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On the Matrix Version of Extended Struve Function and its Application on Fractional Calculus

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Abstract. The main goal of this article is to study the extend Struve and extended modified Struve matrix functions by making use of extended Beta matrix function. In particular, we investigate certain important properties of these extended matrix functions such as integral representation, differentiation formula and hypergeometric representation of these functions. Finally, we obtain some results on the transform and fractional calculus of these extended Struve and extended modified Struve matrix functions.

1. Introduction and preliminaries

A wide range of special functions in applied sciences are defined via improper integrals or infinite series. During last decades, several special functions become essential tools for scientists and engineering due to their applications in mathematical physics, engineering, and Lie theory. This inspire the study of the extensions of the special functions. In last few years, many extensions of gamma function, beta function, and Gauss hypergeometric functions have been studied by many researchers(see [4, 9, 27]).

Struve functions are mainly investigated because of their intrinsic mathematical importance in various problems in many branches of physics and mathematics, because these functions are shown to be natural particular solutions of a set of ordinary and partial differential equations. The Struve functions $H_v(z)$, which have close relationship with Bessel functions, appeared as special solutions of the inhomogeneous Bessel second-order differential equations (see[32–34])

$$z^2\frac{d^2W}{dz^2}+z\frac{dW}{dz}+(z^2-v^2)W=\frac{4}{\sqrt{\pi}\Gamma(v+\frac{1}{2})}(\frac{z}{2})^{v+1}, \tag{1}$$

where $W=W(z)=H_v(z)+c_1J_v(z)+c_2Y_v(z)$, c_1 and c_2 are arbitrary constants, such that $J_v(z)$ is the Bessel functions of the first kind and $Y_v(z)$ is Bessel function of the second kind as follow

$$J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{2k+v}}{k!\Gamma(v+k+1)},\quad Y_v(z)=\frac{J_v(z)\cos(v\pi)-J_{-v}(z)}{\sin(v\pi)}.$$

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The modified Struve $L_A(z)$ which closely related to the modified Bessel functions are solutions of the equations

$$z^2\frac{d^2W}{dz^2}+z\frac{dW}{dz}-(z^2+v^2)W=\frac{4}{\sqrt{\pi}\Gamma(v+\frac{1}{2})}(\frac{z}{2})^{v+1}, \tag{2}$$

where $W=W(z)=L_A(z)+c_1I_v(z)+c_2M_v(z)$, c_1 and c_2 are arbitrary constants, such that $I_v(z)$ is the modified Bessel functions of the first kind and $M_v(z)$ is the modified Bessel function of the second kind as follow

$$I_v(z)=\sum_{k=0}^{\infty}\frac{(\frac{z}{2})^{2k+v}}{k!\Gamma(v+k+1)},\quad M_v(z)=\frac{\pi}{2}\frac{I_{-v}(z)-I_v(z)}{\sin(v\pi)}.$$

Nowadays, matrix generalization of special functions has become very important during last years. One of the motivations is that special matrix functions provide solutions to some physical problems, another is that special matrix functions are closely related to orthogonal matrix polynomials(see [10–13]). Special matrix functions like gamma, beta and Bessel matrix function are frequently used in statistics[20, 28], Lie groups theory [20, 29] and in the solution of matrix differential equations [5, 8, 15, 16, 21–23]. Recently, a generalization of Gamma, Psi and Beta matrix functions and some properties are established in [1–3]. Very recently, Goyalet al. [14, 19] introduced an extension of the Beta matrix function using the Wiman matrix function, thus studying various properties and relation-ships of that function and another study was presented generalized hypergeometric Matrix functions via two-parameter Mittag-Leffler matrix function.

To discuss our main results, we need definition and results of some special matrix functions. Throughout this paper, for $\mathbb{C}^N$ denote the N-dimensional complex vector space and $\mathbb{C}^{N\times N}$ denote all square matrices with N rows and N columns with entries are complex numbers, $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$ denote the real and imaginary parts of a complex number z, respectively. For any matrix A in $\mathbb{C}^{N\times N}$, $\sigma(A)$ is the spectrum of A, the set of all eigenvalues of A, which will be denoted by $||A||$, is defined by

$$||A||=\sup_{x\neq 0}\frac{||Ax||_2}{||x||_2},$$

where for a vector y in $\mathbb{C}^N$, $||y||_2=(y^Hy)^{\frac{1}{2}}$ is Euclidean norm of y. I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A)\subset\Omega$, then, from the properties of the matrix functional calculus (see [6, 17, 22]), it follows that $f(A)g(A)=g(A)f(A)$. Furthermore, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B)\subset\Omega$ and if $AB=BA$, then $f(A)g(B)=g(B)f(A)$. The logarithmic norm of a matrix A in $\mathbb{C}^{N\times N}$ is defined as (see [8, 18])

$$\mu(A)=\lim_{h\to 0}\frac{||I+hA||-1}{h}=\max\{z:z\in\sigma(\frac{A+A^*}{2})\}. \tag{3}$$

