Filomat 36:10 (2022), 3479–3490
https://doi.org/10.2298/FIL2210479B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Equivalence of Multifractal Measures on Moran Sets

Anouar Ben Mabrouk[a,b,c], Bilel Selmi[d]

[a] *Laboratory of Algebra, Number Theory and Nonlinear Analysis, LR18ES15,*
Department of Mathematics, Faculty of Sciences, University of Monastir,
Avenue of the Environment, 5000 Monastir, Tunisia.
[b] *Department of Mathematics, Higher Institute of Applied Mathematics and Computer Sciences,*
University of Kairouan, Street of Assad Ibn Alfourat, 3100 Kairouan, Tunisia.
[c] *Department of Mathematics, Faculty of Sciences, University of Tabuk,*
King Faisal road, 47512 Tabuk, Saudi Arabia.
[d] *Laboratory of Analysis, Probability and Fractals, LR18ES17,*
Department of Mathematics, Faculty of Sciences, University of Monastir,
Avenue of the Environment, 5000 Monastir, Tunisia.

Abstract. In this paper, the equivalence of the multifractal centered Hausdorff measure and the multifractal packing measure is investigated. Furthermore, for the Moran sets satisfying the strong separation condition, the equivalence of the mutual multifractal Hausdorff and packing measures is discussed. A concrete example of fractal sets satisfying the above property is developed.

1. Introduction and statement of the main result

Multifractal (Relative multifractal, Mutual multifractal) analysis of measures is one important research direction in fractal geometry. It has been widely applied in many fields such as dynamical systems, turbulence analysis, rainfall modeling, earthquake analysis, and financial time series modeling. For these reasons multifractal analysis still fascinates researchers and developing the mathematical theory and methods of multifractal measures is of utmost importance.

One of the main problems in the multifractal analysis of measures is to understand the multifractal spectrum and the Rényi dimensions, and their relationship with each other. During the past 30 years, there has been an enormous interest in computing the multifractal spectra of measures. Rigorous computations have been done for various classes of measures in Euclidean space $\mathbb{R}^n$ exhibiting some degree of self-similarity (see for example the papers [28, 34, 38, 43] and the references therein). In an attempt to develop a general theoretical framework for studying the multifractal structure of arbitrary measures, Olsen [34] and Pesin [37] suggested various ways of defining an auxiliary measure in some general settings.

Based on some ideas of multifractal formalism given by Olsen [34], Svetova introduced in [47–51] a new formalism for multifractal analysis of one measure with respect to another, known as mutual multifractal formalism. The set of points with a given local dimension has been investigated with respect to an arbitrary

2020 *Mathematics Subject Classification*. Primary: 28A78, 28A80; Secondary: 28A20, 28A75, 49Q15
Keywords. Mutual multifractal analysis, Density Theorem, Moran sets.
Received: 21 July 2020; Accepted: 25 August 2022
Communicated by Dragan S. Djordjević
Email addresses: anouar.benmabrouk@fsm.rnu.tn (Anouar Ben Mabrouk), bilel.selmi@fsm.rnu.tn, bilel.selmi@isetgb.rnu.tn (Bilel Selmi)

probability measure. More specifically, given two compactly supported Borel probability measures μ and ν on $\mathbb{R}^n$ and $\alpha, \beta \geq 0$, the size of the iso-Hölder set

$$E_{\mu,\nu}(\alpha,\beta) = \Big\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu;\ \ \alpha_\mu(x) = \alpha \quad \text{and} \quad \alpha_\nu(x) = \beta\Big\},$$

has been estimated, where $\alpha_\mu(x) = \lim_{r\to 0} \dfrac{\log \mu(B_x(r))}{\log r}$ and $B_x(r)$ is the closed ball of center x and radius r. Later, in [22, 23, 42], Selmi et al. justified the mutual multifractal formalism under less restrictive hypotheses. Recently, the authors in [32, 46] have developed a mixed case of multifractal analysis and proved a mixed variant of multifractal analysis based on a generalization of the well known large deviation formalism to a mixed case. More details and backgrounds on multifractal analysis as well as the mixed generalizations and their applications may be found in [1, 4–12, 16–21, 26, 30–33, 36, 43–46].

In the remaining part of the present section, we aim to introduce the general tools that will be applied next. We will give in brief the notion of mutual multifractal generalizations of Hausdorff and packing measures already introduced by Svetova in [47–51] and also Menceur and Ben Mabrouk [32, 33].

Let $n \geq 1$ be a fixed integer and denote by $\mathscr{P}(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$. Consider two measures μ and ν be in $\mathscr{P}(\mathbb{R}^n)$. Let $q, t, s \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$. For $\mathbf{q} = (q,t)$ and $\boldsymbol{\mu} = (\mu,\nu)$, we define

$$\overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E) = \sup\left\{\sum_i \mu(B_{x_i}(r_i))^q \nu(B_{x_i}(r_i))^t (2r_i)^s\right\},$$

where the supremum is taken over all the centered δ-packings of E. The mutual packing pre-measure is given by

$$\overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E).$$

In a similar way, we define

$$\overline{\mathscr{H}}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E) = \inf\left\{\sum_i \mu(B_{x_i}(r_i))^q \nu(B_{x_i}(r_i))^t (2r_i)^s\right\},$$

where the infinimum is taken over all centered δ-coverings of E. The mutual Hausdorff pre-measure is defined by

$$\overline{\mathscr{H}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \sup_{\delta>0} \overline{\mathscr{H}}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E).$$

Especially, we use the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

The modified mutual Hausdorff and packing measures $\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ and $\mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ are next defined by

$$\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \sup_{F\subseteq E} \overline{\mathscr{H}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(F) \quad \text{and} \quad \mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E_i).$$

The functions $\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ and $\mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ are metric outer measures and thus measures on the family of Borel subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that

$$\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}} \leq \xi \mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}} \leq \xi \overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}, \tag{1.1}$$

for some constant ξ independent of $\mathbf{q}$, s and $\boldsymbol{\mu}$.

