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Trapezium-Type Inequalities for h-Preinvex Functions and their Applications

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Abstract. In this paper, the integral identity for differentiable functions with the wide range of applications is investigated. As an effect of this result, the paper provides some new Hermite-Hadamard type inequalities for differentiable functions that are in absolute value h-preinvex. Several special cases are also discussed. At the end, some applications for special means are given as well.

1. Introduction

Let I be non degenerate interval in the set of reals. Then a function $f : I \to \mathbb{R}$ is said to be convex provided that the following inequality holds:

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for $x, y \in I$ and $\lambda \in [0,1]$. And for a convex function f the following well known inequality holds, called as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

For concave function f inequality (1) holds in reverse direction. Due to its geometrical significance and applications in different fields of sciences it attracts the researchers, mathematicians and scientists to work on. In recent years, a number of papers have been written relating to this inequality [2, 5, 6, 11, 25, 27, 28, 35] and the references cited therein. In 2007 Varošanec [28] generalized the concept of convex function by defining the concept of $h-$convex functions. This class contains $s-$convex functions [3], Godunova-Levin functions [7] and $P-$functions [6] as special cases. Sarikaya et al. [27] proved some Hermite-Hadamard type inequalities for $h-$convex functions. Another significant generalization of convex function, f, is the preinvex function due to T. Weir et al. [29]. Noor et al. [23] and Matloka [19] recently investigated

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the notion of h-preinvex functions as a generalization of h-convex functions and obtained new refined Hermite-Hadamard type inequalities for h-preinvex functions. Sundas et al. [10] proved some generalized Hermite-Hadamard type inequalities and obtained some Hermite-Hadmard type inequalities for classes of s-preinvex functions of Brckner type, Q-preinvex functions, s-preinvex functions of Godunova-Levin type and P-preinvex functions. This paper is organized in the following way. After this introduction in Section 2 some basic concepts are discussed, in Section 3 main results relating to the topic and in Section 4 applications of the derived results to some special means are discussed.

2. Preliminaries and assumptions

Definition 2.1. *[29] Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if*

$$x + t\eta(y,x) \in K$$

for all $x, y \in K, t \in [0,1]$. The set K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called an η-connected set.

Remark 2.2. *We noted from the above definition that there is a path starting from a point x which is contained in K and we do not require that the point y should be one of the end point of this path. Note that if $\eta(y,x) = y - x$ then y is the end point of the path which is contained in K and consequently invexity reduces to convexity.*

Definition 2.3. *[29] A function $f : K \to \mathbb{R}$ on an invex set $K \subset \mathbb{R}^n$ is said to be preinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq (1-t) f(u) + t f(v)$$

holds for all $u, v \in K, t \in [0,1]$. The function f is said to be preincave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(v,u) = v - u$ but the converse is not true see for instance [1].

Definition 2.4. *[17] A function $f : K \to [0,\infty)$ on an invex set $K \subset [0,\infty)^n$ is said to be s-preinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq (1-t)^s f(u) + t^s f(v)$$

holds for all $u, v \in K, t \in [0,1]$ and for some fixed $s \in (0,1]$. The function f is said to be s-preincave if and only if $-f$ is s-preinvex.

In the paper [28], a larger class of non-negative functions, the so-called $h-$convex functions was considered. This class contains several well-known classes of functions such as non-negative convex functions, $s-$convex in the second sense, Godunova Levin functions and $P-$functions. The definition of $h-$convexity was further generalized by Matloka as follows:

Definition 2.5. *[19] Let J be a real interval such that $(0,1) \subset J$ and let $h : J \to \mathbb{R}$ be a non-negative function with $h \not\equiv 0$. A function $f : K \to \mathbb{R}$ defined on an invex subset K of $\mathbb{R}^n$ is called an $h-$preinvex with respect to η, if for all $x, y \in K$ and $t \in [0,1]$*

$$f(u + t\eta(v,u)) \leq h(1-t) f(u) + h(t) f(v). \tag{2}$$

If the inequality in (2) holds in reversed, then f is called $h-$preincave.

Remark 2.6. *It may be noted that every convex function is h-preinvex function with respect to $\eta(v,u) = v - u$ and h is the identity function.*

For some recent findings about inequalities for preinvex and $h-$preinvex functions, we refer to the references [4, 9, 10],[12]-[24],[26],[29]-[34] and the references therein.

