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Meir-Keeler Condensing Operator to Prove Existence of Solution for Infinite Systems of Differential Equations in the Banach Space and Numerical Method to Find the Solution

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Abstract. In this paper, we establish the existence of solution for infinite systems of differential equations in the Banach sequence space $n\left(\phi\right)$, $\ell_p(1 \leq p < \infty)$ and c by using Meier-Keeler condensing operators. With the help of examples we illustrate our results in the sequence spaces. Also for validity of the results, we find an approximation of solution by using a suitable method with high accuracy.

1. Introduction, Definitions and Preliminaries

The theory of infinite system of ordinary differential equations is an important branch of the theory of differential equations in Banach spaces. Infinite system of integral equations and ordinary differential equations describes many real life problems which can found in the theory of neural nets, the theory of branching processes and mechanics etc. (see [6, 8, 10, 17, 18]).

In functional analysis in the study of infinite dimensional normed spaces and the measure of noncompactness plays a very important role which was introduced by Kuratowski [11]. There are various types of measure of noncompactness in metric and topological spaces. The idea of measure of noncompactness has been used by many authors in obtaining the existence of solutions of infinite system of integral equations and differential equations.

Suppose E is a real Banach space with the norm $\|\ .\ \|$. Let $B(x_0, r)$ be a closed ball in E centered at x_0 and with radius r. If X is a nonempty subset of E then by $\bar{X}$ and $\mathrm{Conv}(X)$ we denote the closure and convex closure of X. Moreover, let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

We consider the definition of the concept of a measure of noncompactness defined by Banaś and Lecko [5].

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Definition 1.1. *A function $\mu : \mathcal{M}_E \to [0, \infty)$ will be called a measure of noncompactness if it satisfies the following conditions:*

(i) *the family ker $\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and ker $\mu \subset \mathcal{N}_E$.*
(ii) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(X)$.
(iv) $\mu(ConvX) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(vi) *if* $X_n \in \mathcal{M}_E,\ X_n = \bar{X}_n,\ X_{n+1} \subset X_n$ *for* $n = 1, 2, 3, ...$ *and* $\lim_{n\to\infty} \mu(X_n) = 0$ *then* $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family *ker*μ is said to be the *kernel of measure* μ.
A measure μ is said to be the sublinear if it satisfies the following conditions:

(1) $\mu(\lambda X) = |\lambda|\,\mu(X)$ for $\lambda \in \mathbb{R}$.
(2) $\mu(X+Y) \leq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition:

$$\mu(X \cup Y) = \max\{\mu(\lambda X), \mu(\lambda Y)\}$$

and such that $ker\mu = \mathcal{N}_E$ is said to be regular.
For a bounded subset S of a metric space X, the Kuratowski measure of noncompactness is defined as

$$\alpha(S) = \inf\left\{\delta > 0 : S = \bigcup_{i=1}^{n} S_i,\ diam(S_i) \leq \delta \text{ for } 1 \leq i \leq n \leq \infty\right\},$$

where $diam(S_i)$ denotes the diameter of the set S_i, that is

$$diam(S_i) = \sup\{d(x,y) : x, y \in S_i\}.$$

The Hausdorff measure of noncompactness for a bounded set S is defines as

$$\chi(S) = \inf\{\epsilon > 0 : S \text{ has finite } \epsilon - \text{net in } X\}.$$

Definition 1.2. *[3] Let E_1 and E_2 be two Banach spaces and let μ_1 and μ_2 be arbitrary measure of noncompactness on E_1 and E_2, respectively. An operator f from E_1 to E_2 is called a (μ_1, μ_2)−condensing operator if it is continuous and $\mu_2(f(D)) < \mu_1(D)$ for every set $D \subset E_1$ with compact closure.*

Remark 1.3. *If $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, then f is called a $\mu-$ condensing operator.*

Theorem 1.4. *[7] Let Ω be a nonempty, closed, bounded and convex subset of a Banach space E and let $f : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property $\mu_2(f(\Omega)) < k\mu_1(\Omega)$. Then f has a fixed point in Ω.*

Definition 1.5. *[14] Let (X, d) be a metric space. Then a mapping T on X is said to be a Meir-Keeler contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq d(x,y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon, \forall x, y \in X.$$

Theorem 1.6. *[14] Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

Some application of Meir-Keeler contraction can be seen in[19]

Definition 1.7. *[1] Let C be a nonempty subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. We say that an operator $T : C \to C$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq \mu(X) < \epsilon + \delta \implies \mu(T(X)) < \epsilon$$

for any bounded subset X of C.

Theorem 1.8. *[1] Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

In our discussion, we study an infinite system of second order differential equation by transforming the system into an infinite system of integral equation with the help of Green's function (see [9]).

Banaś and Mursaleen [4] defined the Hausdorff measure of noncompactness χ (in Theorem 5.18) for the Banach space $\left(\ell_p, \|\cdot\|_{\ell_p}\right), (1 \leq p < \infty)$ as follows.

$$\chi(B) = \lim_{n\to\infty}\left[\sup_{u\in B}\left(\sum_{k=n}^{\infty} | u_k |^p\right)^{1/p}\right]$$

where $u(t) = (u_i(t))_{i=1}^{\infty} \in \ell_p$ for each $t \in [0, T]$ and $B \in \mathcal{M}_{\ell_p}$.