It holds as for the case of the multifractal analysis of a single measure that each of the measures $\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ and $\mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ and the pre-measure $\overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ assign a multifractal dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by $\dim^{\mathbf{q}}_{\boldsymbol{\mu}}(E)$, $\operatorname{Dim}^{\mathbf{q}}_{\boldsymbol{\mu}}(E)$ and $\Delta^{\mathbf{q}}_{\boldsymbol{\mu}}(E)$ (see [48, 51]),

$$\begin{aligned}
\dim^{\mathbf{q}}_{\boldsymbol{\mu}}(E) &= \inf\Big\{s \in \mathbb{R};\quad \mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = 0\Big\},\\
\operatorname{Dim}^{\mathbf{q}}_{\boldsymbol{\mu}}(E) &= \inf\Big\{s \in \mathbb{R};\quad \mathscr{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = 0\Big\},\\
\Delta^{\mathbf{q}}_{\boldsymbol{\mu}}(E) &= \inf\Big\{s \in \mathbb{R};\quad \overline{\mathscr{P}}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = 0\Big\}.
\end{aligned}$$

It follows from (1.1) that

$$\dim_\mu^{\mathbf{q}}(E) \le \mathrm{Dim}_\mu^{\mathbf{q}}(E) \le \Delta_\mu^{\mathbf{q}}(E).$$

We now introduce the mutual multifractal extensions of dimension of measures. Let θ be a probability measure on $\mathbb{R}^n$, we define

$$\dim_\mu^{\mathbf{q}}(\theta) = \inf_E \Big\{ \dim_\mu^{\mathbf{q}}(E);\ \theta(E) = 1 \Big\}$$

and

$$\mathrm{Dim}_\mu^{\mathbf{q}}(\theta) = \inf_E \Big\{ \mathrm{Dim}_\mu^{\mathbf{q}}(E);\ \theta(E) = 1 \Big\}.$$

Definition 1.1. *We say that two Borel measures μ and ν are equivalent and we write $\mu \sim \nu$ if for any Borel set A, we have $\mu(A) = 0$ if and only if $\nu(A) = 0$.*

In this paper, we focus on the mutual multifractal Hausdorff measure and the mutual multifractal packing measure. We precisely prove the existence of equivalence cases of these measures. Moreover, we show that for suitable measures μ, ν and θ, the multifractal dimensions introduced above are equivalent for some general Moran type sets are different. Our main result is the following.

Theorem 1.2. *Let $0 < s < s' < 2$ and $\boldsymbol{q} = (q,t) < \boldsymbol{1} = (1,1)$. There exist a set $E \subset \mathbb{R}^2$, two Borel probability measures μ, ν on $\mathbb{R}^2$ and θ a finite Borel measure on E such that,*

$$\dim_\mu^{\boldsymbol{q}}(E) = s = \dim_\mu^{\boldsymbol{q}}(\theta) \quad \textit{and} \quad \mathrm{Dim}_\mu^{\boldsymbol{q}}(E) = s' = \mathrm{Dim}_\mu^{\boldsymbol{q}}(\theta)$$

and, in addition,

$$\mathscr{H}_{\mu}^{\boldsymbol{q},s}\llcorner_E \sim \mathscr{P}_{\mu}^{\boldsymbol{q},s'}\llcorner_E.$$

Remark 1.3. *Let $\boldsymbol{q} \ge \boldsymbol{1}$. Then, for all $s > 0$, we have*

$$\mathscr{H}_{\mu}^{\boldsymbol{q},s} = \mathscr{P}_{\mu}^{\boldsymbol{q},s} = \overline{\mathscr{P}}_{\mu}^{\boldsymbol{q},s} = 0.$$

In particular, this implies that

$$\dim_\mu^{\boldsymbol{q}} = \mathrm{Dim}_\mu^{\boldsymbol{q}} = \Delta_\mu^{\boldsymbol{q}} = 0.$$

Indeed, for each centred δ-packing $\big(B_{x_i}(r_i)\big)_i$ of $\mathbb{R}^n$, we have

$$\begin{aligned}
\sum_i \mu(B_{x_i}(r_i))^q \nu(B_{x_i}(r_i))^t (2r_i)^s &\le (2\delta)^s \sum_i \mu(B_{x_i}(r_i))^q \nu(B_{x_i}(r_i))^t \\
&\le (2\delta)^s \sum_i \mu(B_{x_i}(r_i))^q \sum_i \nu(B_{x_i}(r_i))^t \\
&\le (2\delta)^s \left(\sum_i \mu(B_{x_i}(r_i))\right)^q \left(\sum_i \nu(B_{x_i}(r_i))\right)^t \\
&= (2\delta)^s \mu\left(\bigcup_i B_{x_i}(r_i)\right)^q \nu\left(\bigcup_i B_{x_i}(r_i)\right)^t \\
&\le (2\delta)^s.
\end{aligned}$$

This implies that $\overline{\mathscr{P}}_{\mu,\delta}^{\mathbf{q},s}(\mathbb{R}^n) \le (2\delta)^s$. Letting $\delta \to 0$ gives $\overline{\mathscr{P}}_{\mu}^{\mathbf{q},s}(\mathbb{R}^n) = 0$. Finally we get

$$0 \le \mathscr{H}_{\mu}^{\mathbf{q},s}(\mathbb{R}^n) \le \mathscr{P}_{\mu}^{\mathbf{q},s}(\mathbb{R}^n) \le \overline{\mathscr{P}}_{\mu}^{\mathbf{q},s}(\mathbb{R}^n) = 0.$$

2. Proof of the main result

The proof of the main result is based on some generalized multifractal density concepts in the mixed case. General and similar density results have been investigated also in [2, 3, 10–12, 24, 39–41]. To prove Theorem 1.2, we need some preliminary results.