Suppose the number $\widetilde{\mu}(A)$ is such that

$$\widetilde{\mu}(A)=-\mu(-A)=\min\{z:z\in\sigma(\frac{A+A^*}{2})\}. \tag{4}$$

Let A and B be two positive stable matrices in $\mathbb{C}^{N\times N}$. The Gamma matrix function $\Gamma(A)$ and the Beta matrix function $\mathfrak{B}(A,B)$ have been defined in [6, 17, 24, 25] as follows

$$\Gamma(A)=\int_0^{\infty}e^{-t}t^{A-I}dt,\quad \mathfrak{B}(A,B)=\int_0^1 t^{A-I}(1-t)^{B-I}dt, \tag{5}$$

where $t^{A-I}=\exp((A-I)\ln t)$. The reciprocal Gamma function denoted by $\Gamma^{-1}(z)=\frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. For $p,q\in\mathbb{Z}^+$, we will denote $\Gamma(A_1)\ldots\Gamma(A_p)\Gamma^{-1}(B_1)\ldots\Gamma^{-1}(B_q)$ by

$$\Gamma\left(\begin{array}{ccc} A_1, & \ldots, & A_p \\ B_1, & \ldots, & B_q \end{array}\right).$$

Furthermore, if

$$A + nI \quad \text{is invertible for all integers } n \geq 0, \tag{6}$$

then, the reciprocal gamma function is defined as [17]

$$\Gamma^{-1}(A) = A(A+I)\dots(A+(n-1)I)\Gamma^{-1}(A+nI), n \geq 1.$$

By application of the matrix functional calculus, for A in $\mathbb{C}^{N\times N}$, then from [8, 15], the Pochhammer symbol of a matrix argument defined by

$$(A)_n = \begin{cases} A(A+I)\dots(A+(n-1)I) = \Gamma^{-1}(A)\Gamma(A+nI), & n \geq 1, \\ I, & n = 0. \end{cases} \tag{7}$$

Jódar and Cortés have proved in [24, 25] that

$$\Gamma(A) = \lim_{n\longrightarrow\infty} (n-1)![(A)_n]^{-1} n^A, \tag{8}$$

where $n \geq 1$is an integer. Let A and B be commuting matrices in $\mathbb{C}^{N\times N}$ such that the matrices $A + nI$, $B + nI$ and $A + B + nI$ are invertible for every integer $n \geq 0$, then , we have (see [6, 17, 25])

$$\mathfrak{B}(A, B) = \Gamma(A)\Gamma(B)[\Gamma(A+B)]^{-1}. \tag{9}$$

Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ and C satisfy condition (6), then the hypergeometric matrix function of 2-numerator and 1-denominator for $|z| < 1$ is defined by the matrix power series (see [8, 25])

$${}_2F_1(A, B; C; z) = \sum_{n\geq 0} \frac{(A)_n(B)_n[(C)_n]^{-1}}{n!} z^n, \tag{10}$$

Let A and B be positive stable matrices in $\mathbb{C}^{N\times N}$ then, the generalized Gamma matrix function $\Gamma(A, B)$ has been defined as follows (see [3])

$$\Gamma(A, B) = \int_0^\infty t^{A-I} e^{-(It+\frac{B}{t})} dt, \quad t^{A-I} = \exp((A-I)\ln t). \tag{11}$$

Let A, B and Q be positive stable and commuting matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (6). Then, the extended Beta matrix function $\mathfrak{B}(A, Q; B)$ is defined in the form (see [1])

$$\mathfrak{B}(A, Q; B) = \int_0^1 t^{A-I}(1-t)^{Q-I} \exp\Big(\frac{-B}{t(1-t)}\Big) dt. \tag{12}$$

Let A, B, C, Q and $C - Q$ be commuting positive stable matrices in $\mathbb{C}^{N\times N}$ and B, C and Q satisfying the condition (6), then, the extended Gauss hypergeometric matrix and extended Kummer hypergeometric matrix functions are defined in the forms (see [2])

$${}_2F_1^{(B)}(A, Q; C; z) = \Gamma(C)\Gamma^{-1}(Q)\Gamma^{-1}(C-Q) \sum_{n=0}^{\infty} (A)_n \mathfrak{B}(Q+nI, C-Q; B) \frac{z^n}{n!}, \tag{13}$$

$${}_1F_1^{(B)}(Q; C; z) = \Gamma(C)\Gamma^{-1}(Q)\Gamma^{-1}(C-Q) \sum_{n=0}^{\infty} \mathfrak{B}(Q+nI, C-Q; B) \frac{z^n}{n!}. \tag{14}$$