Motivated by ongoing research into this subject, the key concern of this research work is to propose generalized Hermite–Hadamard type inequalities for h–preinvex functions as a sequence for $n \in \mathbb{N}$. This sequence of generalized Hermite–Hadamard type inequalities includes some new interesting results for particular values of n.

$$J(f;\eta,\lambda,\mu,n) := (n+\lambda-1)\,f(a) + \mu f(a+\eta(b,a)) + (n-\lambda-\mu+1)\,f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{\eta(b,a)}{2n}\int_a^{a+\eta(b,a)} f(x)\,dx. \quad (3)$$

$$Q_1 := \int_0^n |1-\lambda-t|\left[h\left(\frac{n+t}{2n}\right)|f'(a)|^q + h\left(\frac{n-t}{2n}\right)|f'(b)|^q\right]dt, \quad (4)$$

$$Q_2 := \int_0^n |\mu-t|\left[h\left(\frac{t}{2n}\right)|f'(a)|^q + h\left(\frac{2n-t}{2n}\right)|f'(b)|^q\right]dt, \quad (5)$$

$$Q_3 := \int_0^n \left[h\left(\frac{n-t}{2n}\right)|f'(b)|^q + h\left(\frac{n+t}{2n}\right)|f'(a)|^q\right]dt, \quad (6)$$

$$Q_4 := \int_0^n \left[h\left(\frac{t}{2n}\right)|f'(a)|^q + h\left(\frac{2n-t}{2n}\right)|f'(b)|^q\right]dt, \quad (7)$$

$$A_1 := \int_0^n |1-\lambda-t|\left|f'\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)\right|dt,$$
$$A_2 := \int_0^n |\mu-t|\left|f'\left(a+\left(\frac{2n-t}{2n}\right)\eta(b,a)\right)\right|dt.$$

3. Main Results

Lemma 3.1. *Let $f : K \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on K° (interior of K). If $f' \in L_1[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ for $a, b \in K^\circ$; $\lambda, \mu \in \mathbb{R}$ and $n \in \mathbb{N}$, then*

$$\frac{\eta(b,a)}{2n}\left[\int_0^n (1-\lambda-t)\,f'\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)dt + \int_0^n (\mu-t)\,f'\left(a+\left(\frac{2n-t}{2n}\right)\eta(b,a)\right)dt\right]$$
$$= (n+\lambda-1)\,f(a) + \mu f(a+\eta(b,a)) + (n-\lambda-\mu+1)\,f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{\eta(b,a)}{2n}\int_a^{a+\eta(b,a)} f(x)\,dx. \quad (8)$$

Proof. Consider,

$$\int_0^n (1-\lambda-t)\,f'\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)dt + \int_0^n (\mu-t)\,f'\left(a+\left(\frac{2n-t}{2n}\right)\eta(b,a)\right)dt := I_1 + I_2. \quad (9)$$

Integrating by parts repeatedly, we have

$$I_1 := \int_0^n (1-\lambda-t)\,f'\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)dt$$
$$= -\frac{2n(1-\lambda-t)}{\eta(b,a)}f\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)\Bigg|_0^n - \frac{2n}{\eta(b,a)}\int_0^n f\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)dt$$
$$= -\frac{2n(1-\lambda-n)}{\eta(b,a)}f(a) + \frac{2n(1-\lambda)}{\eta(b,a)}f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{2n}{\eta(b,a)}\int_0^n f\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)dt$$
$$= -\frac{2n(1-\lambda-n)}{\eta(b,a)}f(a) + \frac{2n(1-\lambda)}{\eta(b,a)}f\left(a+\frac{1}{2}\eta(b,a)\right) - \int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx. \quad (10)$$

$$
\begin{aligned}
I_2 &:= \int_0^n (\mu - t) f'\left(a + \left(\frac{2n-t}{2n}\right)\eta(b,a)\right) dt \\
&= -\frac{2n(\mu - t)}{\eta(b,a)} f\left(a + \left(\frac{2n-t}{2n}\right)\eta(b,a)\right)\Big|_0^n - \frac{2n}{\eta(b,a)} \int_0^n f\left(a + \left(\frac{2n-t}{2n}\right)\eta(b,a)\right) dt \\
&= -\frac{2n(\mu - n)}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) + \frac{2n\mu}{\eta(b,a)} f(a + \eta(b,a)) - \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,dx. \qquad (11)
\end{aligned}
$$

A combination of (9)-(11) yields the following:

$$
\begin{aligned}
I_1 + I_2 &= -\frac{2n(1-\lambda-n)}{\eta(b,a)} f(a) + \frac{2n(1-\lambda)}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) - \int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx \\
&- \frac{2n(\mu - n)}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) + \frac{2n\mu}{\eta(b,a)} f(a + \eta(b,a)) - \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,dx \\
&= \frac{2n(n+\lambda-1)}{\eta(b,a)} f(a) + \frac{2n\mu}{\eta(b,a)} f(a+\eta(b,a)) \\
&+ \frac{2n(1+n-\lambda-\mu)}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) - \int_a^{a+\eta(b,a)} f(x)\,dx. \qquad (12)
\end{aligned}
$$