2.1. *Density result*

Consider a finite Borel measure θ on $\mathbb{R}^n$ and $s \in \mathbb{R}$. For $x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu$, we introduce the upper and lower $(\mathbf{q}, s)$-densities of θ with respect to $\boldsymbol{\mu}$ as

$$\overline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta) = \limsup_{r\to 0} \frac{\theta\Big(B_x(r)\Big)}{\mu\Big(B_x(r)\Big)^q \nu\Big(B_x(r)\Big)^t (2r)^s}$$

and

$$\underline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta) = \liminf_{r\to 0} \frac{\theta\Big(B_x(r)\Big)}{\mu\Big(B_x(r)\Big)^q \nu\Big(B_x(r)\Big)^t (2r)^s}.$$

Some analogous cases of the densities $\overline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta)$ and $\underline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta)$ may be found [3, 10, 11, 13, 24, 34, 35].

For $\mu \in \mathscr{P}(\mathbb{R}^n)$ and $a > 1$, we write

$$P_a(\mu) = \limsup_{r\searrow 0} \left(\sup_{x\in\operatorname{supp}\mu} \frac{\mu\Big(B_x(ar)\Big)}{\mu\Big(B_x(r)\Big)} \right).$$

We will now say that the measure μ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter a is unimportant: $P_a(\mu) < \infty$, for some $a > 1$ if and only if $P_a(\mu) < \infty$, for all $a > 1$. Also, we will write $\mathscr{P}_D(\mathbb{R}^n)$ for the set of Borel probability measures on $\mathbb{R}^n$ which satisfy the doubling condition (see [34]).

The following result deals with lower and upper bounds for the mutual multifractal density introduced above by means of the mutual multifractal generalizations of Hausdorff and packing measures. We will see that such bounds permit to obtaining the multifractal formalism already introduced in [34] and reconsidered next in [3, 10, 11, 13, 25, 35].

Proposition 2.1. *Let $\boldsymbol{\mu} = (\mu, \nu) \in \Big(\mathscr{P}_D(\mathbb{R}^n)\Big)^2$, E be a Borel subset of* $\operatorname{supp}\mu \cap \operatorname{supp}\nu$ *and θ be a finite Borel measure on $\mathbb{R}^n$. Then, there exists a constant $C > 0$ for which the following inequalities are true*

$$\mathscr{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(E) \inf_{x\in E} \overline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta) \le \theta(E) \le \mathscr{H}_{\boldsymbol{\mu}}^{\mathbf{q},s}(E) \sup_{x\in E} \overline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta) \tag{2.1}$$

and

$$\mathscr{P}_{\boldsymbol{\mu}}^{\mathbf{q},s}(E) \inf_{x\in E} \underline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta) \le \theta(E) \le C \ \mathscr{P}_{\boldsymbol{\mu}}^{\mathbf{q},s}(E) \sup_{x\in E} \underline{d}_{\boldsymbol{\mu}}^{\mathbf{q},s}(x,\theta). \tag{2.2}$$

Proof. All of the elements needed to prove this lemma can be found in [11, 13, 14, 27, 41].

2.2. *On the equivalence of the mutual multifractal measures*

We will start by defining the Moran sets (for more details, we may see for example [3, 14, 15]). Let $\{n_k\}_k$ and $\{\Phi_k\}_{k\ge 1}$ be respectively two sequences of positive integers, and positive vectors such that

$$\Phi_k = \Big(c_{k_1}, c_{k_2}, \ldots, c_{k_{n_k}}\Big), \quad n_k \ge 2, \quad 0 < c_k < 1, \quad n_k c_k \le 1 \quad \text{for} \quad k \ge 1.$$

For any $m, k \in \mathbb{N}$, such that $m \leq k$, let

$$D_{m,k} = \Big\{(i_m, i_{m+1}, \dots, i_k) : 1 \leq i_j \leq n_j, m \leq j \leq k\Big\}$$

and

$$D_k = D_{1,k} = \Big\{(i_1, i_2, \dots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\Big\}.$$

We also set

$$D_0 = \emptyset \qquad \text{and} \qquad D = \underset{k\geq 0}{\cup} D_k,$$

Considering $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in D_k$, $\tau = (\tau_1, \tau_2, \dots, \tau_m) \in D_{k+1,m}$, we set

$$\sigma * \tau = (\sigma_1, \sigma_2, \dots, \sigma_k, \tau_1, \tau_2, \dots, \tau_m).$$

Definition 2.2. *Let I be a compact subset with non-empty interior in a complete metric space X (For convenience, we assume that the diameter of I is 1). The collection $\mathscr{F} = \{I_\sigma, \sigma \in D\}$ of closed subsets of I is said to have a Moran structure if*

