Filomat 36:10 (2022), 3433–3447
https://doi.org/10.2298/FIL2210433B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Positive Periodic Solutions of Delay Differential System at Resonance

Noureddine Bouteraa[a]

[a] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Benbella. Algeria*
Oran Graduate School of Economics. Algeria.

Abstract. This paper is devoted to the existence of positive periodic solutions for sytem of a class of nonlinear delay differential equations with periodic conditions. Our analysis is based on Mawhin coincidence degree theorem. An example is also presented to illustrate the effectiveness of the main result.

1. Introduction

In this paper, we first consider the nonlinear nonautonomous delayed differential system

$$\begin{cases} \frac{d^2x}{dt^2} = F\left(t, x\left(t-\tau_1\left(t\right)\right), \ldots, x\left(t-\tau_n\right), u\left(t-\delta\left(t\right)\right)\right), & t \in [0,\omega], \\ \frac{d^4u}{dt^4} = a\left(t\right)u\left(t\right) + \eta\left(t\right) f(x\left(t-\sigma\left(t\right)\right)), & t \in [0,\omega], \end{cases} \tag{1}$$

subject with the following periodic boundary conditions

$$\begin{cases} x^{(i)}\left(0\right) = x^{(i)}\left(\omega\right) = 0, & i \in \{0,1\}, \\ u^{(i)}\left(0\right) = u^{(i)}\left(\omega\right) = 0, & i \in \{0,1,2,3\}, \end{cases} \tag{2}$$

where $f\left(t, y_1, y_2, \ldots, y_n, y_{n+1}\right) \in C\left([0,\omega] \times \mathbb{R}^{n+1}, \mathbb{R}\right)$,
$\delta\left(\cdot\right) \in C\left(\mathbb{R}, \mathbb{R}\right)$, $a\left(\cdot\right)$, $\eta\left(\cdot\right) \in C\left(\mathbb{R}, \mathbb{R}_+^*\right)$ are $\omega-$periodic functions with respect to t and $\omega > 0$.
It is well-known that second-order and fourth-order equation (1) under consideration can be reduced in an obvious way to a systems of the first-order

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ F\left(t, x\left(t-\tau_1\left(t\right)\right), \ldots, x\left(t-\tau_n\left(t\right)\right), u\left(t-\delta\left(t\right)\right)\right) \end{pmatrix},$$

and

$$\frac{du}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a(t) & 0 & 0 & 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta\left(t\right) f(x\left(t-\sigma\left(t\right)\right)) \end{pmatrix}.$$

2020 *Mathematics Subject Classification*. Primary 34B15; Secondary 35F55
Keywords. periodic solutions, Green's function, coicidence degree theorem, delay.
Received: 30 June 2020; Accepted: 29 August 2022
Communicated by Dragan S. Djordjević
Email address: bouteraa-27@homail.fr, bouteraanooreddine@gmail.com (Noureddine Bouteraa)

with boundary conditions $x(0) = x(\omega)$, $u(0) = u(\omega)$ respectively.
Delay differential equations (DDEs) are used to model biological, physical, and sociological processes, as well as naturally occurring oscillatory systems (see, for example [1, 32]. It is known that, in delay differential equations, the presence of the delay term causes the difficulties in analysis of differential equations. In many biological phenomena and engineering applications the dynamics of the system is determined in part by a feedback loop. When this feedback is delayed signicantly compared to the time scale of the dynamics, such systems are often described by DDEs. The analysis of DDEs is considerably more difficult than that of ordinary differential equations (ODEs), since the phase space of the dynamics of DDEs is effectively infinite dimensional. Much progress has been made in studying DDEs, and we refer to [23] for overviews and highlights. Nevertheless, it is fair to say that even for the study of relatively simple dynamic structures such as periodic solutions a great desire for new exible, generally applicable techniques remains.

In recent years, there has been considerable interest in the existence of periodic solutions of the following equation

$$u'(t) - a(t) g(u(t)) u(t) = \lambda b(t) f(t, u(t - \tau(t))), \ t \in [0, \omega],$$

where $a, \ b \in C(\mathbb{R}, [0, +\infty))$ are $\omega-$periodic functions with $\int_0^\omega a(t) dt > 0$, $\int_0^\omega b(t) dt > 0$ and τ is a continuous $\omega-$periodic function, see [21]. It is usually difficult to solve these kinds of delay differential equations analytically or to show the existence of periodic solutions for the problem associeted with it (for more details, see [36]). This equation has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias, see, for instence [24] and the references therein. For example, Mackey-Glass equation [32], a scalar DDE with a single delay and a nonpolynomial nonlinearity

$$u'(t) = -\beta u(t) + \alpha \frac{u(t-\tau)}{1 + u(t-\tau)^\rho},$$

this DDE, which models the concentration of white blood cells in a subject, is one of the first scalar DDEs that was conjectured to exhibit chaotic behaviour. In this equation, α is the production rate of new cells and β is the rate at which the cells die. The delay parameter $\tau > 0$ models the time it takes for the subject's body to observe the concentration and react, by either increasing or decreasing cell production and the positive real (i.e., not necessarily integer) parameter ρ models the assumption that the production of new cells will abruptly stop if the concentration is higher than the critical concentration, for more details, see [36].

Boundary value problems at resonance have been investigated for many years. The existence results obtained by different approaches, as example, the following methods have devoloped : coicidence degree theory due to Mawhin [33, 34], coinsidence theorem of Schauder type [41], topological degree [37], a Leggett-Williams typ theorem for coincidences [25], fixed point index theorem [28], a method refered as the shift argument combined with krasnoselskii fixed point or with monotone method coupled with, upper and lower method [4, 6, 7, 29]. For some developement on the existence results of the boundary value problems at resonance for the case that dimension of the kernel can be arbitrary, we can refer to [2, 5, 18, 30, 36]. In recent years, many different kinds of nonlinear boundary value problems, using different approches have been studied by many researchers, see for example [8–17, 20–22].

To the best of our knowledge, few results can be found in the literature conserning boundary value problem for differential equations with delay at resonance and nonlinear delay differential system at resonance [19, 25, 26]. From this point of view, it is imperative to study nonlinear delay differential system. Moreover, it has been noticed that most works existing in the literature and [19, 26] on the topic are based on the nonlinear homogenuous delay systems (i.e., the differential equations are the same order). But in this paper we focus to study nonlinear mixed delay system.