The Struve matrix function is defined in the form (see [31])

$$\mathbf{H}_A(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma^{-1}(A + (n+\frac{3}{2})I)}{\Gamma(n+\frac{3}{2})} \Big(\frac{z}{2}\Big)^{A+(2n+1)I}, \tag{15}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying $\widetilde{\mu}(A) > \frac{-3}{2}$. The modified Struve matrix function $\mathbf{L}_A(z)$ is defined as follows

$$\mathbf{L}_A(z) = \sum_{n=0}^{\infty} \frac{\Gamma^{-1}(A + (n + \frac{3}{2})I)}{\Gamma(n + \frac{3}{2})} (\frac{z}{2})^{A+(2n+1)I}, \tag{16}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ such that $\widetilde{\mu}(A) > \frac{-3}{2}$. Later, Bakhet et al. [7] used $\mathfrak{B}(A, Q; B)$ to extended Bessel matrix function: let A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (6)and $\widetilde{\mu}(A) > -\frac{1}{2}$, then, the extended Bessel matrix function $J_{(A,B)}(z)$ is defined as

$$J_{(A,B)}(z) = \frac{(\frac{1}{2})^A\ \Gamma^{-1}(A + \frac{I}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mathfrak{B}((k + \frac{1}{2})I, A + \frac{I}{2}; B) z^{A+2kI}. \tag{17}$$

The main object of this paper is to investigate various and study the extended Struve and extended modified Struve matrix functions by using of extended beta matrix function, then the integral representations, differentiation formula and hypergeometric representation for the extended Struve and extended modified Struve matrix functions are discussed. We also give connections with certain generalized gamma matrix function under certain conditions. As an application, we present some results on the transform and fractional calculus by the extended Struve and extended modified Struve matrix functions.

The section-wise treatment is as follows. In Section 2, we introduce the extended Struve and extended modified Struve matrix functions, then establish the integral representations, differentiation formula and hypergeometric representation of function. In Section 3, we study operators that are of the Riemann-Liouville fractional integral, and of the Riemann-Liouville and Caputo fractional derivatives of extended Struve and extended modified Struve matrix functions. Finally, we give conclusions in Section 4.

2. Extended Struve matrix function

In this section, we study and introduce extended Struve matrix function by extended Beta matrix function. From (15), we can write the struve matrix function as follows

$$\mathbf{H}_A(z) = \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A + \frac{I}{2})}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \mathfrak{B}((n + 1)I, A + \frac{I}{2}) \frac{z^{2n}}{(2n + 1)!}. \tag{18}$$

Definition 2.1. *Let A and B are commuting matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (6) and $\widetilde{\mu}(A) > \frac{-1}{2}$, then, the extended Struve matrix function $\boldsymbol{H}_{(A,B)}(z)$ is defined in the form*

$$\boldsymbol{H}_{(A,B)}(z) = \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A + \frac{I}{2})}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \mathfrak{B}((n + 1)I, A + \frac{I}{2}, B) \frac{z^{2n}}{(2n + 1)!}. \tag{19}$$

Remark 2.2. *Further, we note the following special cases of the extended Struve matrix function as follows*
i*- Putting $B = \mathbf{0}$ in (19) and using Properties of Pochhammer symbol, we get the Struve matrix function in (15)*
ii*- If taking $A = \alpha \in \mathbb{C}^{1\times 1}$ and $B = \beta \in \mathbb{C}^{1\times 1}$, in (19), we find the Struve function in [34].*

2.1. *Integral representation and differentiation formula*

In this subsection, we study and show some integral representations and differentials of extended Struve matrix functions as follows

Theorem 2.3. *Let A and B be commuting matrices in $\mathbb{C}^{N\times N}$ where A and A + nI are invertible for every integer $n \geq 0$ and $\widetilde{\mu}(A) > \frac{-1}{2}$, then. The integral representation of extended Struve matrix function is as follows*

$$\boldsymbol{H}_{(A,B)}(z) = \frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A + \frac{I}{2})}{\sqrt{\pi}} \int_0^1 (1 - u^2)^{A-\frac{I}{2}} \sin(zu) \exp\Big(\frac{-B}{u^2(1 - u^2)}\Big) du. \tag{20}$$

Proof. *Putting $t = u^2$ in the (12) and using the properties of Beta matrix function, we get*

$$\mathfrak{B}(A, Q; B) = 2\int_0^1 u^{2A-I}(1-u^2)^{Q-I}\exp\Big(\frac{-B}{u^2(1-u^2)}\Big)du. \tag{21}$$

By using (21) in (19), we find that

$$\begin{aligned} \boldsymbol{H}_{(A,B)}(z) &= \frac{4(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^1 u^{(2n+1)I}(1-u^2)^{A-\frac{I}{2}}\exp\Big(\frac{-B}{u^2(1-u^2)}\Big)z^{2n}du \\ &= \frac{4(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^1 u^{(2n+1)I}(1-u^2)^{A-\frac{I}{2}}\exp\Big(\frac{-B}{u^2(1-u^2)}\Big)z^{2n}du \\ &= \frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\int_0^1 (1-u^2)^{A-\frac{I}{2}}\exp\Big(\frac{-B}{u^2(1-u^2)}\Big)\sum_{n=0}^{\infty}\frac{(-1)^n(zu)^{2n+1}}{(2n+1)!}du \\ &= \frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\int_0^1 (1-u^2)^{A-\frac{I}{2}}\sin(zu)\exp(\frac{-B}{u^2(1-u^2)})du. \end{aligned} \tag{22}$$