1. *for any $(i_1, i_2, \dots, i_k) \in D_k$, $I_{i_1 i_2 \dots i_k}$ is similar to I. That is, there exists a similarity transformation*

$$S_{i_1 i_2 \dots i_k} : \begin{array}{ccc} X & \to & X \\ I & \mapsto & I_{i_1 i_2 \dots i_k}, \end{array}$$

   with the convention $I_\emptyset = I$.
2. *For all $k \geq 1$, $(i_1, i_2, \dots, i_{k-1}) \in D_{k-1}$, $I_{i_1 i_2 \dots i_k}$ are subsets of $I_{i_1 i_2 \dots i_{k-1}}$ for all $i_k \in \{1, 2, \dots, n_k\}$, and*

$$\overset{\circ}{I}_{i_1 i_2 \dots i_{k-1}, i_k} \cap \overset{\circ}{I}_{i_1 i_2 \dots i_{k-1}, i'_k} = \emptyset, \qquad \forall \ 1 \leq i_k < i'_k \leq n_k,$$

   where $\overset{\circ}{I}$ denotes the interior of I.
3. *For all $k \geq 1$ and $1 \leq j \leq n_k$, taking $(i_1, i_2, \dots, i_{k-1}, j) \in D_k$, we have*

$$0 < c_{kj} = c_{i_1 i_2 \dots i_{k-1} j} = \frac{|I_{i_1 i_2 \dots i_{k-1} j}|}{|I_{i_1 i_2 \dots i_{k-1}}|} < 1, \quad k \geq 2,$$

   where $|I|$ denotes the diameter of I.

Suppose that $\mathscr{F}$ is a collection of subsets of I having Moran structure. We call $E = \bigcap_{k\geq 1} \bigcup_{\sigma \in D_k} I_\sigma$, a Moran set determined by $\mathscr{F}$, and call $\mathscr{F}_k = \Big\{I_\sigma, \sigma \in D_k\Big\}$ the k-order fundamental sets of E. I is called the original set of E. We assume finally that $\lim_{k\to\infty} \max_{\sigma \in D_k} |I_\sigma| = 0$. Then, for all $i \in D$, the set $\Big(\bigcap_{n\geq 1} I_{i_1 i_2 \dots i_n}\Big)$ is a single point. We shall denote it by $\varphi(i)$. We use the abbreviation $w|_k$ for the first k elements of the sequence

$$w = (i_1, i_2, \dots, i_k, \dots) \in D, \qquad I_k(w) = I_{w|_k} = I_{i_1 i_2 \dots i_k}.$$

Now, we consider a class of Moran sets E which satisfy a special property called the strong separation condition (**SSC**), i.e., take any $I_\sigma \in \mathscr{F}$. Let $I_{\sigma*1}, I_{\sigma*2}, \dots, I_{\sigma*n_{k+1}}$ be the $(k+1)$-order fundamental subsets. We say that I_σ satisfies the (**SSC**) if

$$\text{dist}(I_{\sigma*i}, I_{\sigma*j}) \geq \delta_k |I_\sigma|, \qquad \text{for all } i \neq j,$$

where $(\delta_k)_k$ is a sequence of positive real numbers, such that

$$0 < \delta = \inf_k \delta_k < 1.$$

Lemma 2.3. *Let $E \subset I$ be a Moran set satisfying (**SSC**), and θ be a finite Borel measure such that* $\operatorname{supp}\theta \subset E$*. Then, there exist positive constants $A_i, B_i, 1 \le i \le 2$ depending on δ, q, t and s for whihc the following inequalities hold for any $\varphi(i) \in E$,*

$$A_1 \liminf_{n\to+\infty} \frac{\mu(I_n(i))^q \nu(I_n(i))^t |I_n(i)|^s}{\theta(I_n(i))} \le \liminf_{r\to 0} \frac{\mu\Big(B_{\varphi(i)}(r)\Big)^q \nu\Big(B_{\varphi(i)}(r)\Big)^t r^s}{\theta\Big(B_{\varphi(i)}(r)\Big)} \le B_1 \liminf_{n\to+\infty} \frac{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^s}{\theta\Big(I_n(i)\Big)},$$

and

$$A_2 \limsup_{n\to+\infty} \frac{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^s}{\theta\Big(I_n(i)\Big)} \le \limsup_{r\to 0} \frac{\mu\Big(B_{\varphi(i)}(r)\Big)^q \nu\Big(B_{\varphi(i)}(r)\Big)^t r^s}{\theta\Big(B_{\varphi(i)}(r)\Big)} \le B_2 \limsup_{n\to+\infty} \frac{\mu(I_n(i))^q \nu(I_n(i))^t |I_n(i)|^s}{\theta(I_n(i))}.$$

Proof. The proof may be easily inspired from that of [3, Lemma 6], and is therefore omitted.

Proposition 2.4. *Let $E \subset I$ be a Moran set satisfying (**SSC**). Let $\boldsymbol{\mu} = (\mu, \nu)$, where $\mu, \nu \in \mathscr{P}_D(X)$, and θ be a finite Borel measure such that* $\operatorname{supp}\theta \subset E$*.*

1. *Assume that for some $\alpha \in \mathbb{R}$, we have*

$$\liminf_{n\to+\infty} \frac{\theta\Big(I_n(i)\Big)}{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^s} = \begin{cases} 0 & if \quad s < \alpha, \\ +\infty & if \quad s > \alpha, \end{cases} \quad \textit{for all } \; i \in D,$$

then $\operatorname{Dim}_{\boldsymbol{\mu}}^q(E) = \alpha = \operatorname{Dim}_{\boldsymbol{\mu}}^q(\theta)$.