In the Banach space $(c, \|\cdot\|_c)$, the most convenient measure of noncompactness μ can be formulated as follows (see [5]).

$$\mu(B) = \lim_{p\to\infty}\left[\sup_{u\in B}\{\sup\{| x_n - x_m |: n, m \geq p\}\}\right]$$

where $u(t) = (u_i(t))_{i=1}^{\infty} \in c$ for each $t \in [0, T]$ and $B \in \mathcal{M}_c$. The measure μ is regular.

Let C denote the space whose elements are finite sets of distinct positive integers. Given any element σ of C, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further

$$C_s = \left\{\sigma \in C : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\},$$

that is, C_s is the set of those σ whose support has cardinality at most s, and define

$$\Phi = \left\{\phi = \left(\phi_k\right) \in \omega : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1}, (n+1)\phi_n \geq n\phi_{n+1}\right\}.$$

For $\phi \in \Phi$, the following sequence spaces were introduced by Sargent [20] and further studied in ([12, 13]).

$$n(\phi) = \left\{x = (x_k) \in \omega : \| x \|_{n(\phi)} = \sup_{u\in S(x)}\left(\sum_{k=1}^{\infty} | u_k | \Delta\phi_k\right) < \infty\right\},$$

where $S(x)$ denotes the set of all sequences that are rearrangements of x.

In the Banach space $\left(n(\phi), \|\cdot\|_{n(\phi)}\right)$, the Hausdorff measure of noncompactness χ can be formulated as follows (see [16]).

$$\chi(B) = \lim_{k\to\infty}\sup_{u\in B}\left(\sup_{v\in S(u)}\left(\sum_{n=k}^{\infty} | v_n | \Delta\phi_n\right)\right)$$

where $u(t) = (u_i(t))_{i=1}^{\infty} \in n(\phi)$ for each $t \in [0, T]$ and $B \in \mathcal{M}_{n(\phi)}$.

Consider the infinite system of second order differential equations

$$u_i^{''}(t) = -f_i\,(t, u_1, u_2, u_3, ...) \tag{1}$$

where $u_i(0) = u_i(T) = 0,\ t \in [0, T]$ and $i = 1, 2, 3, ...$

Let $C(I, \mathbb{R})$ denote the space of all continuous real functions on the interval $I = [0, T]$ and let $C^2(I, \mathbb{R})$ be the class of functions with second continuous derivatives on I. A function $u \in C^2(I, \mathbb{R})$ is a solution of (1) if and only if $u \in C(I, \mathbb{R})$ is a solution of the infinite system of integral equation

$$u_i(t) = \int_0^T G(t, s) f_i(s, u(s))ds, \tag{2}$$

where $f_i(t, u) \in C(I, \mathbb{R}),\ i = 1, 2, 3, ...$ and $t \in I$ and the Green's function associated to (1) is given by (see [9, 15])

$$G(t, s) = \begin{cases} \frac{t}{T}\,(T - s)\,, & 0 \leq t \leq s \leq T, \\ \frac{s}{T}\,(T - t)\,, & 0 \leq s \leq t \leq T. \end{cases} \tag{3}$$

The solution of the infinite system (1) in the sequence space ℓ_1 has been studied by Aghajani and Pourhadi [2] and in the sequence spaces c_0 and ℓ_1 has been studied by Mursaleen and Rizvi [15]. In our study, we establish the existence of solution of the infinite system (1) for the sequence spaces $\ell_p\ (1 \leq p < \infty)\,, n(\phi)$ and c.

2. Solvability of infinite system of second order differential equations in ℓ_p $(1 \leq p < \infty)$

Assume that

(i) The functions f_i are defined on the set $I \times \mathbb{R}^\infty$ and take real values. The operator f defined on the space $I \times \ell_p$ into ℓ_p as

$$(t, u) \to (fu)\,(t) = (f_1(t, u), f_2(t, u), f_3(t, u), ...)$$

is such that the class of all functions $((fu)\,(t))_{t\in I}$ is equicontinuous at every point of the space ℓ_p.
(ii) The following inequality holds:

$$\mid f_n\,(t, u_1, u_2, u_3, ...) \mid^p \leq g_n(t) + h_n(t) \mid u_n(t) \mid^p$$

where $g_n(t)$ and $h_n(t)$ are real functions defined on I, such that $\sum\limits_{k\geq1} g_k(t)$ converges uniformly on I and the sequence $(h_n(t))$ is equibounded on I.

Let us introduce

$$G = \sup_{t\in I} \left\{ \sum_{k\geq1} g_k(t) \right\}$$

and

$$H = \sup_{n\in\mathbb{N}, t\in I} \{h_n(t)\}\,.$$

Theorem 2.1. *Under the hypothesis (i)-(ii), infinite system (1) has at least one solution* $u(t) = (u_i(t)) \in \ell_p$ *for all* $t \in [0, T]\,.$

Proof. By using (2) and (ii), we have for all $t \in I$,

$$\begin{aligned}
\| u(t) \|_{\ell_p}^p &= \sum_{i=1}^{\infty} \left| \int_0^T G(t,s) f_i(s,u(s)) ds \right|^p \\
&\leq \sum_{i=1}^{\infty} \int_0^T |G(t,s)|^p \left\{ g_i(s) + h_i(s) |u_i|^p \right\} ds \\
&= \int_0^T |G(t,s)|^p \left(\sum_{i=1}^{\infty} g_i(s) \right) ds + \int_0^T |G(t,s)|^p \left(\sum_{i=1}^{\infty} h_i(s) |u_i|^p \right) ds
\end{aligned}$$

Since $u(t) \in \ell_p$ therefore $\sum\limits_{i=1}^{\infty} |u_i(t)|^p < M < \infty$(say) and $\int_0^T |G(t,s)|^p \, ds \leq \frac{T^{p+1}}{(p+1)4^p}$.