This completes the proof. □

Remark 2.4. *Further, we note the following special cases of the integral representations of the extended Struve matrix function as follows*
i- *Putting $B = \boldsymbol{0}$ in (20), we have the integral representations of Struve matrix function in [31]*
ii - *If taking $A = \alpha \in \mathbb{C}^{1\times 1}$ and $B = \beta \in \mathbb{C}^{1\times 1}$, in (20), we find the integral representations of Struve function in [34].*

Theorem 2.5. *Let A and B are commuting positive stable matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (6), we have the differential of extended Struve matrix function as follows*

$$\Big(\frac{d}{dz}\Big)^2\Big[z^{-A}\boldsymbol{H}_{(A,B)}(z)\Big] = z^{-A}\boldsymbol{H}_{(A,B)}(z) + (A+\frac{I}{2})z^{-(A+I)}\boldsymbol{H}_{(A+I,B)}(z). \tag{23}$$

Proof. *By multiplying both members of (19) by z^{-A} and differentiate each member with respect to z, we get*

$$\begin{aligned} \Big(\frac{d}{dz}\Big)^2\Big[z^{-A}\boldsymbol{H}_{(A,B)}(z)\Big] &= \Big(\frac{d}{dz}\Big)^2\Big[\frac{2(\frac{1}{2})^{-(A+I)}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\mathcal{B}\Big((n+1)I, A+\frac{I}{2}, B\Big)\frac{z^{2n+1}}{(2n+1)!}\Big] \\ &= \frac{2^{-A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\mathcal{B}\Big((n+1)I, A+\frac{I}{2}, B\Big)\frac{z^{2n-1}}{(2n-1)!}. \end{aligned} \tag{24}$$

By properties of extended beta matrix function in [1], we get

$$\mathfrak{B}(A, Q+I; B) + \mathfrak{B}(A+I, Q; B) = \mathfrak{B}(A, Q; B),$$

we find

$$\Big(\frac{d}{dz}\Big)^2\Big[z^{-A}\boldsymbol{H}_{(A,B)}(z)\Big] = \frac{2^{-A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}z^{2n-1}\Big[\mathfrak{B}(nI, A+\frac{I}{2}, B) - \mathfrak{B}(nI, A+\frac{3I}{2}, B)\Big]. \tag{25}$$

Replacing n by k + 1 in the above equation, we get

$$\begin{aligned}\Big(\frac{d}{dz}\Big)^2\Big[z^{-A}\boldsymbol{H}_{(A,B)}(z)\Big] &= -\frac{2^{-A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^n}{(2k+1)!}z^{2k+1}\Big[\mathfrak{B}((k+1)I, A+\frac{I}{2}, B)-\mathfrak{B}((k+1)I, A+\frac{3I}{2}, B)\Big].\\ &= z^{-A}\Big[\frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^n}{(2k+1)!}z^{2k}\mathfrak{B}\Big((k+1)I, A+\frac{I}{2}, B\Big)\Big]\\ &+ (A+\frac{I}{2})z^{-(A+I)}\Big[\frac{2(\frac{z}{2})^{A+2I}\Gamma^{-1}(A+\frac{3I}{2})}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^n}{(2k+1)!}z^{2k}\mathfrak{B}\Big((k+1)I, A+\frac{3I}{2}, B\Big)\Big]\\ &= z^{-A}\boldsymbol{H}_{(A,B)}(z)+(A+\frac{I}{2})z^{-(A+I)}\boldsymbol{H}_{(A+I,B)}(z),\end{aligned} \tag{26}$$

which completes the proof of the required result. □

2.2. Connections with certain generalized gamma matrix function

In the subsection, we give connections with certain generalized gamma matrix function.

Let A and B be commuting matrices in $\mathbb{C}^{N\times N}$ satisfying the condition $\widetilde{\mu}(A) > \frac{-3}{2}$, then, the extended Struve matrix function $\mathbf{H}_{(A,B)}(z)$ is defined in the form

$$\mathbf{H}_{(A,B)}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma^{-1}(A+(n+\frac{3}{2})I, B)}{\Gamma(n+\frac{3}{2})}\,(\frac{z}{2})^{A+(2n+1)I}, \tag{27}$$

Now, We give some properties of extended Struve matrix function as follows.