Multiplying both sides of (12) by $\frac{\eta(b,a)}{2n}$, we obtain

$$
\begin{aligned}
&\frac{\eta(b,a)}{2n}\left[\int_0^n (1-\lambda-t) f'\left(a + \left(\frac{n-t}{2n}\right)\eta(b,a)\right) dt + \int_0^n (\mu - t) f'\left(a + \left(\frac{2n-t}{2n}\right)\eta(b,a)\right) dt\right] \\
&= (n+\lambda-1) f(a) + \mu f(a+\eta(b,a)) + (n-\lambda-\mu+1) f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{\eta(b,a)}{2n}\int_a^{a+\eta(b,a)} f(x)\,dx.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2. *Let the conditions of Lemma 3.1 be satisfied. Moreover, if $|f'|^q$ is h-preinvex function on $[a, a+\eta(b,a)]$ for $q \geq 1$, then*

$$
\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\left(\frac{n^2}{2} + (1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}} \sqrt[q]{Q_1} + \left(\frac{n^2}{2} - n\mu + \mu^2\right)^{1-\frac{1}{q}} \sqrt[q]{Q_2}\right], \qquad (13)
$$

where Q_1 and Q_2 are given by (4) *and* (5) *respectively.*

Proof. By properties of modulus and Lemma 3.1, we have

$$
\begin{aligned}
\left|J(f;\eta,\lambda,\mu,n)\right| &\leq \frac{\eta(b,a)}{2n}\left[\int_0^n |1-\lambda-t| \left|f'\left(a + \left(\frac{n-t}{2n}\right)\eta(b,a)\right)\right| dt\right. \\
&\left. + \int_0^n |\mu - t| \left|f'\left(a + \left(\frac{2n-t}{2n}\right)\eta(b,a)\right)\right| dt\right] := \frac{\eta(b,a)}{2n}(A_1 + A_2). \qquad (14)
\end{aligned}
$$

Repeated applications of the power mean inequality and h-preinvexity of $|f'|^q$ yield the following:

$$
\begin{aligned}
A_1 &\leq \left(\int_0^n |1-\lambda-t| dt\right)^{1-\frac{1}{q}} \left(\int_0^n |1-\lambda-t| \left[h\left(\frac{n+t}{2n}\right)|f'(a)|^q + h\left(\frac{n-t}{2n}\right)|f'(b)|^q\right] dt\right)^{\frac{1}{q}} \\
&= \left(\frac{n^2}{2} + (1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}} Q_1^{\frac{1}{q}} \qquad (15)
\end{aligned}
$$

$$A_2 \le \left(\int_0^n |\mu - t| dt\right)^{1-\frac{1}{q}} \left(\int_0^n |\mu - t| \left[h\left(\frac{t}{2n}\right)|f'(a)|^q + h\left(\frac{2n-t}{2n}\right)|f'(b)|^q\right] dt\right)^{\frac{1}{q}}$$
$$= \left(\frac{n^2}{2} - n\mu + \mu^2\right)^{1-\frac{1}{q}} Q_2^{\frac{1}{q}}. \tag{16}$$

A combination of (14)-(16) yields the desired result (13). □

Corollary 3.3. *Let the conditions of Theorem 3.2 be satisfied for $h \equiv 1$, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \le \frac{\eta(b,a)}{2n}\left(n^2 + (1-\lambda)(1-n-\lambda) - n\mu + n\mu^2\right)\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{1}{q}}. \tag{17}$$

Corollary 3.4. *Let the conditions of Theorem 3.2 be satisfied for $h \equiv$ identity function, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \le \frac{\eta(b,a)}{2n}\left[\left(\frac{n^2}{2} + (1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left\{\left(\frac{(3n+\lambda-1)(\lambda-1)^2 + (1+5n-\lambda)(\lambda+n-1)^2}{12n}\right)|f'(a)|^q + \left(\frac{(n+\lambda-1)^3 + (\lambda-1)^2(\lambda+3n-1)}{12n}\right)|f'(b)|^q\right\}^{\frac{1}{q}}$$
$$\left. + \left(\frac{n^2}{2} - n\mu + \mu^2\right)^{1-\frac{1}{q}}\left\{\left(\frac{2n^3 - 3n^2\mu - 2\mu^3}{12n}\right)|f'(a)|^q + \left(\frac{4n^3 - 9n^2\mu + 12n\mu^2 - 2\mu^3}{12n}\right)|f'(b)|^q\right\}^{\frac{1}{q}}\right]. \tag{18}$$

