

if and only if there exists a unique ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $L = \underline{S_e^0}$.

Proof. By Definition 3.3, and Propositions 3.4 and 3.5, the proof is similar to Theorem 4.1. We thus omit the details. □

For the axiomatic characterization of the 0-type edge graph upper approximation operator, we have the following.

Theorem 5.2. An operator $H : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $H(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
(2) $H(G_1 \cup G_2) = H(G_1) \cup H(G_2)$,
(3) $H(G_1) = (V(H((V_1,\emptyset))) \cup V'(E(H((U,E_1)))), E(H((U,E_1))))$

if and only if there exists a unique ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $H = \overline{S_e^0}$.

Proof. "⇐" It is easy to get by Definition 3.3, and Propositions 3.4 and 3.5.

"⇒" We construct $S = (S_U, S_R)$ by the same way as Theorem 4.2. By Proposition 3.5, we have $\overline{S_e^0}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset} = H(\boldsymbol{\emptyset})$. For any $x \in U$, by the construction and (2), we obtain that $x \in \overline{S_U^0}(V_1)$ if and only if $(S_U)_s(x) \cap V_1 \neq \emptyset$ if and only if $x \in V(H((V_1,\emptyset)))$, which implies that $\overline{S_U^0}(V_1) = V(H((V_1,\emptyset)))$. Analogously, we obtain $\overline{S_R^0}(E_1) = E(H((U,E_1)))$. Hence we have $(\overline{S_U^0}(V_1) \cup V'(\overline{S_R^0}(E_1)), \overline{S_R^0}(E_1)) = (V(H((V_1,\emptyset))) \cup V'(E(H((U,E_1)))), E(H((U,E_1))))$, i.e., $\overline{S_e^0}(G_1) = H(G_1)$. By Definition 3.3 and Theorem 6 of [39] again, we get the uniqueness of S. □

5.2. On axiomatic characterizations of 1*-type edge graph approximation operators*

In this subsection, the axiomatic characterization of the 1-type edge graph lower approximation operator is provided, and in the future, we will study the 1-type edge graph upper approximation operator.

Theorem 5.3. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L(G_1) \subseteq G_1$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $V(L((U,\emptyset))) = U$,
(4) if $V(L((V_1,\emptyset))) = V_1'$, then $V(L((V_1',\emptyset))) = V_1'$,
(5) if $E(L((U,E_1))) = E_1'$, then $E(L((U,E_1'))) = E_1'$,
(6) $L(G_1) = (V(L((V_1,\emptyset))) \cup V'(E(L((U,E_1)))), E(L((U,E_1))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $L = \underline{S_e^1}$.

Proof. By Definition 3.3 and Propositions 3.4 and 3.6, it can be obtained in a similar way as Theorem 4.3. We thus omit the details. □

5.3. On axiomatic characterizations of the 2-type edge graph approximation operators

In a similar way as axiomatic characterizations of 2-type vertex graph approximation operators, we can obtain axiomatic characterizations of 2-type edge graph approximation operators.

Theorem 5.4. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L(G_1) \subseteq G_1$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) for any $x \in U$ and $V \subseteq U$, $x \in V(L((V,\emptyset)))$ if and only of $\exists y \in V$ such that $V(L((\{x,y\},\emptyset))) = \{x,y\}$,
(4) for any $r_1 \in R$ and $E \subseteq R$, $r_1 \in E(L((U,E)))$ if and only of $\exists r_2 \in E$ such that $E(L((U,\{r_1,r_2\}))) = \{r_1,r_2\}$
(5) $L(G_1) = (V(L((V_1,\emptyset))) \cup V'(E(L((U,E_1)))), E(L((U,E_1))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $L = \underline{S_e^2}$.

Proof. It follows immediately in a similar way to that of Theorem 4.4 by Definition 3.3, and Propositions 3.4 and 3.6. Hence we omit the proof. □

Theorem 5.5. An operator $H : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $H(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$,
(2) $H(G_1 \cup G_2) = H(G_1) \cup H(G_2)$,
(3) for any $x, y \in U$, $x \in V(H((\{y\},\emptyset)))$ if and only if $y \in V(H((\{x\},\emptyset)))$, for any $r_1, r_2 \in R$, $r_1 \in E(H((U,\{r_2\})))$ if and only if $r_2 \in E(H((U,\{r_1\})))$,
(4) for any $x \in U$ and for any $V \subseteq U$, $x \in V(H((V,\emptyset)))$ if and only if $\exists y \in V$ such that $x \in V(H((\{y\},\emptyset)))$, for any $r_1 \in R$ and for any $E \subseteq R$, $r_1 \in V(H((U,E)))$ if and only if $\exists r_2 \in R$ such that $r_1 \in V(H((U,\{r_2\})))$,
(5) $H(G_1) = (V(H((V_1,\emptyset))) \cup V'(E(L(U,E_1))), E(H((U,E_1))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $H = \overline{S_e^2}$.