Theorem 2.6. *Let A, B, C, and $A-B$ are positive stable matrices in $\mathbb{C}^{N\times N}$, such that A, B, C commute with each other, we have*

$$\boldsymbol{H}_{(A,C)}(z) = 2\Gamma^{-1}(A-B, C)(\frac{z}{2})^{A-B}\int_0^1(1-t^2)^{A-B-I}t^{B+I}\exp\Big((\frac{-C}{t^2(1-t^2)}\Big)\boldsymbol{H}_{(B,C)}(zt)dt. \tag{28}$$

Proof. *Consider the integral*

$$F = \int_0^1(1-t^2)^{A-B-I}t^{B+I}\exp\Big(\frac{-C}{t^2(1-t^2)}\Big)\boldsymbol{H}_{(B,C)}(zt)dt,$$

we have

$$F = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})}\Gamma^{-1}(B+(n+\frac{3}{2})I, C)(\frac{z}{2})^{B+(2n+1)I}\int_0^1(1-t^2)^{A-B-I}t^{2B+(2n+2)I}\exp\Big((\frac{-C}{t^2(1-t^2)}\Big)dt. \tag{29}$$

Putting $u=t^2$ in (29), we get

$$\begin{aligned}F &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2\Gamma(n+\frac{3}{2})}\Gamma^{-1}(B+(n+\frac{3}{2})I, C)(\frac{z}{2})^{B+(2n+1)I}\int_0^1(1-u)^{A-B-I}u^{B+(n+\frac{1}{2})I}\exp\Big(\frac{-C}{u(1-u)}\Big)du.\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})}(\frac{z}{2})^{B+(2n+1)I}\Gamma^{-1}(B+(n+\frac{3}{2})I, C)\mathfrak{B}(A-B, B+(n+\frac{3}{2})I; C)\\ &= \frac{1}{2}\Gamma^{-1}(B+(n+\frac{3}{2})I, C)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})}(\frac{z}{2})^{B+(2n+1)I}\Gamma(A-B, C)\Gamma(B+(n+\frac{3}{2})I, C)\Gamma^{-1}(A+(n+\frac{3}{2})I, C)\\ &= \frac{1}{2}\Gamma(A-B, C)(\frac{z}{2})^{B-A}\boldsymbol{H}_{(A,C)}(z).\end{aligned} \tag{30}$$

This completes the proof. □

Theorem 2.7. *Let A, B are commuting positive stable matrices in $\mathbb{C}^{N\times N}$, then each of the following properties holds true*

(i) $\frac{d}{dz}[z^{A}\boldsymbol{H}_{(A,B)}(z)] = z^{A}\boldsymbol{H}_{(A-I,B)}(z);$

(ii) $\frac{d}{dz}[z^{-A}\boldsymbol{H}_{(A,B)}(z)] = \frac{2^{-A}\Gamma^{-1}(A+\frac{3}{2}I,B)}{\sqrt{\pi}} - z^{-A}\boldsymbol{H}_{(A+I,B)}(z).$

Proof. (*i*) *From (27), we have*

$$\begin{aligned}\frac{d}{dz}[z^{A}\boldsymbol{H}_{(A,B)}(z)] &= \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{3}{2})I,B)z^{2A+(2n+1)I}}{2^{A+(2n+1)I}\Gamma(n+\frac{3}{2})}\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{3}{2})I,B)(A+(n+\frac{1}{2})I)z^{2A+2nI}}{2^{A+(2n+1)I}\Gamma(n+\frac{3}{2})}\\ &= z^{A}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{1}{2})I,B)z^{A+2nI}}{2^{A+2nI}\Gamma(n+\frac{3}{2})}\\ &= z^{A}\boldsymbol{H}_{(A-I,B)}(z).\end{aligned}$$

(*ii*) *From (27), we have*

$$\begin{aligned}\frac{d}{dz}[z^{-A}\boldsymbol{H}_{(A,B)}(z)] &= \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{3}{2})I,B)z^{(2n+1)I}}{2^{A+(2n+1)I}\Gamma(n+\frac{3}{2})}\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{3}{2})I,B)z^{2nI}}{2^{A+(2n+1)I}\Gamma(n+\frac{1}{2})}\\ &= z^{-A}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma^{-1}(A+(n+\frac{3}{2})I,B)z^{A+2nI}}{2^{A+2nI}\Gamma(n+\frac{1}{2})}.\end{aligned}$$

A shift of index from n to $k+1$ yields

$$\begin{aligned}\frac{d}{dz}[z^{-A}\boldsymbol{H}_{(A,B)}(z)] &= z^{-A}\sum_{k=-1}^{\infty}\frac{(-1)^{k+1}\Gamma^{-1}(A+(k+\frac{5}{2})I,B)z^{A+I+(2k+1)I}}{2^{A+I+(2k+1)I}\Gamma(k+\frac{3}{2})}\\ &= z^{-A}\frac{\Gamma^{-1}(A+\frac{3}{2}I,B)z^{A}}{\Gamma(\frac{1}{2})2^{A}} - z^{-A}\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma^{-1}(A+(k+\frac{5}{2})I,B)z^{A+I+(2k+1)I}}{2^{A+I+(2k+1)I}\Gamma(k+\frac{3}{2})} \qquad (31)\\ &= \frac{2^{-A}\Gamma^{-1}(A+\frac{3}{2}I,B)}{\sqrt{\pi}} - z^{-A}\boldsymbol{H}_{(A+I,B)}(z),\end{aligned}$$

which completes the proof relation (ii). □

2.3. Extended modified Struve matrix function

Definition 2.8. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (6) and $\widetilde{\mu}(A) > \frac{-1}{2}$, then, the extended modified Struve matrix function $\boldsymbol{L}_{(A,B)}(z)$ is defined in the form*

$$\boldsymbol{L}_{(A,B)}(z) = \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\mathfrak{B}((n+1)I, A+\frac{I}{2},B)\frac{z^{2n}}{(2n+1)!}. \qquad (32)$$