2. *Assume that θ satisfies*

$$0 < \liminf_{n\to+\infty} \frac{\theta\Big(I_n(i)\Big)}{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^\alpha} < +\infty, \quad \textit{for all } \; i \in D, \tag{2.3}$$

then $\theta_{\llcorner E} \sim \mathscr{P}^{q,\alpha}_{\boldsymbol{\mu}}{}_{\llcorner E}$.

Proof. This follows immediately from Lemma 2.3, equation (2.2) and Proposition 2.1.

Proposition 2.5. *Assume that the hypotheses of Proposition 2.4 hold.*

1. *Assume further that there exists $\beta \in \mathbb{R}$ with*

$$\limsup_{n\to+\infty} \frac{\theta\Big(I_n(i)\Big)}{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^s} = \begin{cases} 0 & if \quad s < \beta, \\ +\infty & if \quad s > \beta, \end{cases} \quad \textit{for all } \; i \in D,$$

then $\dim_{\boldsymbol{\mu}}^q(E) = \beta = \dim_{\boldsymbol{\mu}}^q(\theta)$.

2. *Assume that θ satisfies*

$$0 < \limsup_{n\to+\infty} \frac{\theta\Big(I_n(i)\Big)}{\mu\Big(I_n(i)\Big)^q \nu\Big(I_n(i)\Big)^t |I_n(i)|^\beta} < +\infty, \quad \textit{for all } \; i \in D, \tag{2.4}$$

then $\theta_{\llcorner E} \sim \mathscr{H}^{q,\beta}_{\boldsymbol{\mu}}{}_{\llcorner E}$.

Proof. It follows from (2.1) and similar techniques as in the proof of Proposition 2.4.

Corollary 2.6. *Let $E \subset I$ be a Moran set satisfying (**SSC**). Let also $\boldsymbol{\mu} = (\mu, \nu)$, where $\mu, \nu \in \mathscr{P}_D(X)$, θ a finite Borel measure such that* $\operatorname{supp}\theta \subset E$, *and $\beta \leq \alpha$ satisfying (2.3) and (2.4). It holds that*

$$\operatorname{Dim}^q_{\boldsymbol{\mu}}(E) = \alpha = \operatorname{Dim}^q_{\boldsymbol{\mu}}(\theta), \quad \dim^q_{\boldsymbol{\mu}}(E) = \beta = \dim^q_{\boldsymbol{\mu}}(\theta),$$

and

$$\theta_{\llcorner E} \sim \mathscr{H}^{q,\beta}_{\boldsymbol{\mu}}{}_{\llcorner E} \sim \mathscr{P}^{q,\alpha}_{\boldsymbol{\mu}}{}_{\llcorner E}.$$

Proof. It is a direct consequence of Propositions 2.4 and 2.5.

Remark 2.7. *In the special case "$t = 0$ or $q = 0$" and "$q = t = 0$", the previous results are treated by M. Dai in [14, 15]. By using similar techniques of [29] then the above results hold if we replace the Moran set with perturbed Cantor sets.*

2.3. Proof of Theorem 1.2

As $q, t < 1$ and $0 < s < s' < 2$, we can choose $m_1 > 0$, $m_2 > 0$ such that

$$0 < m_1 < \frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} < \frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} < m_2 < 1. \tag{2.5}$$

Let u_1 be such that

$$m_1 < \frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} < u_1 < \frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} < m_2.$$

Since

$$\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, u_1} < 1, \quad \text{and} \quad \frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, m_1} > 1,$$

the equation (2.5) yields the existence of a unique $n_1 \in \mathbb{N}$ for which

$$\begin{aligned}
\frac{\left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}}\right)^{n_1}}{u_1 m_1^{n_1-1}} &= \frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, u_1} \times \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, m_1}\right)^{n_1-1} \\
&< 1 \leq \frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, u_1} \times \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, m_1}\right)^{n_1} \\
&= \frac{\left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}}\right)^{n_1+1}}{u_1 m_1^{n_1}}.
\end{aligned} \tag{2.6}$$

We thus define the sequence $\{u_n\}_{n=2}^{n_1+1}$ by

$$u_n = \begin{cases} m_1, & \text{if} \quad n = 2, \ldots, n_1 \\ \left(\displaystyle\prod_{j=1}^{n_1} u_j\right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}}\right)^{n_1+1}, & \text{if} \quad n = n_1 + 1. \end{cases}$$

It follows from (2.6) that

$$\frac{u_{n_1+1}}{m_1} = \frac{1}{m_1} \times \left(\prod_{j=1}^{n_1} u_j\right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}}\right)^{n_1+1},$$

and

$$4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\, u_{n_1+1} = \frac{\left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}}\right)^{n_1}}{u_1 m_1^{n_1-1}} < 1.$$

Therefore,

$$u_{n_1+1} \geq m_1 \quad \text{and} \quad u_{n_1+1} < \left(4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}\right)^{-1} < m_2.$$

Thus, $m_1 \leq u_n < m_2$, $n = 2, \ldots, n_1 + 1$. Note that the sequence

$$\left\{ \left(\prod_{j=1}^{n} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n}; \quad n = 1, 2, \ldots, n_1 + 1 \right\}$$

is clearly increasing according to (2.5) and (2.6). That is to say

$$\begin{aligned}
\left(\prod_{j=1}^{1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right) &< \left(\prod_{j=1}^{2} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{2} \\
&\vdots \\
&< \left(\prod_{j=1}^{n_1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n_1} \\
&= \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n_1} \left(u_1 m_1^{n_1-1} \right)^{-1} \\
&< 1 = \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n_1+1}.
\end{aligned}$$