Proof. The proof is analogous to that of Theorem 4.5 by Definition 3.3, and Propositions 3.4 and 3.6. We thus omit the details. □

5.4. On axiomatic characterizations of 3-type edge graph approximation operators

The axiomatic characterizations of 3-type edge graph approximation operators are also analogous to axiomatic characterizations of 3-type vertex graph approximation operators, and hence we omit the proofs.

Theorem 5.6. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L((U,R)) = (U,R)$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $L(G_1) \subseteq G_1$,
(4) $L(G_1) = (V(L((V_1,\emptyset))) \cup V'(E(L((U,E_1)))), E(L((U,E_1))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $L = \underline{S_e^3}$.

Proof. By Definition 3.3, and Propositions 3.4 and 3.6, the proof is similar to that of Theorem 4.6. We thus omit the details. □

Theorem 5.7. An operator $H : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $H(\emptyset) = \emptyset$,
(2) $H(G_1 \cup G_2) = H(G_1) \cup H(G_2)$,
(3) $G_1 \subseteq H(G_1)$,
(4) for any $x, y \in U$, $x \in V(H((\{y\}, \emptyset)))$ if and only if $y \in V(H((\{x\}, \emptyset)))$, for any $r_1, r_2 \in R$, $r_1 \in E(H((U, \{r_2\})))$ if and only if $r_2 \in E(H((U, \{r_1\})))$,
(5) for any $x \in U$ $V \subseteq U$, $x \in V(H((V, \emptyset)))$ if and only if $\exists y \in V$ such that $x \in V(H((\{y\}, \emptyset)))$, for any $r_1 \in R$ and for any $E \subseteq R$, $r_1 \in V(H((U, E)))$ if and only if $\exists r_2 \in R$ such that $r_1 \in V(H((U, \{r_2\})))$,
(6) $H(G_1) = (V(H((V_1, \emptyset))) \cup V'(E(H((U, E_1)))), E(H((U, E_1))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of tolerance binary relation S_U on U and tolerance binary relation S_R on R such that $H = \overline{S_e^3}$.

Proof. By Definition 3.3, and Propositions 3.4 and 3.6, the proof can be obtained in a similar way to that of Theorem 4.6. Hence we omit the proof. □

Remark 5.1. Since 0-type, 1-type, and 2-type approximation operators are equivalent to each other with respect to equivalence relations [16], and Theorems 5.1 and 5.2, we have that there exists a unique ordered pair of equivalence relations such that the 1-type edge graph lower approximation operator is the same as the abstract edge graph lower approximation operator, and a unique ordered pairs of equivalence relations such that 2-type edge graph lower and upper approximation operators are the same as the abstract edge graph lower and upper approximation operators, respectively. The uniqueness of ordered pair of tolerance binary relations which has the same 3-type edge graph lower approximation operator as the abstract edge graph lower approximation operator can be obtained by Theorem 3 in [16] and Theorem 4.1.

6. On axiomatic characterizations of four types of graph approximation operators

In this section, we give axiomatic characterizations of 0-type, 1-type, and 3-type graph lower approximation operators. The axiomatic characterizations of 2-type graph approximation operators, and 0-type, 1-type, and 3-type graph upper approximation operators have not been solved, and we will study in the future.

6.1. *On the axiomatic characterization of the* 0*-type graph lower approximation operator*

Theorem 6.1. An operator $L : \mathcal{G}((U, R)) \to \mathcal{G}((U, R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U, R))$,

(1) $L((U, R)) = (U, R)$,
(2) $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$,
(3) $L(G_1) = (V'(E(L((U, E_1))) \cap E'(V(L((V_1, E'(V_1)))))), E(L((U, E_1))) \cap E'(V(L((V_1, E'(V_1))))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $L = \underline{S^0}$.

Proof. "⇐" It follows immediately from Definition 3.4, and Propositions 3.7 and 3.8.