Remark 2.9. *Further, we note the following special cases of the extended modified Struve matrix function as follows*
i- *Putting $B = \boldsymbol{0}$ in (32), by using the properties of the Pochhammer symbol, we get the modified Struve matrix function in (16)*
ii- *If taking $A = \alpha \in \mathbb{C}^{1\times 1}$ and $B = \beta \in \mathbb{C}^{1\times 1}$ in (32), we find the modified Struve function in [34].*

Now, we give the integral representations of the extended modified Struve matrix function as

Theorem 2.10. *Let A and B be commuting matrices in $\mathbb{C}^{N\times N}$ where A, $A+nI$ and B are invertible and $\widetilde{\mu}(A) > \frac{-1}{2}$, then*

$$\boldsymbol{L}_{(A,B)}(z) = \frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\int_0^1 (1-t^2)^{A-\frac{I}{2}}\sinh(zt)\exp\Big(\frac{-B}{t^2(1-t^2)}\Big)dt. \tag{33}$$

Proof. By substitute the series for $\sinh(zt)$ in the right hand side (33), we have

$$\begin{aligned} R =&\frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\int_0^1 (1-t^2)^{A-\frac{I}{2}}\sinh(zt)\exp\Big(\frac{-B}{t^2(1-t^2)}\Big)dt \\ =&\frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(z)^{2n+1}}{(2n+1)!}\int_0^1 (1-t^2)^{A-\frac{I}{2}}t^{(2n+1)I}\exp\Big(\frac{-B}{t^2(1-t^2)}\Big)dt. \end{aligned} \tag{34}$$

Putting $t^2 = u$ in (34) and using the properties of the extended Beta matrix function, we have

$$\begin{aligned} R =&\frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(z)^{(2n+1)}}{2(2n+1)!}\int_0^1 (1-u)^{A-\frac{I}{2}}u^{nI}\exp\Big(\frac{-B}{u(1-u)}\Big)du \\ =&\frac{2(\frac{z}{2})^{A}\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(z)^{(2n+1)}}{2(2n+1)!}\mathfrak{B}(n+1)I, A+\frac{I}{2}, B) \\ =&\mathbf{L}_{(A,B)}(z), \end{aligned} \tag{35}$$

which completes the proof relation (33). □

2.4. Hypergeometric representation

Form the definition of the extended Struve matrix function in (19), we can write the extended hypergeometric matrix function form of extended Struve matrix function

$$\begin{aligned} \mathbf{H}_{(A,B)}(z) =& \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{3I}{2})}{\sqrt{\pi}}\Big(\Gamma^{-1}(A+\frac{I}{2})\Gamma(A+\frac{3I}{2})\sum_{n=0}^{\infty}\frac{[(\frac{3}{2}I)_n]^{-1}\mathfrak{B}((n+1)I, A+\frac{I}{2}, B)}{n!}(-\frac{z^2}{4})^n\Big) \\ =& \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{3I}{2})}{\sqrt{\pi}}\, {}_1F_2^{(B)}(I; \frac{3I}{2}, A+\frac{3I}{2}; -\frac{z^2}{4}), \end{aligned} \tag{36}$$

where ${}_1F_2^{(B)}(I; \frac{3I}{2}, A+\frac{3I}{2}; -\frac{z^2}{4})$ is called extended hypergeometric matrix function as (13).

In similar manner, we can write the extended hypergeometric matrix function form of extended modified Struve matrix function

$$\mathbf{L}_{(A,B)}(z) = \frac{2(\frac{z}{2})^{A+I}\Gamma^{-1}(A+\frac{3I}{2})}{\sqrt{\pi}}\, {}_1F_2^{(B)}(I; \frac{3I}{2}, A+\frac{3I}{2}; \frac{z^2}{4}). \tag{37}$$

3. Fractional calculus of extended Struve matrix function

In this section, we introduce and study operators that are of the Riemann- Liouville fractional integral, and of the Riemann-Liouville and Caputo fractional derivatives of the extended Struve and extended Modified Struve matrix functions. The fractional order integral and derivative of Riemann-Liouville operator of order μ and $x > 0$ such that $\mathbf{Re}(\mu) > 0$, which are given as follows (see[26, 30])

$$(\mathbf{I}_a^{\mu} f)(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)dt \tag{38}$$

and

$$\mathbf{D}_a^{\mu} f(x) = \frac{1}{\Gamma(-\mu)} \int_a^x (x-t)^{-\mu-1} f(t)dt. \tag{39}$$

For the fractional order integral and derivative, we have the following definition (see[8]).