Also, since $s < s'$, we deduce that

$$1 = \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n_1+1} < \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} \right)^{n_1+1}. \tag{2.7}$$

Observing now that $\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}} m_2} < 1$, it follows from (2.7) that there exists a unique $n_2 \in \mathbb{N}$ such that

$$m_2^{-n_2} \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} \right)^{n_1+n_2+1} \leq 1 < m_2^{1-n_2} \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\, 2^{\frac{1-t}{s}}} \right)^{n_1+n_2}, \tag{2.8}$$

which gives that

$$\left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}} m_2} \right)^{n_2} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} \right)^{n_1+1} \leq 1 < \left(\prod_{j=1}^{n_1+1} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}} m_2} \right)^{n_2-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} \right)^{n_1+1}.$$

We therefore define the sequence $\{u_n\}_{n=n_1+2}^{n_1+n_2+1}$ by

$$u_n = \begin{cases} m_2, & \text{if} \quad n = n_1 + 2, \ldots, n_1 + n_2 \\[2ex] \left(\prod_{j=1}^{n_1+n_2} u_j \right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s'}}\, 2^{\frac{1-t}{s'}}} \right)^{n_1+n_2+1}, & \text{if} \quad n = n_1 + n_2 + 1. \end{cases} \tag{2.9}$$

Next, note that $m_1 < \dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}$, and that

$$\begin{array}{rcl}
4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}\ u_{n_1+n_2+1} & = & \left(\prod_{j=1}^{n_1+n_2} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2} \\
& = & \left(\prod_{j=1}^{n_1+1} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2} \quad m_2^{1-n_2} \\
& > & 1 \\
& \geq & \left(\prod_{j=1}^{n_1+1} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2+1} \quad m_2\ m_2^{1-n_2} \\
& = & m_2^{-1} u_{n_1+n_2+1}.
\end{array}$$

This implies that

$$m_1 < \frac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}} < u_{n_1+n_2+1} \leq m_2.$$

Consequently, we get

$$m_1 < u_n \leq m_2, \quad \forall\ n = n_1 + 2, \ldots, n_1 + n_2 + 1.$$

It is clear from $\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}} m_2} < 1$, (2.8) and (2.9) that the sequence

$$\left\{\left(\prod_{j=1}^{n} u_j\right)^{-1} \left(\frac{1}{4^{\frac{1-q}{s}}\ 2^{\frac{1-t}{s'}}}\right)^{n}; \quad n = n_1 + 1, \ldots, n_1 + n_2 + 1\right\}$$

is monotone decreasing. This gives that

$$\begin{array}{rcl}
\left(\prod_{j=1}^{n_1+1} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s}}\ 2^{\frac{1-t}{s}}}\right)^{n_1+1} & > & \left(\prod_{j=1}^{n_1+2} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s}}\ 2^{\frac{1-t}{s}}}\right)^{n_1+2} \\
& \vdots & \\
& > & \left(\prod_{j=1}^{n_1+n_2} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s}}\ 2^{\frac{1-t}{s}}}\right)^{n_1+n_2} \\
& = & \left(\prod_{j=1}^{n_1+1} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2} \quad m_2^{1-n_2} \\
& > & 1 \\
& = & \left(\prod_{j=1}^{n_1+n_2} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2+1} \quad u_{n_1+n_2+1}^{-1} \\
& = & \left(\prod_{j=1}^{n_1+n_2+1} u_j\right)^{-1} \left(\dfrac{1}{4^{\frac{1-q}{s'}}\ 2^{\frac{1-t}{s'}}}\right)^{n_1+n_2+1}.
\end{array}$$

Accordingly, we can choose n_3 and $\{u_n\}_{n=n_1+n_2+2}^{n_1+n_2+n_3+1}$ in the same way, and by repeating the same choice, we finally get a sequence $\left(u_n\right)_{n\in\mathbb{N}}$ satisfying

$$\limsup_{n\to+\infty} \frac{\left(\frac{1}{8}\right)^n}{\left(\prod_{j=1}^{n} u_j\right)^s \left(\frac{1}{4}\right)^{nq}\left(\frac{1}{2}\right)^{nt}} = 1 = \liminf_{n\to+\infty} \frac{\left(\frac{1}{8}\right)^n}{\left(\prod_{j=1}^{n} u_j\right)^{s'} \left(\frac{1}{4}\right)^{nq}\left(\frac{1}{2}\right)^{nt}}, \tag{2.10}$$

$$0 = \limsup_{n\to+\infty}(m_1)^n \le \limsup_{n\to+\infty}\left(\prod_{j=1}^{n} u_j\right) \le \limsup_{n\to+\infty}(m_2)^n = 0,$$

and the bounds' inequalities

$$0 < m_1 \le u_n \le m_2 < 1, \ \ \forall\, n \in \mathbb{N}. \tag{2.11}$$

Now, using the sequence $\left(u_n\right)_n$ above, we introduce four families $\left(h_{i,n}\right)_{i=1,2,3,4}$, $n \in \mathbb{N}$ on the unit square S in $\mathbb{R}^2$ as follows,

$$h_{1,n}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_n x_1 \\ u_n x_2 \end{pmatrix}, \ \ h_{2,n}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 - u_n x_1 \\ u_n x_2 \end{pmatrix},$$

$$h_{3,n}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_n x_1 \\ 1 - u_n x_2 \end{pmatrix}, \ \ h_{4,n}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 - u_n x_1 \\ 1 - u_n x_2 \end{pmatrix}.$$