Here is brief outline of the paper. The next section provides the definitions and preliminaries results. Then, we present the existence results in Section 3 for problem (1) – (2). The argument are based on coincidence degree theorem of Mawhin. An illustrative example is also presented.

2. Preliminaries

Let $X = C[0,\omega]$ equipped with the norm $\|u\| = \max_{t\in[0,\omega]} |u(t)|$. We denote

$$C_{\omega}^{-} = \{u(t) \in X,\ u(t) < 0,\ u(\omega + t) = u(t),\ t \in [0,\omega]\}. \tag{3}$$

Lemma 2.1. *For $\rho > 0$ and $h \in X$, the equation*

$$\begin{cases} v^{(4)}(t) - \rho^4 v(t) = h(t), & t \in [0,\omega], \\ v^{(i)}(0) = v^{(i)}(\omega), & i \in \{0,1,2,3\}, \end{cases} \tag{4}$$

has a unique ω-periodic solution which is of the form

$$v(t) = \int_0^{\omega} G(t,s)(-h(s))\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{exp(\rho(t-s))+exp(\rho(s+\omega-t))}{4\rho^3(exp(\rho\omega)-1)} + \frac{\sin\rho(t-s)-\sin\rho(t-s-\omega)}{4\rho^3(1-\cos\rho\omega)}, & 0 \le s \le t \le \omega, \\ \frac{exp(\rho(t+\omega-s))+exp(\rho(s-t))}{4\rho^3(exp(\rho\omega)-1)} + \frac{\sin\rho(s-t)-\sin\rho(s-\omega-t)}{4\rho^3(1-\cos\rho\omega)}, & 0 \le t \le s \le \omega. \end{cases} \tag{5}$$

Proof. It is easy to check that the associated homogeneous equation of (4) has the solution

$$v(t) = c_1 exp(\rho t) + c_2 exp(-\rho t) + c_3\cos(\rho t) + c_4\sin(\rho t).$$

Applying the method of the variation of parameters, we have

$$c_1'(t) = \frac{exp(-\rho t)h(t)}{4\rho^3},\ c_2'(t) = \frac{-exp(\rho t)h(t)}{4\rho^3},$$

$$c_3'(t) = \frac{\sin(\rho t)h(t)}{2\rho^3},\ c_4'(t) = \frac{-\cos(\rho t)h(t)}{2\rho^3},$$

and then

$$c_1(t) = c_1(0) + \int_0^t \frac{exp(-\rho s)h(s)}{4\rho^3}ds,\ c_2(t) = c_2(0) + \int_0^t \frac{-exp(\rho)h(s)}{4\rho^3}ds.$$

$$c_3(t) = c_3(0) + \int_0^t \frac{\sin(\rho s)h(s)}{2\rho^3}ds,\ c_4(t) = c_4(0) + \int_0^t \frac{-\cos(\rho s)h(s)}{2\rho^3}ds.$$

Noting that $v^{(i)}(0) = v^{(i)}(\omega),\ i \in \{0,1,2,3\}$, we obtain

$$c_1(0) = \int_0^{\omega} \frac{exp(\rho(\omega - s))}{4\rho^3(1 - exp(\rho\omega))}h(s)\,ds,\ c_2(0) = \int_0^{\omega} \frac{exp(\rho s)}{4\rho^3(1 - exp(\rho\omega))}h(s)\,ds.$$

$$c_3(0) = -\int_0^\omega \frac{\sin(\rho s) - \sin(\rho(s-\omega))}{4\rho^3(1-\cos(\rho\omega))} h(s)\,ds,\; c_4(0)) = -\int_0^\omega \frac{\cos(\rho(s-\omega)) - \cos(\rho s)}{4\rho^3(1-\cos\rho\omega)} h(s)\,ds.$$

Therefore

$$\begin{aligned}
v(t) &= c_1(t)\,exp(\rho t) + c_2(t)\,exp(-\rho t) + c_3(t)\cos(\rho t) + c_4(t)\sin(\rho t),\\
&= \int_0^t \left[\frac{exp(\rho(t-s)) + exp(\rho(s+\omega-t))}{4\rho^3(exp(\rho\omega)-1)} + \frac{\sin(\rho(t-s)) - \sin(\rho(t-s-\omega))}{4\rho^3(1-\cos(\rho\omega))}\right](-h(s))\,ds\\
&+ \int_t^\omega \left[\frac{exp(\rho(t+\omega-s)) + exp(\rho(s-t))}{4\rho^3(exp(\rho\omega)-1)} + \frac{\sin(\rho(s-t)) - \sin(\rho(s-t-\omega))}{4\rho^3(1-\cos(\rho\omega))}\right](-h(s))\,ds\\
&= \int_0^\omega G(t,s)(-h(s))\,ds,
\end{aligned}$$

where $G(t,s)$ is defined as in (5).
By a direct calculation, we obtain the solution u satisfies the periodic boundary value condition of the problem (4). □

We shall using in the sequel the following property. Note that, from the property of periodicity, for all $h \in X$, we know

$$\int_0^\omega h(s)\,ds = \int_t^{t+\omega} h(s)\,ds,\; t \in [0,\omega].$$

Now, we compute the lower and an upper bound of the Green's function defined in (5).

Lemma 2.2. *The Green's function satisfies the equality* $\int_0^\omega G(t,s)\,ds = \frac{1}{\rho^4}$ *and if* $\rho < \frac{\pi}{\omega}$ *holds, then* $0 < l \le G(t,s) \le L$ *for all* $t \in [0,\omega]$ *and* $s \in [0,\omega]$, *where*

$$l = \frac{1}{4\rho^3(exp(\rho\omega)-1)},$$

and

$$L = \frac{1+exp(\rho\omega)}{4\rho^3(exp(\rho\omega)-1)} + \frac{1}{2\rho^3(1-\cos\rho\omega)}.$$