Hence we get

$$\| u(t) \|_{\ell_p}^p \leq (G + HM) \frac{T^{p+1}}{(p+1)4^p} = r^p$$

i.e. $\| u(t) \|_{\ell_p} \leq r$.

Let $u^0(t) = \left(u_i^0(t) \right)$ where $u_i^0(t) = 0 \ \forall \ t \in I$.

Consider $\bar{B} = \bar{B}\left(u^0, r_1\right)$, the closed ball centered at u_0 and radius $r_1 \leq r$, thus $\bar{B}$ is an non-empty, bounded, closed and convex subset of ℓ_p. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \bar{B})$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{ \int_0^T G(t,s) f_i(s,u(s)) ds \right\},$$

where $u(t) = (u_i(t)) \in \bar{B}$ and $u_i(t) \in C(I, \mathbb{R})$.

We have that $(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} \in \ell_p$ for each $t \in I$. Since $(f_i(t, u(t))) \in \ell_p$ for each $t \in I$, we have,

$$\sum_{i=1}^{\infty} |(\mathcal{F}_i u)(t)|^p = \sum_{i=1}^{\infty} \left| \int_0^T G(t,s) f_i(s,u(s)) ds \right|^p \leq r^p < \infty.$$

Also $(\mathcal{F}_i u)(t)$ satisfies boundary conditions i.e.

$$(\mathcal{F}_i u)(0) = \int_0^T G(0,s) f_i(s,u(s)) ds = \int_0^T 0.f_i(s,u(s)) ds = 0$$

and

$$(\mathcal{F}_i u)(T) = \int_0^T G(T,s) f_i(s,u(s)) ds = \int_0^T 0.f_i(s,u(s)) ds = 0$$

Since $\| (\mathcal{F}u)(t) - u^0(t) \|_{\ell_p} \leq r$ thus $\mathbb{F}$ is self mapping on $\bar{B}$.

The operator $\mathcal{F}$ is continuous on $C(I, \bar{B})$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir-Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \chi(\bar{B}) < \epsilon + \delta \implies \chi(\mathcal{F}\bar{B}) < \epsilon$.

We have

$$
\begin{aligned}
\chi\left(\mathcal{F}\bar{B}\right) &= \lim_{n\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sum_{k\geq n}\left|\int_0^T G(t,s)f_k(s,u(s))ds\right|^p\right\}^{\frac{1}{p}}\right] \\
&\leq \lim_{n\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sum_{k\geq n}\int_0^T |G(t,s)|^p\left(g_k(s)+h_{k(s)}\left|u_k(s)\right|^p\right)ds\right\}^{\frac{1}{p}}\right] \\
&\leq \lim_{n\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\int_0^T |G(t,s)|^p\left(\sum_{k\geq n} g_k(s)+H\sum_{k\geq n}|u_k(s)|^p\right)ds\right\}^{\frac{1}{p}}\right] \\
&\leq H^{1/p}\left(\frac{T^{1+1/p}}{4(p+1)^{1/p}}\right)\chi\left(\bar{B}\right).
\end{aligned}
$$

Hence $\chi\left(\mathcal{F}\bar{B}\right)\leq H^{1/p}\left(\frac{T^{1+1/p}}{4(p+1)^{1/p}}\right)\chi\left(\bar{B}\right)<\epsilon \implies \chi\left(\bar{B}\right)<\frac{4\epsilon(p+1)^{1/p}}{H^{1/p}.T^{1+1/p}}$.

Taking $\delta=\frac{\left(4(p+1)^{1/p}-H^{1/p}.T^{1+1/p}\right)}{H^{1/p}.T^{1+1/p}}\epsilon$, we get $\epsilon\leq\chi\left(\bar{B}\right)<\epsilon+\delta$. Therefore, $\mathcal{F}$ is a Meir-Keeler condensing operator defined on the set $\bar{B}\subset\ell_p$. So $\mathcal{F}$ satisfies all the conditions of Theorem1.8 which implies $\mathcal{F}$ has a fixed point in $\bar{B}$. This is a required solution of the system (1). □

3. Solvability of infinite system of second order differential equations in $n(\phi)$

We assume that

(i) The functions f_i are defined on the set $I\times\mathbb{R}^\infty$ and take real values. The operator f defined on the space $I\times n\left(\phi\right)$ into $n\left(\phi\right)$ as

$$(t,u)\to(fu)(t)=(f_1(t,u),f_2(t,u),f_3(t,u),...)$$

is such that the class of all functions $((fu)(t))_{t\in I}$ is equicontinuous at every point of the space $n\left(\phi\right)$.