Definition 3.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ and $\mu \in \mathbb{C}$ such that $\mathbf{Re}(\mu) > 0$. Then, the Riemann-Liouville fractional integral of order μ is defined*

$$\mathbf{I}^{\mu}(x^A) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^A dt. \tag{40}$$

The Riemann-Liouville fractional derivative of order μ is defined by

$$\mathbf{D}^{\mu}(x^A) = \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} t^A dt. \tag{41}$$

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$, such that $\mathbf{Re}(\mu) > 0$. Then, the Riemann-Liouville fractional integral of order μ is defined (see[8])

$$\mathbf{I}^{\mu}(x^{A-I}) = \Gamma(A)\Gamma^{-1}(A+\mu I)x^{A+(\mu-1)I}. \tag{42}$$

Lemma 3.2. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$, such that $\mathbf{Re}(\mu) > 0$, we get the Riemann-Liouville fractional derivative of order μ as follows*

$$\mathbf{D}^{\mu}(x^{A-I}) = \Gamma(A)\Gamma^{-1}(A-\mu I)x^{A-(\mu+1)I}. \tag{43}$$

Proof. *From (41), we find that*

$$\mathbf{D}^{\mu}(x^{A-I}) = \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} t^{A-I} dt = \frac{1}{\Gamma(-\mu)} \int_0^x x^{-\mu-1}(1-\frac{t}{x})^{-\mu-1} t^{A-I} dt. \tag{44}$$

putting $u = \frac{t}{x}$, we obtain

$$\begin{aligned}
\mathbf{D}^{\mu}(x^{A-I}) &= \frac{1}{\Gamma(-\mu)} \int_0^1 x^{-\mu-1}(1-u)^{-\mu-1}(ux)^{A-I} du = \frac{x^{A-(\mu+1)I}}{\Gamma(-\mu)} \int_0^1 (1-u)^{(-\mu-1)I} u^{A-I} du \\
&= \frac{x^{A-(\mu+1)I}}{\Gamma(-\mu)} \mathfrak{B}(-\mu I, A) = \Gamma(A)\Gamma^{-1}(A-\mu I)x^{A-(\mu+1)I},
\end{aligned}$$

which yields assertion (43). □

Thus we can rewrite (19) in this form

$$\mathbf{H}_{(A,B)}(z) = 2^{-A}(z)^{A+I}\, \Gamma^{-1}(A+\frac{3}{2}I) \sum_{n=0}^{\infty} \Gamma\Big(\begin{matrix} A+\frac{3}{2}I \\ I, \quad A+\frac{1}{2}I \end{matrix} \Big) \mathfrak{B}((n+1)I, A+\frac{I}{2}; B)\, \frac{(\frac{-z}{2})^2}{(\frac{3}{2})_n n!}. \tag{45}$$

Theorem 3.3. *Let A, B and C are positive stable matrices in $\mathbb{C}^{N\times N}$ and A, B , C commute with each other, such that $\mathbf{Re}(\mu) > 0$, we find integral operators representation of extended Struve matrix function as follows*

$$\begin{aligned}
\mathbf{I}^{\mu}\Big[x^{C-I}\, \boldsymbol{H}_{(A,B)}(wx)\Big] &= \sqrt{\pi} x^{A+C+\mu I}\, \Gamma^{-1}(A+\frac{3}{2}I)_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(\frac{-wx}{2})^2) \\
&\quad \times \sum_{n=0}^{\infty} \Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I, \quad (\frac{3}{2}+k)I \end{matrix} \Big) (\frac{-wx}{2})^{2n}, \quad |wx| < 1.
\end{aligned} \tag{46}$$

Proof. *Form (45), we find*

$$\mathbf{I}^{\mu}\Big[x^{C-I}\,\boldsymbol{H}_{(A,B)}(wx)\Big] = 2^{-A}(w)^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+\frac{3}{2}I \\ I,\quad A+\frac{1}{2}I\end{matrix}\Big)\mathfrak{B}((n+1)I, A+\frac{I}{2};B)\,\frac{(\frac{-wx}{2})^2}{(\frac{3}{2})_n n!}\mathbf{I}^{\mu}\Big[x^{A+C+2nI}\Big], \quad (47)$$

and using (42), we have

$$\begin{aligned}\mathbf{I}^{\mu}\Big[x^{C-I}\,\boldsymbol{H}_{(A,B)}(wx)\Big] &= \sqrt{\pi}x^{A+C+\mu I}(\frac{w}{2})^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\\ &\times\sum_{n=0}^{\infty}\frac{(I)_n\Gamma\Big(\begin{matrix} A+\frac{3}{2}I \\ I,\quad A+\frac{1}{2}I\end{matrix}\Big)}{n!}\,\mathfrak{B}((n+1)I, A+\frac{I}{2};B)\,\Gamma\Big(\begin{matrix} A+C+(2n+1)I \\ \frac{3}{2}I,\quad A+C+\mu I\end{matrix}\Big)(\frac{-wx}{2})^{2n}\\ &= \sqrt{\pi}x^{A+C+\mu I}\,\Gamma^{-1}(A+\frac{3}{2}I)\,{}_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(\frac{-wx}{2})^2)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I,\quad (\frac{3}{2}+k)I\end{matrix}\Big)\,(\frac{-wx}{2})^{2n}.\end{aligned} \quad (48)$$