The set E will be defined by

$$E = \bigcap_{n=1}^{+\infty} \bigcup_{(i_1,i_2,\ldots,i_n)\in\{1,2,3,4\}^n} h_{i_1,1} \circ h_{i_2,2} \circ \cdots \circ h_{i_n,n}(S),$$

which consists of uncountable elements. Using (2.11), we deduce that $E \subset D$ is a homogeneous Moran set. Let next, for any $I_k \in \mathscr{F}$, $\delta_k = 1 - 2u_{k+1}$. It follows from (2.11) that

$$\delta = \inf_k \delta_k \ge 1 - 2m_2 > 0.$$

So, there exists a sequence of positive real numbers δ_k that satisfies

$$\operatorname{dist}\left(I_{\sigma * i}, I_{\sigma * j}\right) \ge 1 - 2u_{k+1}|I_\sigma| = \delta_k |I_\sigma|, \quad \text{ for all } i \ne j.$$

This guarantees that the set E satisfies (**SSC**). For $i \in D = \left\{(i_1, i_2, \ldots),\ i_j = 1,2,3,4\right\}$, put

$$I_n(i) = h_{i_1,1} \circ h_{i_2,2} \circ \cdots \circ h_{i_n,n}(S).$$

It is clear that $|I_n(i)| = \prod_{j=1}^{n} u_j$. Let μ, ν be tow Borel probability measures on S and θ be a finite Borel measure on S such that

$$\mu(I_n(i)) = 4^{-n}, \ \ \nu(I_n(i)) = 2^{-n}, \ \text{ and } \ \theta(I_n(i)) = 8^{-n}, \ \ \forall\, n \in \mathbb{N}.$$

Following [14, Proposition 4.1] we prove that $\mu, \nu \in \mathscr{P}_D(X)$. It follows from (2.10) that

$$\limsup_{n\to+\infty} \frac{\theta\left(I_n(i)\right)}{\mu\left(I_n(i)\right)^q \nu\left(I_n(i)\right)^t |I_n(i)|^s} = 1 = \liminf_{n\to+\infty} \frac{\theta\left(I_n(i)\right)}{\mu\left(I_n(i)\right)^q \nu\left(I_n(i)\right)^t |I_n(i)|^{s'}}.$$

Finally, Theorem 1.2 is a direct consequence of Propositions 2.4 and 2.5, and Corollary 2.6. This yields the desired result.

Acknowledgments

We would like to thank the anonymous referees for their valuable comments and suggestions that led to the improvement of the manuscript. This work was supported by Analysis, Probability & Fractals Laboratory LR18ES17, and the Laboratory of Algebra, Number Theory and Nonlinear Analysis LR18ES15.