(ii) The following inequality holds:

$$\mid f_n(t,u_1,u_2,u_3,...)\mid\leq g_n(t)+h_n(t)\mid u_n(t)\mid$$

where $g_n(t)$ and $h_n(t)$ are real functions defined and continuous on I, such that $\sum\limits_{k\geq1} g_k(t)\Delta\phi_k$ converges uniformly on I and the sequence $(h_n(t))$ is equibounded on I.

Let us consider

$$
\begin{aligned}
G &= \sup_{t\in I}\left\{\sum_{k\geq1} g_k(t)\Delta\phi_k\right\} \\
H &= \sup_{n\in\mathbb{N},t\in I}\left\{h_n(t)\right\}.
\end{aligned}
$$

Theorem 3.1. *Under the hypothesis (i)-(ii), infinite system (1) has at least one solution $u(t)=(u_i(t))\in n\left(\phi\right)$ for all $t\in[0,T]$.*

Proof. Let $S(u(t))$ denotes the set of all sequences that are rearrangements of $u(t)$. If $v(t)\in S(u(t))$ then $\sum\limits_{i=1}^{\infty}\mid v_i(t)\mid\Delta\phi_i\leq M<\infty$ where M is a finite positive real number for all $u(t)=(u_i(t))\in n\left(\phi\right)$ and $t\in I$.

By using (2) and (ii), we have for all $t \in I$,

$$\begin{aligned}
\| u(t) \|_{n(\phi)} &= \sup_{v \in S(u(t))} \left[\sum_{i=1}^{\infty} \left| \int_0^T G(t,s) f_i(s,u(s)) ds \right| \Delta\phi_i \right] \\
&\leq \sup_{v \in S(u(t))} \left[\sum_{i=1}^{\infty} \int_0^T \left| G(t,s) f_i(s,u(s)) \right| ds \Delta\phi_i \right] \\
&\leq \sup_{v \in S(u(t))} \left[\sum_{i=1}^{\infty} \int_0^T |G(t,s)| \{ g_i(s) + h_i(s) \mid v_i(s) \mid \} ds \Delta\phi_i \right] \\
&= \sup_{v \in S(u(t))} \left[\sum_{i=1}^{\infty} \int_0^T G(t,s) g_i(s) \Delta\phi_i ds + \sum_{i=1}^{\infty} \int_0^T G(t,s) h_i(s) \mid v_i(s) \mid \Delta\phi_i ds \right] \\
&\leq \sup_{v \in S(u(t))} \left[\int_0^T G(t,s) \left\{ \sum_{i=1}^{\infty} g_i(s) \Delta\phi_i \right\} ds + H \int_0^T G(t,s) \left\{ \sum_{i=1}^{\infty} \mid v_i(s) \mid \Delta\phi_i \right\} ds \right] \\
&\leq G \sup_{v \in S(u(t))} \int_0^T G(t,s) ds + H \sup_{v \in S(u(t))} \int_0^T G(t,s) M ds \leq \frac{GT^2}{8} + \frac{HMT^2}{8} = r \text{ (say)} \\
\text{i.e.} \quad & \| u(t) \|_{n(\phi)} \leq r.
\end{aligned}$$

Let $u^0(t) = \left(u_i^0(t)\right)$ where $u_i^0(t) = 0 \; \forall \; t \in I$.

Consider $\bar{B} = \bar{B}\left(u^0, r_1\right)$, the closed ball centered at u_0 and radius $r_1 \leq r$, thus $\bar{B}$ is an non-empty, bounded, closed and convex subset of $n\left(\phi\right)$. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \bar{B})$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{ \int_0^T G(t,s) f_i(s,u(s)) ds \right\}$$

where $u(t) = (u_i(t)) \in \bar{B}$ and $u_i(t) \in C(I, \mathbb{R})$.

We have that $(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} \in n\left(\phi\right)$ for each $t \in I$. Since $(f_i(t, u(t))) \in n\left(\phi\right)$ for each $t \in I$, we have,

$$\sup_{v \in S(u(t))} \left[\sum_{i=1}^{\infty} |(\mathcal{F}_i u)(t)| \Delta\phi_i \right] \leq r < \infty.$$

Also $(\mathcal{F}_i u)(t)$ satisfies boundary conditions i.e.

$$\begin{aligned}
(\mathcal{F}_i u)(0) &= \textstyle\int_0^T G(0,s) f_i(s,u(s)) ds = \int_0^T 0.f_i(s,u(s)) ds = 0, \\
(\mathcal{F}_i u)(T) &= \textstyle\int_0^T G(T,s) f_i(s,u(s)) ds = \int_0^T 0.f_i(s,u(s)) ds = 0
\end{aligned}$$

Since $\| (\mathcal{F}u)(t) - u^0(t) \|_{n(\phi)} \leq r$ thus $\mathcal{F}$ is self mapping on $\bar{B}$.

The operator $\mathcal{F}$ is continuous on $C(I, \bar{B})$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir-Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \chi(\bar{B}) < \epsilon + \delta \implies \chi(\mathcal{F}\bar{B}) < \epsilon$.