Corollary 3.5. *Let the conditions of Theorem 3.2 be satisfied for $h(t) = t^s$, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \le \frac{\eta(b,a)}{2n}\left[\left(\frac{n^2}{2} + (1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left\{\left(\frac{n^2\left(2^{1+s}s + 2\left(\frac{1+n-\lambda}{n}\right)^s - 1\right) + 2\left(\frac{1+n-\lambda}{n}\right)^s(\lambda-1)^2}{2^s(1+s)(2+s)} + \frac{n\left(2 + 2^{2+s} + s + 2^{1+s}s - 4\left(\frac{1+n-\lambda}{n}\right)^s\right)(\lambda-1)}{2^s(1+s)(2+s)}\right)|f'(a)|^q\right.$$
$$\left. + \left(\frac{2(\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^s + n^2\left(2\left(\frac{-1+n+\lambda}{n}\right)^s - 1\right)}{2^s(1+s)(2+s)} + \frac{n(\lambda-1)\left(4\left(\frac{-1+n+\lambda}{n}\right)^s - s - 2\right)}{2^s(1+s)(2+s)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}$$
$$+ \left(\frac{n^2}{2} - n\mu + \mu^2\right)^{1-\frac{1}{q}}\left\{\left(\frac{n^2(1+s) - n(2+s)\mu + 2\mu^2\left(\frac{\mu}{n}\right)^s}{2^s(1+s)(2+s)}\right)|f'(a)|^q\right.$$
$$\left.\left. + \left(\frac{2\mu^2\left(2-\frac{\mu}{n}\right)^s - n^2\left(3 + 2^{2+s} + s - 8\left(2-\frac{\mu}{n}\right)^s\right)}{2^s(1+s)(2+s)} + \frac{n\mu\left(2 + 2^{2+s} + s + 2^{1+s}s - 8\left(2-\frac{\mu}{n}\right)^s\right)}{2^s(1+s)(2+s)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}\right]. \tag{19}$$

Corollary 3.6. *Let the conditions of Theorem 3.2 be satisfied for $h(t) = t^{-s}$, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\left(\frac{n^2}{2}+(1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left\{\left(\frac{2^s\left((1+n-\lambda)^2\left(\frac{1+n-\lambda}{n}\right)^{-s}-2^{1-s}n(2-2\lambda+s(n+\lambda-1))\right)}{(s-2)(s-1)}\right.\right.$$
$$\left.-\frac{2^s\left(n(n+(s-2)(\lambda-1))-(1+n-\lambda)^2\left(\frac{1+n-\lambda}{n}\right)^{-s}\right)}{(s-2)(s-1)}\right)|f'(a)|^q$$
$$+\left(-\frac{2^s\left(n(n-(s-2)(\lambda-1))-(n+\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^{-s}\right)}{(s-2)(s-1)}\right.$$
$$\left.\left.\frac{2^s(n+\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^{-s}}{(s-2)(s-1)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}+\left(\frac{n^2}{2}-n\mu+\mu^2\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{2^s\mu^2\left(\frac{\mu}{n}\right)^{-s}-2^s\left(n^2(s-1)-n(s-2)\mu-\mu^2\left(\frac{\mu}{n}\right)^{-s}\right)}{(s-2)(s-1)}\right)|f'(a)|^q\right.$$
$$+\left(\frac{(\mu-2n)^2\left(1-\frac{\mu}{2n}\right)^{-s}-2^sn(-n(s-3)+(s-2)\mu)}{(s-2)(s-1)}\right.$$
$$\left.\left.\left.+\frac{(\mu-2n)^2\left(1-\frac{\mu}{2n}\right)^{-s}-2n(2n+(s-2)\mu)}{(s-2)(s-1)}\right)|f'(b)|^q\right\}^{\frac{1}{q}}\right]. \tag{20}$$

Corollary 3.7. *Let the conditions of Theorem 3.2 be satisfied for $h(t) = t(1-t)$, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\left(\frac{n^2}{2}+(1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\left(\frac{3n^4+8n^3(\lambda-1)+12n^2(\lambda-1)^2-2(\lambda-1)^4}{48n^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{n^2}{2}-n\mu+\mu^2\right)^{1-\frac{1}{q}}\left(\frac{n^4-2n^3\mu+6n\mu^3-2\mu^4}{48n^2}\right)^{\frac{1}{q}}\right]\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}. \tag{21}$$

Theorem 3.8. *Let the conditions of Lemma 3.1 be satisfied. Moreover, if $q > 1$ is such that $p = \frac{q}{q-1}$ and $|f'|^q$ is h-preinvex on $[a, a+\eta(b,a)]$ with $n \leq \min\{1-\lambda, \mu\}$, then*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}}\sqrt[q]{Q_3}+\sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\sqrt[q]{Q_4}\right], \tag{22}$$

where Q_3 and Q_4 are given by (6) *and* (7) *respectively.*

Proof. By properties of modulus and Lemma 3.1, we have

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\int_0^n |1-\lambda-t|\left|f'\left(a+\left(\frac{n-t}{2n}\right)\eta(b,a)\right)\right|dt + \int_0^n |\mu-t|\left|f'\left(a+\left(\frac{2n-t}{2n}\right)\eta(b,a)\right)\right|dt\right]. \tag{23}$$