Proof. From (5), we obtain $\int_0^\omega G(t,s)\,ds = \frac{1}{\rho^4}$. If $\rho < \frac{\pi}{\omega}$, we get $G(t,s) > 0$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$. Next, we compute a lower and an upper bound for $G(t,s)$, $s \in [0,\omega]$. We have

$$\frac{1}{4\rho^3(exp(\rho\omega)-1)} \le \frac{exp(\rho(t-s)) + exp(\rho(s+\omega-t))}{4\rho^3(exp(\rho\omega)-1)} \le G(t,s),$$

and

$$G(t,s) = \frac{exp(\rho(t-s)) + exp(\rho(s+\omega-t))}{4\rho^3(exp(\rho\omega)-1)} + \frac{\sin(\rho(t-s)) - \sin(\rho(t-s-\omega))}{4\rho^3(1-\cos(\rho\omega))},$$

$$\leq \frac{1+exp\left(\rho\left(\omega\right)\right)}{4\rho^{3}\left(exp\left(\rho\omega\right)-1\right)}+\frac{\sin\left(\rho\left(t-s-\frac{1}{2}\omega+\frac{1}{2}\omega\right)\right)-\sin\left(\rho\left(t-s-\frac{1}{2}\omega-\frac{1}{2}\omega\right)\right)}{4\rho^{3}\left(1-\cos\left(\rho\omega\right)\right)},$$

$$\leq \frac{1+exp\left(\rho\omega\right)}{4\rho^{3}\left(exp\left(\rho\omega\right)-1\right)}+\frac{2\cos\left(\rho\left(t-s-\frac{1}{2}\omega\right)\right)\sin\left(\rho\left(\frac{1}{2}\omega\right)\right)}{4\rho^{3}\left(1-\cos\left(\rho\omega\right)\right)},$$

$$\leq \frac{1+exp\left(\rho\omega\right)}{4\rho^{3}\left(exp\left(\rho\omega\right)-1\right)}+\frac{1}{2\rho^{3}\left(1-\cos\left(\rho\omega\right)\right)}.$$

□

Consider the existence of positive periodic solutions for the fourth-order nonlinear differential equation with ω-periodic boundary value conditions in (4),

$$v^{(4)}\left(t\right)-a\left(t\right)v\left(t\right)=\eta\left(t\right)f\left(u\left(t-\tau\left(t\right)\right)\right),$$

where $f\in C\left([0,\omega]\times[0,\infty),[0,\infty)\right)$, $a\in C\left([0,\omega],(0,\infty)\right)$ and $f\left(t,u\right)>0$ for $u>0$.

We introduce the following abbreviations (for more details, see [3]).

$$a^{*}=\max\left\{a\left(t\right)\,:\,t\in[0,\omega]\right\},\ a_{*}=\min\left\{a\left(t\right)\,:\,t\in[0,\omega]\right\},\ \rho=\sqrt[4]{a^{*}}.$$

It is easy to see that, for any $h\in C_{\omega}^{-}$, the following boundary value problem

$$\begin{cases} v^{(4)}\left(t\right)-a\left(t\right)v\left(t\right)=h\left(t\right), & t\in\left(0,\omega\right),\\ v^{(i)}\left(0\right)=v^{(i)}\left(\omega\right), & i\in\left\{0,1,2,3\right\}, \end{cases} \tag{6}$$

is equivalent to the following boundary value problem

$$\begin{cases} v^{(4)}\left(t\right)-a^{*}v\left(t\right)=\left(a\left(t\right)-a^{*}\right)v\left(t\right)+h\left(t\right),\\ v^{(i)}\left(0\right)=v^{(i)}\left(\omega\right), \qquad i\in\left\{0,1,2,3\right\}, \end{cases} \tag{7}$$

Define operators $A,\ B\,:\,X\to X$ by

$$\left(Ah\right)\left(t\right)=\int_{0}^{\omega}G\left(t,s\right)\left(-h\left(s\right)\right)ds,\ \left(Bu\right)\left(t\right)=\left(a\left(t\right)-a^{*}\right)u\left(t\right). \tag{8}$$

Let $\Theta\,:\,X\to X$ be a operator defined by

$$\left(\Theta h\right)\left(t\right)=\left(I-AB\right)^{-1}\left(Ah\right)\left(t\right),$$

where $h\in C_{\omega}^{-}$.

Lemma 2.3. *(See [3]) Let $h\in C_{\omega}^{-}$. Then, $v\in C[0,\omega]$ is a solution of*

$$\begin{cases} v^{(4)}\left(t\right)-a^{*}v\left(t\right)=\left(a\left(t\right)-a^{*}\right)v\left(t\right)+h\left(t\right),\\ v^{(i)}\left(0\right)=v^{(i)}\left(\omega\right), \qquad i\in\left\{0,1,2,3\right\}, \end{cases}$$

if and only if

$$v\left(t\right)=\left(I-AB\right)^{-1}\left(Ah\right)\left(t\right)=\left(\Theta h\right)\left(t\right),\ t\in[0,\omega],$$

where A and B defined by (8).

Lemma 2.4. *(see [3]) Let $\rho<\frac{\pi}{\omega}$ holds. Then, Θ is a continuous mapping satisfying the following condition:*

$$\left(\Theta u\right)\left(t\right)>0,\ u\in C[0,\omega].$$

3. Main Results

In this section, we state and prove the existence results. The proof is based on the following theorem, wich can be found in [31].

We first recall some notations and an abstract existence results of coincidence degree theory due to Mawhin [7, 9, 15]. Let

$$X = Z = C([0,\omega]) = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t)\},$$

be two real Banach spaces endowed with the norm

$$\|x\| = \max_{t\in\mathbb{R}} |x(t)| = \max_{t\in[0,\omega]} |x(t)|.$$

Definition 3.1. *A linear operator $L : domL \subset X \to Z$ is called to be a Fredholm operator provided that*
(i) *$KerL$ is finite dimensional,*
(ii) *ImL is closed and has finite codimension.*
In addition the Fredholm of idex of L is defined by the integer number

$$indL = \dim KerL - co\dim ImL.$$

From Definition 3.1, it follows that if $L : domL \subset Y \to Z$ is a Fredholm of index zero (that is, ImL, the image of L, $KerL$, the kernel of L is finite dimensional with the same dimension as the Z/ImL) and it is a linear operator, then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $ImP = KerL$, $KerQ = ImL$ and $Y = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L|_{domL\cap KerP} \to ImL$ is invertible, we denote the inverse of that map by K_P.
Let Ω be an open bounded subset of Y such that $domL \cap \Omega \neq \phi$, the map $N : Y \to Z$ is said to be $L-compact$ on $\overline{\Omega}$ if the map $QN : \overline{\Omega} \to Z$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Definition 3.2. *Let L be a Fredholm operator of index zero. The operator $N : X \to Z$ is said to be $L-compact$ in $\overline{\Omega}$ provided that*
(i) *the map $QN : \overline{\Omega} \to Z$ is cotinuous $QN\left(\overline{\Omega}\right)$ is bounded in Z,*
(ii) *the map $K_{P,Q}N : \overline{\Omega} \to X$ is completely continuous.*
In addition, we say that N is $L-$completely continuous if it is $L-$compact on every bounded set in X.