We have

$$\begin{aligned}
\chi(\mathcal{F}\bar{B}) &= \lim_{k\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{v\in S(u(t))}\left(\sum_{n\geq k}\left|\int_0^T G(t,s)f_n(s,v(s))ds\right|\Delta\phi_n\right)\right\}\right]\\
&\leq \lim_{k\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{v\in S(u(t))}\left(\sum_{n\geq k}\int_0^T \left|G(t,s)f_n(s,v(s))\right|ds\Delta\phi_n\right)\right\}\right]\\
&\leq \lim_{k\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{v\in S(u(t))}\left(\sum_{n\geq k}\int_0^T G(t,s)g_n(s)\Delta\phi_n ds+\sum_{n\geq k}\int_0^T G(t,s)h_n(s)\mid v_n(s)\mid\Delta\phi_n ds\right)\right\}\right]\\
&\leq \lim_{k\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{v\in S(u(t))}\left(\int_0^T G(t,s)\left(\sum_{n\geq k}g_n(s)\Delta\phi_n\right)ds+H\int_0^T G(t,s)\left(\sum_{n\geq k}\mid v_n(s)\mid\Delta\phi_n\right)ds\right)\right\}\right]\\
&\leq H\chi(\bar{B})\int_0^T G(t,s)ds\leq\frac{HT^2}{8}\chi(\bar{B})
\end{aligned}$$

Hence $\chi(\mathcal{F}\bar{B})\leq\frac{HT^2}{8}\chi(\bar{B})<\epsilon\implies\chi(\bar{B})<\frac{8\epsilon}{HT^2}$.

Taking $\delta=\frac{\epsilon}{HT^2}\left(8-HT^2\right)$ we get $\epsilon\leq\chi(\bar{B})<\epsilon+\delta$. Therefore, $\mathcal{F}$ is a Meir-Keeler condensing operator defined on the set $\bar{B}\subset n\left(\phi\right)$. So $\mathcal{F}$ satisfies all the conditions of Theorem1.8 which implies $\mathcal{F}$ has a fixed point in $\bar{B}$. This is a required solution of the system (1). □

4. Solvability of infinite system of second order differential equations in *c*

Suppose that

(i) The functions f_i are defined on the set $I\times\mathbb{R}^\infty$ and take real values. The operator f defined on the space $I\times c$ into c as

$$(t,u)\to(fu)(t)=(f_1(t,u),f_2(t,u),f_3(t,u),...)$$

is such that the class of all functions $((fu)(t))_{t\in I}$is equicontinuous at every point of the space c.

(ii) The following inequality holds:

$$\mid f_n(t,u_1,u_2,u_3,...)\mid\leq p_n(t)+q_n(t)\sup_{i,j\geq n}\mid u_i(t)-u_j(t)\mid$$

where $p_n(t)$ and $q_n(t)$ are real functions and continuous defined on I, such that the sequence $\{p_n(t)\}$ converges uniformly to a function identically vanishing on I and the sequence $(q_n(t))$ is equibounded on I.

Let us assume

$$Q=\sup\nolimits_{t\in I,n\in\mathbb{N}}\{p_n(t)\}$$
$$P=\sup\nolimits_{t\in I,n\in\mathbb{N}}\{q_n(t)\}$$

Theorem 4.1. *Under the hypothesis (i)-(ii), infinite system (1) has at least one solution $u(t)=(u_i(t))\in c$ for all $t\in[0,T]$.*

Proof. We have $\sup_{i\in\mathbb{N}}\mid u_i(t)\mid\leq M<\infty$, where M is a finite positive real number for all $u(t)=(u_i(t))\in c$ and $t\in I$.

By using (2) and (ii), we have for all $t \in I$,

$$\begin{aligned}
\| u(t) \|_c &= \max_{k\geq 1}\left|\int_0^T G(t,s) f_k(s,u(s))ds\right| \\
&\leq \max_{k\geq 1}\int_0^T \left|G(t,s) f_k(s,u(s))\right| ds \\
&\leq \max_{k\geq 1}\int_0^T G(t,s)\left\{p_k(t) + q_k(t)\sup_{i,j\geq k} | u_i(t) - u_j(t) |\right\} ds \\
&\leq \max_{k\geq 1}\left\{P\int_0^T G(t,s)ds + Q\int_0^T G(t,s)\sup_{i,j\geq k} | u_i(t) - u_j(t) | ds\right\} \\
&\leq \frac{PT^2}{8} + \frac{MQT^2}{4} = r \\
i.e. \quad & \| u(t) \|_c \leq r.
\end{aligned}$$

Let $u^0(t) = \left(u_i^0(t)\right)$ where $u_i^0(t) = 0 \; \forall \; t \in I$.

Consider $\bar{B} = \bar{B}\left(u^0, r_1\right)$, the closed ball centered at u_0 and radius $r_1 \leq r$, thus $\bar{B}$ is an non-empty, bounded, closed and convex subset of c. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \bar{B})$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{\int_0^T G(t,s) f_i(s,u(s))ds\right\}$$

where $u(t) = (u_i(t)) \in \bar{B}$ and $u_i(t) \in C(I, \mathbb{R})$.

Since $(f_i(t, u(t))) \in c$ for each $t \in I$, we have,

$$\lim_{i\to\infty}(\mathcal{F}_i u)(t) = \lim_{i\to\infty}\int_0^T G(t,s) f_i(s,u(s))ds = \int_0^T G(t,s)\lim_{i\to\infty} f_i(s,u(s))ds$$

is finite and unique. Hence $(\mathcal{F}u)(t) \in c$.