This completes the proof. □

Theorem 3.4. *Let A, B and C are positive stable matrices in* $\mathbb{C}^{N\times N}$ *and A, B , C commute with each other, such that* $\mathbf{Re}(\mu) > 0$, *we have differential operators representation of extended Struve matrix function as follows*

$$\begin{aligned}\mathbf{D}^{\mu}\Big[x^{C-I}\,\boldsymbol{H}_{(A,B)}(wx)\Big] &= \sqrt{\pi}x^{A+C-2I}(\frac{w}{2})^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\,{}_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(-\frac{wx}{2})^2)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I,\quad (\frac{3}{2}+k)I\end{matrix}\Big)\,(-\frac{wx}{2})^{2},\quad |wx|<1.\end{aligned} \quad (49)$$

Proof. *By using (45) , we get*

$$\begin{aligned}\mathbf{D}^{\mu}\Big[x^{C-I}\,\boldsymbol{H}_{(A,B)}(wx)\Big] &= 2^{-A}(w)^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+\frac{3}{2}I \\ I,\quad A+\frac{1}{2}I\end{matrix}\Big)\mathfrak{B}((n+1)I, A+\frac{I}{2};B)\,\frac{(\frac{-wx}{2})^2}{(\frac{3}{2})_n n!}\mathbf{D}^{\mu}\Big[x^{A+C+(2n-1)I}\Big],\end{aligned} \quad (50)$$

from (43), we have

$$\begin{aligned}\mathbf{D}^{\mu}\Big[x^{C-I}\,\boldsymbol{H}_{(A,B)}(wx)\Big] &= \sqrt{\pi}x^{A+C-2I}(\frac{w}{2})^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\\ &\times\sum_{n=0}^{\infty}\frac{(I)_n\Gamma\Big(\begin{matrix} A+\frac{3}{2}I \\ I,\quad A+\frac{1}{2}I\end{matrix}\Big)}{n!}\,\mathfrak{B}((n+1)I, A+\frac{I}{2};B)\,\Gamma\Big(\begin{matrix} A+C+(2n+1)I \\ \frac{3}{2}I,\quad A+C+\mu I\end{matrix}\Big)(\frac{-wx}{2})^{2}\\ &= \sqrt{\pi}x^{A+C-2I}(\frac{w}{2})^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)\,{}_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(-\frac{wx}{2})^2)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I,\quad (\frac{3}{2}+k)I\end{matrix}\Big)\,(-\frac{wx}{2})^{2}.\end{aligned} \quad (51)$$

This completes the proof. □

Similar to Theorem 3.3 and Theorem 3.4, we can get the result about the fractional integral and the fractional derivatives of extended modified Struve matrix function $\mathbf{L}_{(A,B)}(z)$, we give the following statements without proofs.

Theorem 3.5. *Let A, B and C are positive stable matrices in $\mathbb{C}^{N\times N}$ and A, B , C commute with each other, such that* $\mathbf{Re}(\mu) > 0$*, we find integral operators representation of extended modified Struve matrix function as follows*

$$\begin{aligned}\mathbf{I}^{\mu}\Big[x^{C-I}\,\boldsymbol{L}_{(A,B)}(wx)\Big] &= \sqrt{\pi}x^{A+C+\mu I}\,\Gamma^{-1}(A+\frac{3}{2}I)_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(\frac{wx}{2})^2)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I, \quad (\frac{3}{2}+k)I\end{matrix}\Big)(\frac{wx}{2})^{2n},\ \ |wx|<1.\end{aligned} \tag{52}$$

Theorem 3.6. *Let A, B and C are positive stable matrices in $\mathbb{C}^{N\times N}$ and A, B , C commute with each other, such that* $\mathbf{Re}(\mu) > 0$*, we get differential operators representation of extended modified Struve matrix function as follows*

$$\begin{aligned}\mathbf{D}^{\mu}\Big[x^{C-I}\,\boldsymbol{L}_{(A,B)}(wx)\Big] &= \sqrt{\pi}x^{A+C-2I}(\frac{w}{2})^{A+I}\,\Gamma^{-1}(A+\frac{3}{2}I)_2F_1^{(B)}(I,I;A+\frac{3}{2}I;(\frac{wx}{2})^2)\\ &\times\sum_{n=0}^{\infty}\Gamma\Big(\begin{matrix} A+C+(2k+1)I \\ A+C+(\mu+2k+1)I, \quad (\frac{3}{2}+k)I\end{matrix}\Big)(\frac{wx}{2})^{2},\ \ |wx|<1.\end{aligned} \tag{53}$$

4. Conclusions

We conclude our analysis by remarking that the results presented in this article are new and very potential for the extension of other special matrix functions. First, we have generalized extended Struve and modified Struve matrix functions, then we have studied several basic properties like integral representations, differentiation formulas, and transformations hypergeometric functions. The results presented here articulating an interesting application in fractional calculus of these extended Struve and extended modified Struve matrix functions.

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