We will formulate the boundary value problem $(1)-(2)$ as $Lu = Nu$ where L and N are approriate operators. To obtain our existence results we use the following fixed point theorem of (Mawhin 1979).

Theorem 3.3. *(See [31]) Let L be a Fredholm operator of index zero and N be $L-compact$ on $\overline{\Omega}$. Assume that the following conditions are satisfied:*
(i) *$Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0,1)$.*
(ii) *$Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$.*
(iii) *$deg\left(QN|_{KerL}, \Omega \cap KerL, 0\right) \neq 0$,*
where $Q : Z \to Z$ is a projection as above with $ImL = KerQ$.
Then the abstract equation $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$.

Let us list the following assumptions.
(H_1) For $t \in [0,\omega]$ and $u \in C[0,\omega]$,

$$\eta(t) f(x(t-\sigma(t))) < 0. \tag{9}$$

(H_2) There exists a constant $C > 0$ such that, if $x(t)$ and $u(t)$ are continuous $\omega-$periodic function and satisfy

$$\int_0^{\omega} F(t, x(t-\tau_1(t)), \ldots, x(t-\tau_n), u(t-\delta(t)))\, dt = 0,$$

then we have

$$\int_0^{\omega} F(t, x(t-\tau_1(t)), \ldots, x(t-\tau_n), u(t-\delta(t)))\, dt \leq C.$$

(H_3) There exists a constant $H > 0$ such that when $v_i \geq H,\ i = 1, 2, \ldots, n+1$,

$$F(t, v_1, v_2, \ldots, v_{n+1}) > 0,\ F(t, -v_1, -v_2, \ldots, -v_{n+1}) < 0,$$

uniformly hold for $[0, \infty)$.

Theorem 3.4. *Let the conditions* (H_1), (H_2) *and* (H_3) *hold. Then, the system* $(1)-(2)$ *has at least one positive* $\omega-$*periodic solution.*

For the Proof of Theorem 3.4, we shall apply Theorem 3.3 and the following Lemmas. Before we state our lemmas, we say that L is a Fredholm operator of index zero, that is, ImL is closed and $\dim KerL = co\dim ImL$. This implies that there exist a continuous projections $P : Y \to Y$ and $Q : Z \to Z$ such that $ImP = KerL$ and $KerQ = ImL$. For this purpose, we must define P by (12) (see later), by setting

$$Ry = a\omega + \int_0^{\omega} (\omega - s)\, y(s)\, ds,\ \forall a \in \mathbb{R},$$

the linear continuous projector operator Q defined by

$$Qy(t) = \frac{1}{C} \cdot (Ry) \cdot t,$$

where $C = a\omega + \int_0^{\omega} (\omega - t)\, t dt \neq 0,\ t \in (0, \omega]$ and the generalized inverse operator $K_P : ImL \to X \cap KerP$ of L defined by

$$K_P y(t) = \int_0^{t} (t-s)\, y(s)\, ds.$$

Lemma 3.5. (i) *The operator* $L : domL \subset Y \to Z$ *is a Fredholm operator of index zero.*
(ii) *For every* $y \in ImL$, *we have*

$$\left\|K_P y\right\| \leq \left\|y\right\|_1.$$

Proof. It is obvious that $u(t)$ is the unique $\omega-$periodic solution of second equation in (1) for $x \in C[0, \omega]$. Therefore, the existence problem of $\omega-$periodic solution of $(1)-(2)$ is equivalent to that of $\omega-$periodic solutions of the equation

$$\frac{d^2 x(t)}{dt^2} = F(t, x(t-\tau_1(t)), \ldots, x(t-\tau_n), \Theta(t-\delta(t))),$$

where Θ defined in Lemma 2.3.
Let $X = C^1[0,1]$, $Z = L^1[0, \omega]$, define the linear operator $L : domL \subset X \to Z$ by

$$Lx = x'',\quad x \in domL,$$

where

$$domL = \left\{u \in W^{2,1}(0,1) : \ u(0) = u(\omega) = 0\right\},$$

and define $N : X \to Z$ by

$$Nx = F\left(t, x\left(t - \tau_1(t)\right), \ldots, x\left(t - \tau_n\right), \Theta\left(t - \delta(t)\right)\right). \tag{10}$$

It is clear that

$$\begin{aligned} KerL &= \{x \in domL \subset X : \ Lx = 0\} \\ &= \{x \in domL \subset X \ : \ x'' = 0\} \\ &= \{x \in domL : \ x(t) = at, \ a \in \mathbb{R}\} \simeq \mathbb{R}. \end{aligned}$$

Now, we show that

$$ImL = \left\{ y \in Z : \ at + \int_0^{\omega} (\omega - s)\, y(s)\, ds = 0, \ a \in \mathbb{R}, \ t \in [0, \omega] \right\}. \tag{11}$$

Since the problem

$$x'' = y,$$

has a solution $x(t)$ that satisfies the conditions $x^{(i)}(0) = x^{(i)}(\omega) = 0, \ i \in \{0, 1\}$ if and only if

$$at + \int_0^{\omega} (\omega - s)^3\, y(s)\, ds = 0.$$

Let $C = a\omega + \int_0^{\omega} (\omega - t)\, t dt \neq 0, \ t \in (0, \omega]$. By simple calculation, we get $C = \frac{\omega}{6}\left(6a + \omega^2\right)$.
Now, we need to show that the operator Q is projector. From $Qy(t) = \frac{1}{C} \cdot (Ry) \cdot t$, it is clear that $\dim ImQ = 1$. We have

$$\begin{aligned} \left(Q^2 y\right)(t) &= \left(Q\left(Qy\right)\right)(t) \\ &= \frac{1}{C}\left(\frac{1}{C} Ry\right)\left(a\omega + \int_0^{\omega} (\omega - t)\, t ds\right) t \\ &= \frac{1}{C}(Ry)\, t \\ &= (Qy)(t), \end{aligned}$$

which implies that the operator Q is projector. Furthermore, $ImL = KerQ$.