Also $(\mathcal{F}_i u)(t)$ satisfies boundary conditions i.e.

$$\begin{aligned}
(\mathcal{F}_i u)(0) &= \int_0^T G(0,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0 \\
(\mathcal{F}_i u)(T) &= \int_0^T G(T,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0
\end{aligned}$$

Since $\| (\mathcal{F}u)(t) - u^0(t) \|_c \leq r$ thus $\mathcal{F}$ is self mapping on $\bar{B}$.

The operator $\mathcal{F}$ is continuous on $C(I, \bar{B})$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir-Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \mu(\bar{B}) < \epsilon + \delta \implies \mu(\mathcal{F}\bar{B}) < \epsilon$.

We have

$$
\begin{aligned}
\mu(\mathcal{F}\bar{B}) &= \lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{m,n\geq p}\left|\int_0^T G(t,s)f_m(s,u(s))ds - \int_0^T G(t,s)f_n(s,u(s))ds\right|\right\}\right] \\
&\leq \lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{m,n\geq p}\int_0^T G(t,s)\left|f_m(s,u(s)) - f_n(s,u(s))\right|ds\right\}\right] \\
&\leq \lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\sup_{m,n\geq p}\int_0^T G(t,s)\left\{\left|f_m(s,u(s))\right| + \left|f_n(s,u(s))\right|\right\}ds\right] \\
&\leq \lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\sup_{m,n\geq p}\int_0^T G(t,s)\left\{\sum_{k=m,n}\left(p_k(t) + q_k(t)\sup_{i,j\geq k}|u_i(t) - u_j(t)|\right)\right\}ds\right] \\
&\leq \frac{T^2Q\mu(\bar{B})}{4}.
\end{aligned}
$$

Hence $\mu(\mathcal{F}\bar{B}) \leq \frac{T^2Q\mu(\bar{B})}{4} < \epsilon \implies \chi(\bar{B}) < \frac{4\epsilon}{T^2Q}$.

Taking $\delta = \frac{\epsilon}{T^2Q}\left(4 - T^2Q\right)$ we get $\epsilon \leq \chi(\bar{B}) < \epsilon + \delta$. Therefore, $\mathcal{F}$ is a Meir-Keeler condensing operator defined on the set $\bar{B} \subset c$. So $\mathcal{F}$ satisfies all the conditions of Theorem1.8 which implies $\mathcal{F}$ has a fixed point in $\bar{B}$. This is a required solution of the system (1). □

5. Applications

In this section we illustrate our results by use of examples.

Example 5.1. *Let us consider the following system of second order differential equations*

$$-\frac{d^2u_n(t)}{dt^2} = f_n(t,u(t)) \tag{4}$$

where $f_n(t,u(t)) = \frac{e^t cos(t)u_n(t)}{n^2}$ $\forall\, n \in \mathbb{N},\ t \in I = [0,T]$.

We have $\sum_{k=1}^{\infty}\left|f_n(t,u(t))\right|^p \leq e^{pT}\sum_{k=1}^{\infty}|u_k(t)|^p < \infty$ *if* $u(t) = (u_i(t)) \in \ell_p$ *where* $1 \leq p < \infty$.

Let us consider a positive arbitrary real number $\epsilon > 0$ *and* $u(t) \in \ell_p$. *Taking* $v(t) \in \ell_p$ *with* $\| u(t) - v(t) \|_{\ell_p} < \delta = \left(\frac{\epsilon}{e^T}\right)^{1/p}$,

$$\left|f_n(t,u(t)) - f_n(t,v(t))\right| = \left|\frac{e^t cos(t)u_n(t)}{n^2} - \frac{e^t cos(t)v_n(t)}{n^2}\right| e^T \leq \| u(t) - v(t) \|_{\ell_p}^p < \epsilon$$

which implies the equicontinuity of $((fu)(t))_{t\in I}$ *on* ℓ_p.

Again, we have for all $n \in \mathbb{N}$ *and* $t \in I$,

$$\left|f_n(t,u(t))\right|^p \leq e^{pt}|u_n(t)|^p = g_n(t) + h_n(t)|u_n(t)|^p$$

where $g_n(t) = 0$ *and* $h_n(t) = e^{pt}$ *are real functions on I and* $\sum_{k\geq 1} g_n(t)$ *converges uniformly on I and the sequence* $\{h_n(t)\}$ *is equibounded on I. Thus, by Theorem 2.1, the system (4) has unique solution in* ℓ_p.

Example 5.2. *Let us consider the following system of second order differential equations*

$$-\frac{d^2u_n(t)}{dt^2} = f_n(t,u(t)) \tag{5}$$

$$u_n(0) = u_n(T) = 0,\ \forall n \in \mathbb{N} \tag{6}$$

where $f_n(t,u(t)) = \frac{t}{n^2} + \sum\limits_{m=n}^{\infty} \frac{u_m(t)-u_n(t)}{m^2}\ \forall\ n \in \mathbb{N},\ t \in I = [0,T]$.

If $u(t) \in c$ *then*

$$\lim_{n\to\infty} f_n(t,u(t)) = \lim_{n\to\infty}\left[\frac{t}{n^2} + \sum_{m=n}^{\infty}\frac{u_m(t)-u_n(t)}{m^2}\right] = 0.$$

Therefore, $f_n(t,u(t)) \in c$.