Repeated applications of Hölder inequality and h-preinvexity of $|f'|^q$, yield:

$$A_1 \leq \left(\int_0^n |1-\lambda-t|^p dt\right)^{\frac{1}{p}}\left\{\int_0^n\left[h\left(\frac{n-t}{2n}\right)|f'(b)|^q + h\left(\frac{n+t}{2n}\right)|f'(a)|^q\right]dt\right\}^{\frac{1}{q}} = \sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}}\sqrt[q]{Q_3} \tag{24}$$

$$A_2 \leq \left(\int_0^n |\mu-t|^p dt\right)^{\frac{1}{p}}\left\{\int_0^n\left[h\left(\frac{t}{2n}\right)|f'(a)|^q + h\left(\frac{2n-t}{2n}\right)|f'(b)|^q\right]dt\right\}^{\frac{1}{q}} = \sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\sqrt[q]{Q_4}. \tag{25}$$

A combination of (23)-(25) yields the desired result (22). □

Corollary 3.9. *Let the conditions of Theorem 3.8 be satisfied. Moreover, if h is the identity function, then the following result for P-preinvex function on* $[a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$ *and* $n \leq \min\{1-\lambda, \mu\}$ *holds*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\sqrt[q]{|f'(a)|^q + |f'(b)|^q}\left[\sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}} + \sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\right]. \tag{26}$$

Corollary 3.10. *Let the conditions of Theorem 3.8 be satisfied. Moreover, if* $h = I$*, identity map, then the following result for preinvex function on* $[a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$ *and* $n \leq \min\{1-\lambda, \mu\}$ *holds*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\left(\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{3n|f'(a)|^q + n|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{n|f'(a)|^q + 3n|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{27}$$

Corollary 3.11. *Let the conditions of Theorem 3.8 be satisfied. Moreover, if* $h(t) = t^s$*, then the following result for s-preinvex function of Breckner type on* $[a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$ *and* $n \leq \min\{1-\lambda, \mu\}$ *holds*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \leq \frac{\eta(b,a)}{2n}\left[\left(\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left\{\frac{n\left(2-s^{-s}\right)|f'(a)|^q + \left(2^{-s}\cdot n\right)|f'(b)|^q}{s+1}\right\}^{\frac{1}{q}} + \left(\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left\{\frac{\left(2^{-s}\cdot n\right)|f'(a)|^q + n\left(2-s^{-s}\right)|f'(b)|^q}{s+1}\right\}^{\frac{1}{q}}\right]. \tag{28}$$

Corollary 3.12. *Let the conditions of Theorem 3.8 be satisfied. Moreover, if $h(t) = t^{-s}$, then the following result for s-preinvex function of Godunova-Levin type on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and $n \le \min\{1-\lambda, \mu\}$ holds*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \le \frac{\eta(b,a)}{2n}\left[\left(\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left\{\frac{n\left(s^{-s}-2\right)|f'(a)|^q-\left(2^{-s}\cdot n\right)|f'(b)|^q}{1-s}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left\{\frac{n\left(s^{-s}-2\right)|f'(b)|^q-\left(2^{-s}\cdot n\right)|f'(a)|^q}{1-s}\right\}^{\frac{1}{q}}\right]. \quad (29)$$

Corollary 3.13. *Let the conditions of Theorem 3.8 be satisfied. Moreover, if $h(t) = t(1-t)$, then the following result for tgs-preinvex function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and $n \le \min\{1-\lambda, \mu\}$ holds*

$$\left|J(f;\eta,\lambda,\mu,n)\right| \le \frac{\eta(b,a)}{2n}\left(\frac{n[|f'(a)|^q+|f'(b)|^q]}{6}\right)^{\frac{1}{q}}\left\{\left(\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}+\left(\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}\right)^{\frac{1}{p}}\right\}. \quad (30)$$

Remark 3.14. *From Theorems 3.2, 3.8 and its related corollaries, against particular choices of $\lambda, \mu \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n \le \min\{1-\lambda, \mu\}$, some interesting results can be concluded. The details are omitted for interested readers.*

4. Applications to special means

Throughout the whole discussion a and b are assumed to be positive. For $p \in \mathbb{R}$, the arithmetic mean $A(a,b)$, generalized logarithmic mean $L_p(a,b)$ are defined as:

$$A(a,b) = \frac{a+b}{2}.$$

$$L_p(a,b) = \begin{cases} a, & a = b; \\ \sqrt[p]{\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}}, & a \ne b, p \ne 0, p \ne -1; \\ \frac{b-a}{\log b - \log a}, & a \ne b, p = -1; \\ \frac{1}{e}\sqrt[b-a]{\frac{b^b}{a^a}}, & a \ne b, p = 0. \end{cases}$$

Let $f : [0, +\infty) \to \mathbb{R}$ be a function defined by:

$$f(x) = \frac{qx^{\frac{s}{q}+1}}{q+s} \text{ for } q \ge 1, \quad (31)$$

so that $\left|f'(x)\right|^q = x^s$. Then, obviously $\left|f'(x)\right|^q$ is s-preinvex function for $0 < s \le 1$ and preinvex function for positive integer $s \in [2, \infty)$ with respect to the bifunction $\eta(b,a) = b - a$ on $[a,b]$.