In order, to show $Z = ImL \oplus ImQ$, it remains to shows two following steps.

Step 1. For $y \in Z$, let $y = (y - Qy) + Qy$, since $Q(y - Qy) = Qy - Q^2 y = 0$, we know $(y - Qy) \in KerQ = ImL$ and $Qy \in ImQ$. Thus $Z = ImL + ImQ$.

Step 2. Let $y \in ImL \cap ImQ$. Since $y \in ImQ$, then there exists $\rho \in \mathbb{R}$ such that $y(t) = \rho t,\ t \in [0,\omega]$. Since $y \in ImL = KerQ$, then

$$0 = \rho\,(Ry)(t) = \rho\left(a\omega + \int_0^{\omega} (\omega - t)\, t ds\right) = \rho C.$$

Since $C \neq 0$, then $\rho = 0$, so we have $y(t) = 0,\ t \in [0,\omega]$. Which implies $ImL \cap ImQ = \{0\}$.

As consequence of Step 1 and Step 2, we deduce that

$$Z = ImL \oplus ImQ,$$

and so

$$dimKerL = co\dim ImL = \dim ImQ = 1.$$

Then $IndL = \dim KerL - co\dim ImL = 1 - 1 = 0$.
Thus L is Fredholm operayor of index zero.

We are now ready to give the other projector employed in the proof of (*ii*). Define the other projector $P : X \to X$ by

$$(Pu)(t) = u''(0)\, t,\ t \in [0,\omega], \tag{12}$$

Note that $KerP = \{u \in X : u'(0)\, t = 0\} = \{u \in X : u'(0) = 0\}$ and $ImP = KerL$.
Similarly, we shall prove that the operator P is projector and $Y = KerP \oplus KerL$. Fistly, since $(Pu)'(t) = u'(0)$, then $\left(P^2u\right)(t) = P(t),\ t \in [0,\omega]$ for all $u \in X$, we have

$$u = (u - Pu) + Pu,$$

and

$$u(t) = (u(t) - u'(0)\, t) + u'(0)\, t.$$

For $u \in X$, let $u = (u - Pu) + Pu$. Since $P(u - Pu) = Pu - P^2u = Pu - Pu = 0$, we know, $(u - Pu) \in KerP$ and $Pu \in ImP = KerL$, thus $X = KerP + KerL$.
Let $u \in KerL \cap KerP$, since $u \in KerL = ImP$, there exists $\mu \in \mathbb{R}$ such that $u(t) = \mu t$ and since $u \in KerP$, then $\mu = u'(0) = 0$ and so $u(t) = 0,\ t \in [0,\omega]$. Consequently, $KerL \cap KerP = \{0\}$. Then $X = KerP \oplus KerL$.

Let $u \in KerL \cap KerP$, since $u \in KerL = ImP$, there exists $\mu \in \mathbb{R}$ such that $u(t) = \mu t$ and since $u \in KerP$, then $\mu = u'(0) = 0$ and so $u(t) = 0,\ t \in [0,\omega]$. Consequently, $KerL \cap KerP = \{0\}$. Then $X = KerP \oplus KerL$.

Before, to estimate the supremum norm of the generalized inverse operator K_P. It remains to prove that the operator K_P is the generalized inverse of L. In fact, if $y \in ImL$, then

$$(LK_P)\, y(t) = [(K_P y)(t)]'' = y(t),$$

and for $u \in domL \cap KerP$, we know

$$(K_P L)\, u(t) = (K_P)\, u''(t) = \int_0^t (t - s)\, u''(s)\, ds = u(t) - u(0) - u'(0)\, t,$$

in view of $u \in X \cap KerP,\ u(0) = 0$ and $Pu = 0$, thus

$$(K_P L)\, u(t) = u(t).$$

This shows that $K_P = (L\mid_{X\cap KerP})^{-1}$.
Lastly, we estimate the supremum norm of the generalized inverse operator K_P.
From the definition of K_P, it follows that

$$\left\|K_P y\right\|_\infty \leq \int_0^1 (1-s)^2 \left|y(s)\right| ds \leq \int_0^1 \left|y(s)\right| ds = \left\|y\right\|_1$$

from $(K_P y)'(t) = \int_0^t y(s)\,ds$, we obtain

$$\left\|(K_P y)'\right\|_\infty \leq \int_0^1 \left|y(s)\right| ds = \left\|y\right\|_1$$

As such we have

$$\left\|K_P y\right\| = \max\left\{\left\|K_P y\right\|, \left\|(K_P y)'\right\|\right\} \leq \left\|y\right\|_1$$

then, we have

$$\left\|K_p y\right\| \leq \left\|y\right\|_1, \tag{13}$$

□

Lemma 3.6. *The operator $N : X \to Z$ given by* (10) *is $L-$ completely continuous.*

Proof. The proof is standard, we omit it.
□

Lemma 3.7. *Let $\Omega_1 = \{x \in domL \setminus KerL : Lx = \lambda Nx, \ for\ some\ \lambda \in [0,1]\}$. Then Ω_1 is buonded.*

Proof. Suppose that $x \in \Omega_1$, and $Lx = \lambda Nx$. Thus $\lambda \neq 0$ and $QNx = 0$, so it yields

$$a\omega + \int_0^\omega (\omega - s) f(s, x(s-\tau_1(s)), \ldots, x(s-\tau_n(s)), \Theta(s-\delta(s)))\,ds = 0.$$

Thus, by condition (H_2), there exist an $i_0 \in \{1,2,\ldots,n\}$, a point $t_1 \in [0,\omega]$, such that $\left|x(t_1 - \tau_{i_0}(t_1))\right| \leq M$ and $|(\Theta x)(t_1 - \delta(t_1))| \leq M$. In view of

$$x(0) = x(t_1 - \tau_{i_0}(t_1)) - \int_0^{t_1} x'(s - \tau_{i_0}(s))\,ds.$$

Denote $t_1 - \tau_{i_0}(t_1) = \zeta_1 + k\omega,\ \zeta_1 \in [0,\omega]$, k being an integer. So

$$x(\zeta_1) < M.$$

In a similar way, there exist an $i_1 \in \{1,2,\ldots,n\}$, a point $\zeta_2 \in [0,\omega]$ and a constants $M_1 = -M$ such that

$$x(\zeta_1) > -M = M_1.$$

Then $|x(\zeta_1)| < M$.