Let $\epsilon > 0$ *arbitrary and* $u(t) \in c$ *with* $\| u(t) - v(t) \|_c \leq \delta = \frac{3\epsilon}{\pi^2}$ *we get*

$$\begin{aligned} \mid f_n(t,u(t)) - f_n(t,v(t)) \mid &= \left|\sum_{m=n}^{\infty}\frac{u_m(t)-u_n(t)}{m^2} - \sum_{m=n}^{\infty}\frac{v_m(t)-v_n(t)}{m^2}\right| \\ &\leq \sum_{m=n}^{\infty}\frac{1}{m^2}\{|u_m(t)-v_m(t)| + |u_n(t)-v_n(t)|\} \\ &\leq 2\delta\sum_{m=n}^{\infty}\frac{1}{m^2} \\ &< 2\delta.\frac{\pi^2}{6} < \epsilon \text{ for any fixed } n. \end{aligned}$$

Therefore, $f_n(t,u(t))$ *is equicontinuous on* c *for all* n*. Again, we have*

$$\begin{aligned} \left|f_n(t,u(t))\right| &\leq \frac{t}{n^2} + \sum_{m=n}^{\infty}\frac{1}{m^2}|u_m(t)-u_n(t)| \\ &\leq \frac{t}{n^2} + \sup_{i,j\geq n}\left|u_i(t)-u_j(t)\right|\sum_{m=n}^{\infty}\frac{1}{m^2} \\ &\leq \frac{t}{n^2} + \frac{\pi^2}{6}\sup_{i,j\geq n}\left|u_i(t)-u_j(t)\right| \\ &= p_n(t) + q_n(t)\sup_{i,j\geq n}\left|u_i(t)-u_j(t)\right|, \end{aligned}$$

where $p_n(t) = \frac{t}{n^2}$ *and* $q_n(t) = \frac{\pi^2}{6}$ *defined and continuous on* I *and* $(p_n(t))$ *converges uniformly on* I *to the function identically vanishing on* I *and* $q_n(t)$ *is equibounded on* I*. Thus, by Theorem 4.1, the system (5) has unique solution in* c.

Example 5.3. *Let us consider to the following system of second order differential equations*

$$\frac{d^2u_n(t)}{dt^2} = f_n(t,u(t)) \tag{7}$$

with $u_n(0) = u_n'(0) = 0$*, where* $f_n(t,u(t)) = \frac{cos(t)u_n(t)\Delta\phi_n}{n^2}\ \forall\ n \in \mathbb{N},\ t \in I = [0,T]$.

We have $0 \leq \Delta\phi_k \leq M$ *for all* k *where* M *is a positive real number.*

Therefore,

$$\sum_{k=1}^{\infty}\left|f_k(t,u(t))\right|\Delta\phi_k \leq \sum_{k=1}^{\infty}\frac{|u_k(t)|\,\Delta\phi_k}{k^2}\Delta\phi_k \leq M\sum_{k=1}^{\infty}|u_k(t)|\,\Delta\phi_k < \infty$$

if $u(t) = (u_i(t)) \in n(\phi)$ *i.e.* $(f_n(t,u(t))) \in n(\phi)$.

Let us consider a positive arbitrary real number $\epsilon > 0$ and $v(t) \in n(\phi)$. Taking $v(t) \in n(\phi)$ with $\| u(t) - v(t) \|_{n(\phi)} < \delta = \epsilon$, then we have

$$\begin{aligned}\left|f_n(t,u(t)) - f_n(t,v(t))\right| &= \left|\frac{cos(t)u_n(t)}{n^2} - \frac{cos(t)v_n(t)}{n^2}\right|\Delta\phi_n\\ &\leq \mid u_n(t) - v_n(t) \mid \Delta\phi_n \leq \| u(t) - v(t) \|_{n(\phi)}\\ i.e. \quad &\left|f_n(t,u(t)) - f_n(t,v(t))\right| < \epsilon\end{aligned}$$

which implies the equicontinuity of $((fu)(t))_{t\in I}$ on $n(\phi)$.

Again, we have for all $n \in \mathbb{N}$ and $t \in I$,

$$\left|f_n(t,u(t))\right| \leq |u_n(t)|.\frac{\Delta\phi_n}{n^2} \leq g_n(t) + h_n(t)\,|u_n(t)|$$

where $g_n(t) = 0$ and $h_n(t) = \frac{M}{n^2}$ are real functions on I and $\sum_{k\geq 1} g_k(t)\Delta\phi_k$ converges uniformly on I and the sequence $\{h_n(t)\}$ is equibounded on I. Thus, by Theorem 3.1, the system (7) has unique solution in $n(\phi)$.