Proposition 4.1. *Let $n \in \mathbb{N}$ be such that $n \ge 2$ and $q \ge 1$, then*

$$\left|(n+\lambda-\mu-1)\frac{qa^{\frac{n}{q}+1}}{q+n}+\frac{2q\mu}{q+n}A\left(a^{\frac{n}{q}+1}, b^{\frac{n}{q}+1}\right)+(n-\lambda-\mu+1)\frac{qA^{\frac{n}{q}+1}(a,b)}{q+n}-\frac{q(b-a)^2}{2n(q+n)}L_{\frac{n}{q}+1}^{\frac{n}{q}+1}(a,b)\right|$$
$$\le \frac{(b-a)}{2n}\left[\left(\frac{n^2}{2}+(1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\left\{\left(\frac{(\lambda+3n-1)(\lambda-1)^2+(1+5n-\lambda)(\lambda+n-1)^2}{12n}\right)a^n\right.\right.$$
$$\left.+\left(\frac{(n+\lambda-1)^3+(\lambda-1)^2(\lambda+3n-1)}{12n}\right)b^n\right\}^{\frac{1}{q}}$$
$$\left.+\left(\frac{n^2}{2}-n\mu+\mu^2\right)^{1-\frac{1}{q}}\left\{\left(\frac{2n^3-3n^2\mu-2\mu^3}{12n}\right)a^n+\left(\frac{4n^3-9n^2\mu+12n\mu^2-2\mu^3}{12n}\right)b^n\right\}^{\frac{1}{q}}\right]. \quad (32)$$

Proof. The proof follows from Corollary 3.4 for the function defined by (31). □

In particular, for $n = 2$, $\mu, q \to 1$ and $\lambda \to 0$, inequality (32) reduces to

$$\left|A\left(a^3, b^3\right) + A^3(a,b) - \frac{(b-a)^2}{8} L_3^3(a,b)\right| \leq \frac{(b-a)[9A\left(a^2, b^2\right) + 4b^2]}{32} \tag{33}$$

Proposition 4.2. *Let $n \in \mathbb{N}$, $s \in (0,1]$ and $q \geq 1$, then*

$$\begin{aligned}
&\left|(n+\lambda-\mu-1)\frac{qa^{\frac{s}{q}+1}}{q+s} + \frac{2q\mu}{q+s} A\left(a^{\frac{s}{q}+1}, b^{\frac{s}{q}+1}\right) + (n-\lambda-\mu+1)\frac{qA^{\frac{s}{q}+1}(a,b)}{q+s} - \frac{q(b-a)^2}{2n(q+s)} L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right| \\
&\leq \frac{(b-a)}{2n}\left[\left(\frac{n^2}{2} + (1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}} \left\{\left(\frac{n^2\left(2^{1+s}s + 2\left(\frac{1+n-\lambda}{n}\right)^s - 1\right) + 2\left(\frac{1+n-\lambda}{n}\right)^s (\lambda-1)^2}{2^s(1+s)(2+s)}\right.\right.\right. \\
&\left.+\frac{n\left(2 + 2^{2+s} + s + 2^{1+s}s - 4\left(\frac{1+n-\lambda}{n}\right)^s\right)(\lambda-1)}{2^s(1+s)(2+s)}\right) a^s + \left(\frac{2(\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^s + n^2\left(2\left(\frac{-1+n+\lambda}{n}\right)^s - 1\right)}{2^s(1+s)(2+s)}\right. \\
&\left.\left.+\frac{n(\lambda-1)\left(4\left(\frac{-1+n+\lambda}{n}\right)^s - s - 2\right)}{2^s(1+s)(2+s)}\right) b^s\right\}^{\frac{1}{q}} + \left(\frac{n^2}{2} - n\mu + \mu^2\right)^{1-\frac{1}{q}} \left\{\left(\frac{n^2(1+s) - n(2+s)\mu + 2\mu^2\left(\frac{\mu}{n}\right)^s}{2^s(1+s)(2+s)}\right) a^s\right. \\
&\left.\left.+\left(\frac{2\mu^2\left(2-\frac{\mu}{n}\right)^s - n^2\left(3 + 2^{2+s} + s - 8\left(2-\frac{\mu}{n}\right)^s\right)}{2^s(1+s)(2+s)} + \frac{n\mu\left(2 + 2^{2+s} + s + 2^{1+s}s - 8\left(2-\frac{\mu}{n}\right)^s\right)}{2^s(1+s)(2+s)}\right) b^s\right\}^{\frac{1}{q}}\right].
\end{aligned} \tag{34}$$

Proof. The proof follows from Corollary 3.5 for the function defined by (31). □

In particular, for $n \to 2$, $\mu, q \to 1$ and $\lambda \to 0$, inequality (34) reduces to

$$\begin{aligned}
&\left|A\left(a^{s+1}, b^{s+1}\right) + A^{s+1}(a,b) - \frac{(b-a)^2}{8} L_{s+1}^{s+1}(a,b)\right| \\
&\leq \frac{(b-a)}{2^{s+2}(s+2)}\left[2\left(4^s \times s + 5 \times 3^s - 2^{2s+1} - 2^{s+1}\right) b^s + \left(1 + 3^{s+2} + 4^{s+1} + (1 + 3 \times 2^s) \times s\right) a^s\right].
\end{aligned} \tag{35}$$