Hence

$$|x(0)| = \left| x(t_1 - \tau_{i_0}(t_1)) - \int_0^{t_1} x'(s - \tau_{i_0}(s))\, ds \right| \leq M + \|x'\|_1 .$$

Thus, we have

$$|x'(0)| \leq M + \int_0^1 \left| x''(s - \tau_{i_0}(s)) \right| ds = M + \|x''\|_1 = M + \|Lx\|_1 \leq M + \|Nx\|_1 \tag{14}$$

Again for $x \in \Omega_1$, then $(I-P)x \in domL \cap KerP = ImK_P$ and $LPu = 0$, $0 < \lambda < 1$ and $Nx = \frac{1}{\lambda} Lx \in ImL$, thus from Lemma 3.5, we know

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \|L(I-P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1 . \tag{15}$$

From (14), (15) and $\|Px\| = |x'(0)|$, we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| = |x'(0)| + \|(I-P)x\| \leq M + 2\|Nx\|_1 . \tag{16}$$

From (9) and (16), we obtain

$$\|x\| \leq 2\left[C + \frac{M}{2}\right]. \tag{17}$$

Again, from (9), we have

$$\|x''\|_1 = \|Lx\|_1 \leq \|Nx\|_1 \leq C.$$

Which shows that Ω_1 is bounded.

□

Lemma 3.8. *The set* $\Omega_2 = \{x \in KerL : Nu \in ImL\}$ *is bounded.*

Proof. Let $x \in \Omega_2$, then $x \in KerL = \{x \in domL : x(t) = at,\ a \in \mathbb{R},\ t \in [0,\omega]\}$. Also, since $KerQ = ImL$, then $QNu = 0$, therefore

$$a\omega + \int_0^{\omega} (\omega - s) f(s, x(s - \tau_1(s)), \ldots, x(s - \tau_n(s), \Theta(s - \delta(s))))\, ds = 0.$$

From condition (H_2), $\|x\|_\infty = |at| \leq M,\ \forall a \in \mathbb{R}$, so $\|x\| \leq M$, thus Ω_2 is bounded.

□

Because we know $Nx \in ImL = KerQ$, so $QNx = 0$.

Before we define the set Ω_3, we must state our isomorphism, $J : KerL \to ImQ$. Let

$$J(at) = at,\ \forall a \in \mathbb{R},\ t \in [0,\omega],$$

and define

$$\Omega_3 = \{u \in KerL : -\lambda Ju + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\}.$$

Lemma 3.9. *If the first part of* (H_3) *holds, then*

$$a\left(\frac{6}{6a+\omega^2}\right)\left[a\omega+\int_0^\omega(\omega-s)\,f\left(s,x\left(s-\tau_1\left(s\right)\right),\ldots,x\left(s-\tau_n\left(s\right),\Theta\left(s-\delta\left(s\right)\right)\right)\right)ds\right]<0,$$

for all $|a|>M^*$ *and* Ω_3 *is bounded.*

Proof. Suppose that $x(t)=a_0t\in\Omega_3$, then we obtain

$$\lambda a_0=(1-\lambda)\left(\frac{6}{6a+\omega^2}\right)\left[a\omega+\int_0^\omega(\omega-s)\,f\left(s,x\left(s-\tau_1\left(s\right)\right),\ldots,x\left(s-\tau_n\left(s\right),\Theta\left(s-\delta\left(s\right)\right)\right)\right)ds\right]<0,$$

where $x(s-\tau_i(s))=a_0(s-\tau_i(s))\,,\ i\in\{1,2,\ldots,n\}$.
If $\lambda=1$, then $a_0=0$, which give Ω_3 bounded.
Otherwise, if $\lambda\neq1$, there exist $M^*>0$ such that $|a_0|>M^*$, then in view of first part of (H_3), we have

$$\lambda a_0^2=(1-\lambda)\,a_0\left(\frac{6}{6a+\omega^2}\right)\left[a\omega+\int_0^\omega(\omega-s)\,f\left(s,x\left(s-\tau_1\left(s\right)\right),\ldots,x\left(s-\tau_n\left(s\right),\Theta\left(s-\delta\left(s\right)\right)\right)\right)ds\right]<0,$$

which contradicts the fact that $\lambda a_0^2\geq0$. Then $|u|=|a_0t|\leq|a_0|\leq M^*$, we obtain $\|u\|\leq M^*$, hence $\Omega_3\subset\{u\in KerL\ :\ \|u\|\leq M^*\}$ is bounded.

If $\lambda=0$, it yields

$$\left(\frac{6}{6a+\omega^2}\right)\left[a\omega+\int_0^\omega(\omega-s)\,f\left(s,x\left(s-\tau_1\left(s\right)\right),\ldots,x\left(s-\tau_n\left(s\right),\Theta\left(s-\delta\left(s\right)\right)\right)\right)ds\right]=0,$$

taking condition (H_2) into account, we obtain $\|u\|=a\leq M^*$.
□

Now, define Ω_3 by

$$\Omega_3=\{u\in KerL:\ \lambda Ju+(1-\lambda)\,QNu=0,\ \lambda\in[0,1]\}\,.$$

Lemma 3.10. *If the second part of* (H_3) *holds, then*

$$a\left(\frac{6}{6a+\omega^2}\right)\left[a\omega+\int_0^\omega(\omega-s)\,f\left(s,x\left(s-\tau_1\left(s\right)\right),\ldots,x\left(s-\tau_n\left(s\right),\Theta\left(s-\delta\left(s\right)\right)\right)\right)ds\right]>0.$$

for all $|a|>M^*$ *and* Ω_3 *is bounded.*

Proof. A similar argument as above shows that Ω_3 is bounded. □

Proof. . Proof of Theorem 3.4. Let Ω to be an open bounded subset of X such that $\overset{3}{\underset{i=1}{\cup}}\overline{\Omega}_i\subset\Omega$. By using the fact that x'' is bounded and Arzela-Ascoli theorem, we can prove that $K_P(I-QN)\ :\ \overline{\Omega}\to X$ is compact, thus N is $L-$compact on $\overline{\Omega}$.
Then by Lemmas 3.7 and 3.8, we have
(*i*) $Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(domL\setminus KerL)\cap\partial\Omega]\times(0,1)$.
(*ii*) $Nx\notin ImL$ for every $x\in KerL\cap\partial\Omega$.