6. Approximation of solution to Eq.(5) in polynomials space

For solving the boundary value problem (5), we consider the solution space $C[0,T]$. A suitable choose of the basis functions in $C[0,T]$ space can be polynomials, thus we suppose that $u_n(t) \in$ *linear span* $\{1, t, t^2, ...\}$,

$$u_n(t) = \sum_{j=0}^{\infty} \alpha_j t^j, \quad t \in [0,T]. \tag{8}$$

Substituting the boundary values $u_n(0) = u_n(T) = 0$ in Eq.(5) concludes that,

$$u_n''(0) = 0, u_n''(T) = -\frac{T}{n^2}, \tag{9}$$

by replacing boundary values and (8-9) in Eq.(5), we can write

$$\alpha_0 = 0,\ \alpha_2 = 0,\ \alpha_1 T + \alpha_3 T^3 + \sum_{j=4}^{\infty} \alpha_j T^j = 0, \tag{10}$$

$$3!\alpha_3 T + \sum_{j=4}^{\infty} \alpha_j j(j-1)T^{j-2} = -\frac{T}{n^2}. \tag{11}$$

To compare both sides of (11) in terms of T powers we obtain,

$$\alpha_3 = -\frac{1}{3!n^2},\ \alpha_j = 0,\ j \geq 4, \tag{12}$$

by (12) and (10), we have

$$\alpha_1 = \frac{T^2}{3!n^2}. \tag{13}$$

Thus, $u_n(t)$ as an approximation of exact solution in $C[0,T]$ space can be given in the form

$$u_n(t) = \frac{1}{3!n^2}(T^2 t - t^3),\ t \in [0,T],\ n \in \mathbb{N}. \tag{14}$$

We compute $u_n(t)$ for $n = 1, 30, 60, 100, ect.$ (computations was down by mathematica10).

- **Case.1** For $T = 1 (0 \le t \le 1)$

$$\begin{aligned} u_1(t) &= 0.166667t - 0.166667t^3, \\ u_{30}(t) &= 0.000185185t - 0.000185185t^3, \\ u_{60}(t) &= 0.0000462963t - 0.0000462963t^3, \\ u_{100}(t) &= 0.0000166667t - 0.0000166667t^3. \end{aligned} \tag{15}$$

For validity every one of the above functions as a solution of differential equations system (5), we substitute $u_i(t),\ (i = 1, 30, 60, 100)$ in Eq.(5) to compute errors in some points by table 1.

Table 1: computing of errors ($T = 1$)

t	$u_1(t)$	$u_{30}(t)$	$u_{60}(t)$	$u_{100}(t)$
0.0	0	0	0	0
0.1	9.2×10^{-3}	4.0×10^{-7}	5.0×10^{-8}	1.0×10^{-8}
0.2	1.8×10^{-2}	7.8×10^{-7}	9.8×10^{-8}	2.1×10^{-8}
0.3	2.5×10^{-2}	1.1×10^{-6}	1.4×10^{-7}	3.0×10^{-8}
0.4	3.1×10^{-2}	1.3×10^{-6}	1.6×10^{-7}	3.7×10^{-8}
0.5	3.5×10^{-2}	1.5×10^{-6}	1.9×10^{-7}	4.1×10^{-8}
0.6	3.6×10^{-2}	1.5×10^{-6}	1.9×10^{-7}	4.2×10^{-8}
0.7	3.3×10^{-2}	1.4×10^{-6}	1.8×10^{-7}	3.9×10^{-8}
0.8	2.7×10^{-2}	1.1×10^{-6}	1.4×10^{-7}	3.1×10^{-8}
0.9	1.6×10^{-2}	7.0×10^{-7}	8.7×10^{-8}	1.8×10^{-8}
1.0	0	0	0	0

- **Case.2** For $T = 2 (0 \le t \le 2)$

$$\begin{aligned} u_2(t) &= 0.166667t - 0.0416667t^3, \\ u_{20}(t) &= 0.00166667t - 0.000416667t^3, \\ u_{100}(t) &= 0.0000666667t - 0.0000166667t^3, \\ u_{200}(t) &= 0.0000166667t - 4.16667 \times 10^{-6}t^3. \end{aligned} \tag{16}$$

Similar to Case 1, for validity of the $u_i(t)$'s in (16), we show errors in some points by table 2. with

Table 2: computing of errors ($T = 2$)

t	$u_2(t)$	$u_{20}(t)$	$u_{100}(t)$	$u_{200}(t)$
0.0	0	0	0	0
0.2	1.0×10^{-2}	1.0×10^{-5}	8.7×10^{-8}	1.0×10^{-8}
0.4	2.0×10^{-2}	2.1×10^{-5}	1.7×10^{-7}	2.1×10^{-8}
0.6	2.8×10^{-2}	3.0×10^{-5}	2.4×10^{-7}	3.0×10^{-8}
0.8	3.5×10^{-2}	3.7×10^{-5}	2.9×10^{-7}	3.7×10^{-8}
1.0	3.9×10^{-2}	4.1×10^{-5}	3.3×10^{-7}	4.1×10^{-8}
1.2	4.0×10^{-2}	4.2×10^{-5}	3.4×10^{-7}	4.2×10^{-8}
1.4	3.7×10^{-2}	3.9×10^{-5}	3.1×10^{-7}	3.9×10^{-8}
1.6	3.0×10^{-2}	3.1×10^{-5}	2.5×10^{-7}	3.1×10^{-8}
1.8	1.7×10^{-2}	1.8×10^{-5}	1.5×10^{-7}	1.8×10^{-8}
2.0	0	0	0	0

consider to table 1 and 2, errors vanish specially for $n \ge 20$, so we have high accuracy. Also by increasing of n such as $n = 1000$ then error is equal to 10^{-14} and this is an illustration of high precision.

Conclusion In this research, we proved the existence of solution for infinite system of differential equations in the Banach spaces by some theorems. we presented some examples to show efficiency of our analytical results. Also we used from suitable basis functions to find solution of the above system with high accuracy.

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