"⇒" We construct $S = (S_U, S_R)$ by the following way: for any $x, y \in U$, $(x, y) \in S_U$ if and only if there exists a subset $V \subseteq U$ such that $y \in V$, $x \in V(L((V, E'(V))))$ and $x \notin V(L((V - \{y\}, E'(V - \{y\}))))$; for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if there exists a subset $E \subseteq R$ such that $r_2 \in E$, $r_1 \in E(L((U, E)))$ and $r_1 \notin E(L((U, E - \{r_2\})))$. It is obvious that $\underline{S^0}((U, R)) = (U, R) = L((U, R))$. For any $x \in U$, by the construction and (2), we have $x \in \underline{S_U^0}(V_1)$ if and only if $(S_U)_s(x) \subseteq V_1$ if and only if $x \in V(L((V_1, E'(V_1))))$. Hence $\underline{S_U^0}(V_1) = V(L((V_1, E'(V_1))))$. Similarly, $\underline{S_R^0}(E_1) = E(L((U, E_1)))$. Therefore, we have $\underline{S^0}(G_1) = (V'(\underline{S_R^0}(E_1) \cap E'(\underline{S_U^0}(V_1))), \underline{S_R^0}(E_1) \cap E'(\underline{S_U^0}(V_1))) = (V'(E(L((U, E_1))) \cap E'(V(L((V_1, E'(V_1)))))), E(L((U, E_1))) \cap E'(V(L((V_1, E'(V_1)))))) = L(G_1)$. □

6.2. On the axiomatic characterization of the 1*-type graph lower approximation operator*

Theorem 6.2. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L(G_1) \subseteq G_1$,
(2) $L(G_1 \cap G_2) \subseteq L(G_1) \cap L(G_2)$,
(3) $L((V_1, E'(V_1))) = L((U, E'(V_1)))$,
(4) if $E(L((U,E_1))) = E'_1$, then $E(L((U,E'_1))) = E'_1$, if $L((V_1,E_1)) = (V'_1, E_1)$, then $L((V'_1,E_1)) = (V'_1,E_1)$,
(5) $L(G_1) = (V'(E(L((U,E_1))) \cap E'(V(L((V_1,E'(V_1)))))), E(L((U,E_1))) \cap E'(V(L((V_1,E'(V_1))))))$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of binary relation S_U on U and binary relation S_R on R such that $L = \underline{S^1}$, where S satisfies the following (a),(b),(c).

(*a*) $\underline{S^1}((V_1, E'(V_1))) = \underline{S^1}((U, E'(V_1)))$,
(*b*) if $E(\underline{S^1}((U,E_1))) = E'_1$, then $E(\underline{S^1}((U,E'_1))) = E'_1$,
(*c*) if $\underline{S^1}((V_1,E_1)) = (V'_1,E_1)$, then $\underline{S^1}((V'_1,E_1)) = (V'_1,E_1)$,

Proof. "⇐" It follows immediately from Definition 3.4, and Propositions 3.7 and 3.9.
"⇒" We first construct $S = (S_U, S_R)$. Let

$$P_1 = \{E \subseteq R \,|\, E(L((U,E))) = E\},$$
$$P_2 = \{E \in P_1 \,|\, \text{for any } E' \in P_1, \text{ if } E' \subseteq E, \text{ then } E' = E\},$$
$$P_3 = \{E \in P_1 - P_2 \,|\, E \text{ is not the union of some elements in } P_2\}.$$

Then $P_1, P_2, P_3 \subseteq \mathcal{P}(R)$. Assume that $|P_2| = m$ and $|P_3| = l$. Let $P_2 = \{E'_1, E'_2, \cdots, E'_m\}$ and $P_3 = \{E'_{m+1}, E'_{m+2}, \cdots, E'_{m+l}\}$. Let us take $m+l$ different elements $r_1, r_2, \cdots, r_{m+l}$ from R. Set $S_R = \{(r_i, r) \in R \times R \,|\, r \in E_i,\ i = 1,2,\cdots,m+l\}$. Then we have $(S_R)_s(r_i) = E_i$, where $i = 1,2,\cdots,m+l$. For the S_U, let

$$O_1 = \{V \subseteq U \,|\, (V,E_i) \in \mathcal{G}((U,R)), L((V,E_i)) = (V,E_i), i = 1,2,\cdots,m+l\},$$
$$O_2 = \{V \in O_1 \,|\, \text{for any } V' \in O_1, \text{ if } V' \subseteq V, \text{ then } V' = V\},$$
$$O_3 = \{V \in O_1 - O_2 \,|\, V \text{ is not the union of some elements in } O_2\}.$$

Then $O_1, O_2, O_3 \subseteq \mathcal{P}(U)$. Assume that $|O_2| = n$ and $|O_3| = q$. Let $O_2 = \{V'_1, V'_2, \cdots, V'_n\}$ and $O_3 = \{V'_{n+1}, V'_{n+2}, \cdots, V'_{n+q}\}$. Taking $n+q$ different elements $y_1, y_2, \cdots, y_{n+q}$ from U. Set $S_U = \{(y_i, x) \in U \times U \,|\, x \in V'_i, i = 1,2,\cdots,n+q\}$.