(*iii*) $H(x,\lambda) = \pm\lambda Jx + (1-\lambda) QNx = 0,\ \lambda \in [0,1]$.
According to Lemmas 3.9 and 3.10, we know that $H(x,\lambda) \neq 0$ for every $x \in KerL \cap \partial\Omega$. Thus, from the property of invariance under a homotopy,

$$\begin{aligned}\deg\left(QN\mid_{KerL}, \Omega \cap KerL, 0\right) &= \deg\left(H(\cdot,0), \Omega \cap KerL, 0\right)\\ &= \deg\left(H(\cdot,1), \Omega \cap KerL, 0\right)\\ &= \deg\left(\pm J, \Omega \cap KerL, 0\right) \neq 0.\end{aligned}$$

Consequently, by Theorem 3.3, $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$, so the boundary value problem (1) − (2) has at least one $\Omega-$periodic solution. The proof is complete. □

Example 3.11. *Consider the following logistic model with several delays and feedback control, see* [25, 26].

$$\begin{cases} \frac{d^2x}{dt^2} = x(t)\left[r(t) - \sum\limits_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\, u(t-\delta(t))\right], \\ \frac{d^4u}{dt^4} = \frac{0{,}05\cos(t)}{1+\cos^2(t)} u(t) + \eta(t)\, x^4(t-\sigma(t)), \end{cases} \tag{18}$$

subject with the periodic boundary conditions (2)*, where* $\sigma,\ \delta \in C(\mathbb{R},\mathbb{R})$ *and for* $i = 1, 2, \ldots, n$ *one has* $c,\ r,\ b,\ a_i,\ \tau_i \in C(\mathbb{R},\mathbb{R})$ *are all* $\omega-$*periodic functions with* $0 < \omega \leq 2\pi$*. Then the system* (18) *has at least one positive* $\omega-$*periodic solution.*
Indeed, Let $x(t)$ *be a continuous* $\omega-$*periodic solution and satisfies*

$$a\omega + \int_0^{\omega}\left[r(t) - \sum_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t))\right] = 0,$$

where Θe^x *as above or in Lemma* 2.3*. Then*

$$a\omega + \int_0^{\omega} r(t)\, dt = \int_0^{\omega}\sum_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t)) = 0.$$

On the other hand

$$\begin{aligned}&\left|a\omega + \int_0^{\omega} r(t) - \sum_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t))\right|\\ &\leq |a|\,\omega + \int_0^{\omega}\left|r(t) - \sum_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t))\right|\\ &\leq \int_0^{\omega}|r(t)|\, dt + |a|\,\omega + \int_0^{\omega}\sum_{i=1}^{n}\left|a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t))\right|\\ &\leq \frac{0{,}05}{2}\omega + 2\int_0^{\omega}|r(t)|\, dt = C > 0.\end{aligned}$$

Moreover,

$$\lim_{(v_1, v_2, \ldots, v_{n+1}) \to +\infty}\left[r(t) - \sum_{i=1}^{n} a_i(t)\, x(t_i - \tau_i(t)) - c(t)\left(\Theta e^x\right)(t-\delta(t))\right] = -\infty,$$

$$\lim_{(v_1,v_2,\ldots,v_{n+1})\to-\infty}\left[r(t)-\sum_{i=1}^{n}a_i(t)\,x(t_i-\tau_i(t))-c(t)\,(\Theta e^x)(t-\delta(t))\right]=r(t)>0,$$

hold uniformly in $t\in[0,\omega]$. *Furthermore,* $a(\cdot)$ *is* $2\pi-$*periodic and*

$$0<a_*=\min\left\{\frac{0,05\cos(t)}{1+\cos^2(t)},\ t\in[0,\omega]\right\}=-\frac{0,05}{2}=-0,025,$$

$$a^*=\max\left\{\frac{0,05\cos(t)}{1+\cos^2(t)},\ t\in[0,\omega]\right\}=\frac{0,05}{2}=0,025<\left(\frac{\pi}{2\pi}\right)^4=\frac{1}{2^4}=0,0625,$$

and

$$\eta(t)\,x^4(t-\sigma(t))>0.$$

By Theorem 3.4*, we see that system* (18) *has at least one positive* $\omega-$*periodic solution.*

4. Conclusion

In this work, we discussed the existence of positive periodic solutions for sytem of a class of nonlinear delay differential equations with periodic conditions, using Mawhin coincidence degree theorem. Furthermore, one of our obtained Lemma 2.3 is applied to set up novel existence of solution of class of integral equations. An example is also presented to illustrate the effectiveness of the main result.

References

[1] A. Arino, Delay Differential Equations and Applications, Springer, Berlin, (2006) 477-517.

[2] N. Bouteraa and S. Benaicha. A class of third-order boundary value problem with integral condition at resonance, Maltepe J. of Math. Volume II, Issue 2,(2020), 43-54..

[3] N. Bouteraa and S. Benaicha, Positive periodic solutions for a class of fourth-order nonlinear differential equations, Siberian Journal of Numerical Mathematics, Volume 22, No. 1 (2019), 1-14.

[4] N. Bouteraa, S. Benaicha and H. Djourdem, Positive solutions for systems of fourth-order two-point boundary value problems with parameter, Journal of Mathematical Sciences and Modeling, 2(1) (2019), 30-38.

[5] N. Bouteraa and S. Benaicha, Nonlinear boundary value problems for higher-order ordinary differential equation at resonance, Romanian Journal of Mathematic and Computer Science. 2018. Vol 8, Issue 2 (2018), p. 83-91.

[6] N. Bouteraa and S. Benaicha, Existence of solutions for third-order three-point boundary value problem, Mathematica. 60 (83), N 0 1, 2018, pp. 21-31.

[7] N. Bouteraa and S. Benaicha, Triple positive solutions of higher-order nonlinear boundary value problems, Journal of Computer Science and Computational Mathematics, Volume 7, Issue 2, June 2017, 25-31.

[8] S. Benaicha and N. Bouteraa, Existence of solutions for three-point boundary value problem for nonlinear fractional dierential equations, Bull. Transil. Univ. Brasov Ser. III: Mathematics, Informatics, Physics, 10(59)(2017), no. 2, 31-48.

[9] N. Bouteraa and S. Benaicha, Existence of solutions for three-point boundary value problem for nonlinear fractional equations, An. Univ. Oradea, Fasc. Mat, 24(2017), no. 2, 109- 119.