Proposition 4.3. *Let $n \in \mathbb{N}$ be such that $n \geq 2$, $s \in (0,1)$ and $q \geq 1$, then*

$$\left|(n+\lambda-\mu-1)\frac{qa^{\frac{s}{q}+1}}{q+s}+\frac{2q\mu}{q+s}A\left(a^{\frac{s}{q}+1},b^{\frac{s}{q}+1}\right)+(n-\lambda-\mu+1)\frac{qA^{\frac{s}{q}+1}(a,b)}{q+s}-\frac{q(b-a)^2}{2n(q+s)}L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right|$$
$$\leq \frac{(b-a)}{2n}\left[\left(\frac{n^2}{2}+(1-\lambda)(1-n-\lambda)\right)^{1-\frac{1}{q}}\left\{\left(\frac{2^s\left((1+n-\lambda)^2\left(\frac{1+n-\lambda}{n}\right)^{-s}-2^{1-s}n(2-2\lambda+s(n+\lambda-1))\right)}{(s-2)(s-1)}\right.\right.\right.$$
$$\left.-\frac{2^s\left(n(n+(s-2)(\lambda-1))-(1+n-\lambda)^2\left(\frac{1+n-\lambda}{n}\right)^{-s}\right)}{(s-2)(s-1)}\right)a^s$$
$$\left.+\left(-\frac{2^s\left(n(n-(s-2)(\lambda-1))-(n+\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^{-s}\right)}{(s-2)(s-1)}\ \frac{2^s(n+\lambda-1)^2\left(\frac{-1+n+\lambda}{n}\right)^{-s}}{(s-2)(s-1)}\right)b^s\right\}^{\frac{1}{q}}$$
$$+\left(\frac{n^2}{2}-n\mu+\mu^2\right)^{1-\frac{1}{q}}\left\{\left(\frac{2^s\mu^2\left(\frac{\mu}{n}\right)^{-s}-2^s\left(n^2(s-1)-n(s-2)\mu-\mu^2\left(\frac{\mu}{n}\right)^{-s}\right)}{(s-2)(s-1)}\right)a^s\right.$$
$$\left.\left.+\left(\frac{(\mu-2n)^2\left(1-\frac{\mu}{2n}\right)^{-s}-2^sn(-n(s-3)+(s-2)\mu)}{(s-2)(s-1)}+\frac{(\mu-2n)^2\left(1-\frac{\mu}{2n}\right)^{-s}-2n(2n+(s-2)\mu)}{(s-2)(s-1)}\right)b^s\right\}^{\frac{1}{q}}\right]. \tag{36}$$

Proof. The proof follows from Corollary 3.6 for the function defined by (31). □

In particular, for $n \to 2$, $\mu, q \to 1$ and $\lambda \to 0$, inequality (36) reduces to

$$\left|A\left(a^{s+1},b^{s+1}\right)+A^{\frac{s}{q}+1}(a,b)-\frac{(b-a)^2}{8}L_{s+1}^{s+1}(a,b)\right|$$
$$\leq \frac{(s+1)(b-a)\left(2^{2s}-2s+3^{2-s}\times 4^s-2^{s+2}-4\right)A(a^s,b^s)}{2(s-1)(s-2)}. \tag{37}$$

Proposition 4.4. *Let $n \leq \min\{1-\lambda,\mu\}$ for $n \in \mathbb{N}$ and let $q > 1$ be such that $p = \frac{q}{q-1}$, then*

$$\left|(n+\lambda-\mu-1)\frac{qa^{\frac{n}{q}+1}}{q+n}+\frac{2q\mu}{q+n}A\left(a^{\frac{n}{q}+1},b^{\frac{n}{q}+1}\right)+(n-\lambda-\mu+1)\frac{qA^{\frac{n}{q}+1}(a,b)}{q+n}-\frac{q(b-a)^2}{2n(q+n)}L_{\frac{n}{q}+1}^{\frac{n}{q}+1}(a,b)\right|$$
$$\leq \frac{(b-a)}{2n}\sqrt[q]{\frac{n}{2}}\left[\sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}}\ \sqrt[q]{A(3a^n,b^n)}+\sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\ \sqrt[q]{A(a^n,3b^n)}\right]. \tag{38}$$

Proof. The proof follows from Corollary 3.10 for the function defined by (31). □

In particular, for $p, q \to 2$, $\mu \to 1$ and $\lambda \to 0$, inequality (38) reduces to

$$\left|4A\left(a^{\frac{3}{2}},b^{\frac{3}{2}}\right)+2A^{\frac{3}{2}}(a,b)-(b-a)^2L_{\frac{3}{2}}^{\frac{3}{2}}(a,b)-a^{\frac{3}{2}}\right| \leq \frac{3(b-a)}{4}\sqrt[q]{\frac{1}{2}}\left[\sqrt{\frac{A(3a,b)}{3}}+\sqrt{A(a,3b)}\right]. \tag{39}$$