It is obvious that $\underline{S^1}(G_1) \subseteq G_1$. Now we prove that $\underline{S^1}(G_1) = L(G_1)$. For any $r \in R$, by the construction and (2), $r \in \underline{S^1_R}(E_1)$ if and only if there exists $r' \in \{r_1, r_2, \cdots, r_{m+l}\}$, assume that $r' = r_k$, such that $r \in (S_R)_s(r_k) = E'_k \subseteq E_1$ if and only if $r \in E(L((U,E_1)))$. Hence $\underline{S^1_R}(E_1) = E(L((U,E_1)))$. For any $x \in U$, by the construction, (2), (3), and (4), $x \in \underline{S^1_U}(V_1)$ if and only if there exists $y' \in \{y_1, y_2, \cdots, y_{n+q}\}$, assume that $y' = y_t$, such that $x \in (S_U)_s(y_t) \subseteq V_1$ if and only if $x \in V(L((V_1, E'_i)))$ for some $E'_i \in P_2 \cup P_3$ if and only if $x \in V(L((V_1, E'(V_1))))$. Hence $\underline{S^1_U}(V_1) = V(L((V_1, E'(V_1))))$. By Definition (5) and 3.4, we obtain that $\underline{S^1}(G_1) = L(G_1)$. □

6.3. On the axiomatic characterization of the 3*-type graph lower approximation operator*

Theorem 6.3. An operator $L : \mathcal{G}((U,R)) \to \mathcal{G}((U,R))$ satisfies the following axioms: for any $G_1, G_2 \in \mathcal{G}((U,R))$,

(1) $L((U,R)) = (U,R)$,
(2) $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$,
(3) $L(G_1) \subseteq G_1$,
(4) $L(G_1) = (V'(E(L((U,E_1))) \cap E'(V(L((V_1,E'(V_1)))))), E(L((U,E_1))) \cap E'(V(L((V_1,E'(V_1)))))).$

if and only if there exists an ordered pair $S = (S_U, S_R)$ of symmetric binary relation S_U on U and symmetric binary relation S_R on R such that $L = \underline{S}^3$.

Proof. "$\Leftarrow$" It is easy to get by Definition 3.4, and Propositions 3.7 and 3.9.

"$\Rightarrow$" We construct $S = (S_U, S_R)$ as follows: for any $x, y \in U$, $(x, y), (y, x) \in S_U$ if and only if there exists a subset $V \subseteq U$ such that $y \in V$, $x \in V(L((V, E'(V))))$ and $x \notin V(L((V - \{y\}, E'(V - \{y\}))))$; for any $r_1, r_2 \in R$, $(r_1, r_2) \in S_R$ if and only if there exists a subset $E \subseteq R$ such that $r_2 \in E$, $r_1 \in E(L((U, E)))$ and $r_1 \notin E(L((U, E - \{r_2\})))$. It is obvious that $\underline{S}^3((U, R)) = (U, R) = L((U, R))$. For any $x \in U$, by the construction, (2) and (3), we have $x \in \underline{S_U}^3(V_1)$ if and only if $(S_U)_p(x) \cup (S_U)_s(x) \cup \{x\} \subseteq V_1$ if and only if $x \in V(L((V_1, E'(V_1))))$. Hence $\underline{S_U}^3(V_1) = V(L((V_1, E'(V_1))))$. Similarly, we obtain that $\underline{S_R}^3(E_1) = E(L((U, E_1)))$. Therefore, we have $\underline{S}^3(G_1) = L(G_1)$. □

Remark 6.1. From Definition 3.4 and the constructions of $S = (S_U, S_R)$, it is easy to obtain that binary relations S in Theorems 6.1, 6.2, and 6.3 are not unique, respectively.

7. Conclusions

As is well known, relational data cannot be neglected, and graphs which contain the data on edges can represent more complex knowledges than vertex sets. This paper developed rough set analysis of graphs. Vertex graph, edge graph, and graph approximation operators were introduced. We explored their properties within the constructive approach, and then within the axiomatic approach.

However, axiomatic characterizations of 0-type graph upper approximation operator, and 1-type vertex graph, edge graph, and graph upper approximation operators, 2-type graph approximation operators, and 3-type graph upper approximation operators are still problems that need further consideration. In the future, we will explore the relationships between pairs of different kinds of relations, including serial, reflexive, symmetric, transitive relations as well as their compositions, and approximation operators of graphs, and investigate that approximation operators of graphs corresponding to pairs of special kind of relations can be characterized by axioms. Then we will explore fuzzy rough graphs based on fuzzy relations, extend the work of this paper to the covering based rough graphs and fuzzy covering based rough graphs, investigate rough set analysis of graphs based on other types of approximation operators of sets, and study applications of the rough graph model in feature selection, decision analysis, and the analysis of graphs.

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