[10] N. Bouteraa and S. Benaicha, The uniqueness of positive solution for higher-order nonlinear fractional differential equation with nonlocal boundary conditions, Advances in the Theory of Nonlinear and it Application, 2(2018) No 2, 74-84.

[11] N. Bouteraa, S. Benaicha and H. Djourdem, Positive solutions for nonlinear fractional differential equation with nonlocal boundary conditions, Universal Journal of Mathematics and Applications. vol.1 (1)(2018), 39-45.

[12] N. Bouteraa, S. Benaicha, Existence of solution for nonlocal boundary value problem for Caputo nonlinear fractional inclusion, Journal of Mathematical Sciences and Modeling. vol.1 (1)(2018), 45-55.

[13] N. Bouteraa and S. Benaicha, The uniqueness of positive solution for nonlinear fractional differential equation with nonlocal boundary conditions, Analele Universitatii Oradea Fasc. Matematica. vol.25 (2018), Issue No. 2, 53-65.

[14] N. Bouteraa and S. Benaicha, Existence results for fractional differential inclusion with nonlocal boundary conditions, Riv. Mat. Univ. Parma, Vol.11(2020),181-206.

[15] N. Bouteraa and S. Benaicha, A study of existence and multiplicity of positive solutions for nonlinear fractional dierential equations with nonlocal boundary conditions, Stud. Univ. Babes-Bolyai Math. 66(2021), No. 2, 361-380.

[16] N. Bouteraa M. Inç, M. A. Akinlar, B. Almohsen, Mild solutions of fractional PDE with noise, Math. Meth. Appl. Sci. 2021;1–15.

[17] N. Bouteraa, Mustafa Inc, M. S. Hashemi and S. Benaicha, Study on the existence and nonexistenceof solutions for a class of nonlinear Erdélyi-Kober type fractional differential equation on unbounded domain, Journal of Geometry and Physics 178(2022)104546. 8 pages, www.elsevier.com/locate/geomphys.

[18] R. Chen and X. Lie, New existence results for nonlinear delayed differential systems at resonance, Journal of Inequalities and Applications, (2018), 10 pages. https://doi.org/10.1186/s13660-018-1912-7.
[19] O. Diekmann, A. Stephan, V. Gils, M. Sjoerd, V. Lunel, and Hans-Otto Walther, Delay equations, volume 110 of Applied Mathematical Sciences. Springer-Verlag, New York, 1995. Functional, complex, and nonlinear analysis.
[20] H. Djourdem, S. Benaicha and N. Bouteraa, Existence and iteration of monotone positive solution for a fourth-order nonlinear boundary value problem, Fundamental Journal of Mathematics and Applications, 1 (2) (2018), 205-211.
[21] H. Djourdem, S. Benaicha, and N. Bouteraa, Two Positive Solutions for a Fourth-Order Three-Point BVP with Sign-Changing Green's Function, Communications in Advanced Mathematical Sciences. Vol. II, No. 1 (2019), 60-68.
[22] H. Djourdem and N. Bouteraa, Mild solution for a stochastic partial differential equation with noise, WSEAS Transactions on Systems, Vol. 19, (2020), 246-256.
[23] H. I. Freedman, J. Wu, Periodic solutions of single-species models with periodic delay, SIAM J. Math. Anal. 23 (1992) 689-701.
[24] A. E. Garcia and J. T. Neugebauer, Solutions of boundary value problems at resonance with periodic and antiperiodic boundary conditions, Involve, Vol.12, no.1, (2019), 170-179.
[25] H. Gabsi, A. Arjouni, A. Djoudi, Existence of positive periodic solutions of a class of delay differential systems, Romanian Journal of Mathematic and Computer Science. 2018. Vol 9, Issue 1 (2019), p. 1-14.
[26] Z. Jin, H. Wang, A note on positive periodic solutions of delayed differential equations, Appl. Math. Lett. 23 (2010), 581-584.
[27] M. A. Krasnosel'skii, Positive solutions of operator equations, Noordhoff, Groningen, 1964.
[28] J. L. Kaplan and J. A. Yorke, On the nonlinear differential delay equation $x'(t) = f(x(t), x(t-1))$J. Differential Equations, 2 23 (2)(1977), 293-314.
[29] J. Mallet-Paret and R. D. Nussbaum, Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation. Ann. Mat. Pura Appl. (4) (1986), 33-128.
[30] L. B. Kong, D. Q. Jiang, Multiple solutions of a nonlinear fourth-order periodic boundary value problem, annal. Polon. Math. LXIV (3) (1998) 265-270.
[31] G. Kurulay, H. Ozbay, Design of first-order controllers for a exible robot arm with time delay, Applied and Computational Mathematics, 16:3 (2017), 48-58.
[32] M. C. Mackey and L. Glass, Oscillation and Chaos in Physiological Control Systems. Science, 197(4300): 287-289, July 1977.
[33] R. Ma, R. Chen and T. Chen, Existence of positive periodic solutions of nonlinear first-order delayed differential equations, J. Math. Anal. Appl. 384 (2011) 527-535.
[34] A. L. Skubachevskii, On the problem of attainment of equilibrium for control-system with delay, Doklady Akademii Nauk, 335:2 (1994), 157-160.
[35] R. Szczelina and P. Zgliczynski, Algorithm for Rigorous Integration of Delay Differential Equations and the Computer-Assisted Proof of Periodic Orbits in the Mackey- Glass Equation. Found. Comput. Math., 2017. DOI: 10.1007/s10208-017-9369-5
[36] V. Lakshmikantham, S. Leela, Existence and monotone method for periodic solutions of first-order differential equations, J. Math. Anal. Appl, 91 (1983) 237-243.
[37] H. Y. Wang, Positive periodic solutions of singular systems with a parameter, J. Differ. Equ. 249, (2010), 2986-3002.
[38] HO. Walther. Topics in delay differential equations. Jahresber. Dtsch. Math.-Ver., 116(2)(2014), 87-114.
[39] E. Zeidler, Applied Functional Analysis. Applied Mathematical Sciences, vol.108. Springer, NewYork(1995).
[40] R. G. Zhang and K. Gopalsamy, Global attractivity and oscillations in a periodic delay-logistic equation, J. Math. Anal. Appl. 150(1) (1990), 274-283.
[41] J. Zhi-Long, Global attractivity of a positive periodic solutions of delay differential equations with feedback control, Discret Dynamics in Nature and Society, Vol. 2007, 1-11.