Proposition 4.5. *Let the conditions of Proposition 4.4 be satisfied for $s \in (0,1]$, then*

$$\left|(n+\lambda-\mu-1)\frac{qa^{\frac{s}{q}+1}}{q+s}+\frac{2q\mu}{q+s}A\left(a^{\frac{s}{q}+1},b^{\frac{s}{q}+1}\right)+(n-\lambda-\mu+1)\frac{qA^{\frac{s}{q}+1}(a,b)}{q+s}-\frac{q(b-a)^2}{2n(q+s)}L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right|$$
$$\leq \frac{(b-a)}{2n}\left[\sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}}\sqrt[q]{\frac{n\left(2-2^{-s}\right)a^s+\left(2^{-s}\cdot n\right)b^s}{s+1}}\right.$$
$$\left.+\sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\sqrt[q]{\frac{\left(2^{-s}\cdot n\right)a^s+n\left(2-2^{-s}\right)b^s}{s+1}}\right]. \tag{40}$$

Proof. The proof follows from Corollary 3.11 for the function defined by (31). □

In particular, for $p,q\to 2$, $\mu\to 1$ and $\lambda\to 0$, inequality (40) reduces to

$$\left|A^{\frac{s}{2}+1}(a,b)+2A\left(a^{\frac{s}{2}+1},b^{\frac{s}{2}+1}\right)-\frac{(b-a)^2}{2}L_{\frac{s}{2}+1}^{\frac{s}{2}+1}(a,b)-a^{\frac{s}{2}+1}\right|$$
$$\leq \frac{(s+2)(b-a)}{8}\left[\sqrt{\frac{2\left(2-2^{-s}\right)a^s+\left(2^{-s}\cdot 2\right)b^s}{3(s+1)}}+\sqrt{\frac{\left(2^{-s}\cdot 2\right)a^s+2\left(2-2^{-s}\right)b^s}{3(s+1)}}\right]. \tag{41}$$

Proposition 4.6. *Let the conditions of Proposition 4.4 be satisfied for $s \in (0,1)$, then*

$$\left|(n+\lambda-\mu-1)\frac{qa^{\frac{s}{q}+1}}{q+s}+\frac{2q\mu}{q+s}A\left(a^{\frac{s}{q}+1},b^{\frac{s}{q}+1}\right)+(n-\lambda-\mu+1)\frac{qA^{\frac{s}{q}+1}(a,b)}{q+s}-\frac{q(b-a)^2}{2n(q+s)}L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right|$$
$$\leq \frac{(b-a)}{2n}\left[\sqrt[p]{\frac{(1-\lambda)^{p+1}+(1-\lambda-n)^{p+1}}{p+1}}\sqrt[q]{\frac{n\left(2^{-s}-2\right)a^s-\left(2^{-s}\cdot n\right)b^s}{1-s}}\right.$$
$$\left.+\sqrt[p]{\frac{\mu^{p+1}+(\mu-n)^{p+1}}{p+1}}\sqrt[q]{\frac{n\left(2^{-s}-2\right)b^s-\left(2^{-s}\cdot n\right)a^s}{1-s}}\right]. \tag{42}$$

Proof. The proof follows from Corollary 3.12 for the function defined by (31). □

In particular, for $p,q\to 2$, $\mu\to 1$ and $\lambda\to 0$, inequality (42) reduces to

$$\left|A\left(a^{\frac{s}{2}+1},b^{\frac{s}{q}+1}\right)+A^{\frac{s}{2}+1}(a,b)-\frac{(b-a)^2}{8}L_{\frac{s}{2}+1}^{\frac{s}{2}+1}(a,b)\right|$$
$$\leq \frac{(s+2)(b-a)}{16}\left[\sqrt{\frac{2\left(2^{-s}-2\right)a^s-\left(2^{-s}\cdot 2\right)b^s}{3(1-s)}}+\sqrt{\frac{2\left(2^{-s}-2\right)b^s-\left(2^{-s}\cdot 2\right)a^s}{3(1-s)}}\right]. \tag{43}$$

Remark 4.7. *For different choices of a function f, several new inequalities for special means can be found. The details are left to the interested readers.*

5. Conclusion

A new generalized integral identity is proved involving differentiable mappings. Using this identity we obtained several new Hermite-Hadamard type integral inequalities for differentiable functions that are h-preinvex in absolute value. Additionally new special cases are discussed in depth. The ideas and strategies of the acquired studies are expected to inspire the interested readers. We suspect that our findings can be extended to obtain various results in convex analysis, special functions, theories related to optimization, mathematical inequalities and can stimulate further research work in various fields of pure and applied sciences.

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