Bibliography

[1] N. Attia and B. Selmi, *Relative multifractal box dimensions*, Filomat 33 (2019) 2841-2859.
[2] N. Attia and B. Selmi, *Regularities of multifractal Hewitt-Stromberg measures*, Commun. Korean Math. Soc. 34 (2019) 213-230.
[3] N. Attia, B. Selmi and Ch. Souissi, *Some density results of relative multifractal analysis*, Chaos, Solitons and Fractals 103 (2017) 1-11.
[4] L. Barreira, B. Saussol and J. Schmeling, *Higher-dimensional multifractal analysis*, J. Math. Pures Appl. 81 (2002) 67-91.
[5] L. Barreira and P. Doutor, *Birkhoff Averages for Hyperbolic Flows: Variational Principles and Applications*, Journal of Statistical Physics 115 (2004) 1567-1603.
[6] L. Barreira and P. Doutor, *Almost additive multifractal analysis*, J. Math. Pures Appl. 92 (2009) 1-17.
[7] L. Barreira and P. Doutor, *Dimension spectra of almost additive sequences*, Nonlinearity 22 (2009) 2761-2773.
[8] L. Barreira, Y. Cao and J. Wang, *Multifractal Analysis of Asymptotically Additive Sequences*, J. Stat. Phys. 153 (2013) 888-910.
[9] A. Ben Mabrouk and A. Farhat, *A mixed multifractal analysis for quasi-ahlfors vector-valued measures,* Fractals 30 (2022) 2240001.
[10] A. Ben Mabrouk and A. Farhat, *Mixed multifractal densities for quasi-ahlfors vector-valued measures,* Fractals 30 (2022) 2240003.
[11] A. Ben Mabrouk and B. Selmi, *A mixed multifractal analysis of vector-valued measures: Review and extension to densities and regularities of non-necessary Gibbs cases,* Frontiers of Fractal Analysis: Recent Advances and Challenges, Taylor & Francis Group, LLC, CRC Press, (2022).
[12] A. Ben Mabrouk, M. Menceur and B. Selmi, *On the mixed multifractal densities and regularities with respect to gauges,* Filomat (accepted).
[13] J. Cole and L. Olsen, *Multifractal Variation Measures and Multifractal Density Theorems*, Real Analysis Exchange 28 (2003) 501-514.
[14] M. Dai, *On the equivalence of the multifractal centred Hausdorff measure and the multifractal packing measure*, Nonlinearity 21 (2008)443-1453.
[15] M. Dai, *The equivalence of measures on Moran set in general metric space*, Chaos, Solitons and Fractals 29 (2006) 55-64.
[16] M. Dai, *Mixed self-conformal multifractal measures*, Analysis in Theory and Applications 25 (2009) 154-165.
[17] M. Dai and Y. shi, *Typical behavior of mixed L^q-dimensions*, Nonlinear Analysis: Theory, Methods & Applications 72 (2010) 2318-2325.
[18] M. Dai and W. Li, *The mixed L^q-spectra of self-conformal measures satisfying the weak separation condition*, J. Math. Anal. Appl. 382 (2011) 140-147.
[19] M. Dai, C. Wang and H. Sun, *Mixed generalized dimensions of random self-similar measures*, Int. J. Nonlinear. Sci. 13 (2012) 123-128.
[20] M. Dai, J. Houa, J. Gaob, W. Suc, L. Xid and D. Ye, *Mixed multifractal analysis of China and US stock index series*, Chaos, Solitons & Fractals 87 (2016) 286-275.
[21] M. Dai, S. Shao, J. Gao, Y. Sun and W. Su, *Mixed multifractal analysis of crude oil, gold and exchange rate series*, Fractals 24 (2016) 1-7.
[22] Z. Douzi and B. Selmi, *Projections of mutual multifractal functions*, Journal of Classical Analysis 19 (2022) 21-37.
[23] Z. Douzi and B. Selmi, *On the projections of the mutual multifractal Rényi dimensions*, Anal. Theory Appl. 37 (2021) 572-592.
[24] Z. Douzi and B. Selmi, *Regularities of general Hausdorff and packing functions,* Chaos, Solitons & Fractals 123 (2019) 240-243.
[25] Z. Douzi and B. Selmi, *A relative multifractal analysis: box-dimensions, densities, and projections*, Quaestiones Mathematicae http://doi.org/10.2989/16073606.2021.1941375
[26] Z. Douzi, B. Selmi and A. Ben Mabrouk, *The refined multifractal formalism of some homogeneous Moran measures*, The European Physical Journal Special Topics 230 (2021) 3815-3834.
[27] G. A. Edgar, *Centered densities and fractal measures,* New York J. Math. 13 (2007) 33-87.
[28] K. J. Falconer, *Techniques in fractal geometry*, Wiley. New York., (1997).
[29] S. Ikeda and M. Nakamura, *Dimensions of measures on perturbed Cantor sets,* Topology Appl. 122 (2002) 223–236.
[30] M. Khelifi, H. Lotfi, A. Samti and B. Selmi, *A relative multifractal analysis*, Choas, Solitons & Fractals 140 (2020) 110091.
[31] Z. Li and B. Selmi, *On the multifractal analysis of measures in a probability space*, Illinois Journal of Mathematics 65 (2021) 687-718.
[32] M. Menceur, A. Ben Mabrouk and K. Betina, *The Multifractal Formalism For Measures, Review and Extension to Mixed Cases*, Anal. Theory Appl. 33 (2016) 77-106.
[33] M. Menceur and A. Ben Mabrouk, *A joint multifractal analysis of vector valued non Gibbs measures*, Solitons and Fractals 126 (2019) 1-15.
[34] L. Olsen, *A multifractal formalism*, Advances in Mathematics 116 (1995) 82-196.
[35] L. Olsen, *Dimension Inequalities of Multifractal Hausdorff Measures and Multifractal Packing Measures*, Math. Scand. 86 (2000) 109-129.
[36] L. Olsen, *Mixed generalized dimensions of self-similar measures*, J. Math. Anal. Appl. 306 (2005) 516-539.
[37] P.Y. Pesin, *Dimension type characteristics for invariant sets of dynamical systems*, Russian Math: Surveys. 43 (1988) 111-151.
[38] P.Y. Pesin, *Dimension theory in dynamical systems, Contemporary views and applications*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, (1997).
[39] B. Selmi, *Some results about the regularities of multifractal measures*, Korean J. Math. 26 (2018) 271-283.
[40] B. Selmi, *On the strong regularity with the multifractal measures in a probability space*, Anal.Math.Phys. 9 (2019) 1525-1534.
[41] B. Selmi, *The relative multifractal densities: a review and application*, Journal of Interdisciplinary Mathematics 24 (2021) 1627-1644.
[42] B. Selmi, *Projections of measures with small supports*, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica 20 (2021) 5-15.

[43] B. Selmi, *The relative multifractal analysis, review and examples*, Acta Scientiarum Mathematicarum 86 (2020) 635-666.
[44] B. Selmi, *Multifractal dimensions of vector-valued non-Gibbs measures*, General Letters in Mathematics 8 (2020) 51-66.
[45] B. Selmi, *Projection estimates for mutual multifractal dimensions*, Journal of Pure and Applied Mathematics: Advances and Applications 22 (2020) 71-89.
[46] B. Selmi and A. Ben Mabrouk, *On the mixed multifractal formalism for vector-valued measures,* Proyecciones 41 (2022) 1015-1032.
[47] N. Yu. Svetova, *Conditional and mutual multifractal spectra. Definition and basic properties*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. 10 (2003) 41-58.
[48] N. Yu. Svetova, *Mutual multifractal spectra I: Exact spectra*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. 11 (2004) 41-46.
[49] N. Yu. Svetova, *Mutual multifractal spectra. II: Legendre and Hentschel-Procaccia spectra, and spectra defined for partitions*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. 11 (2004) 47-56.
[50] N. Yu. Svetova, *An estimate for exact mutual multifractal spectra*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. 14 (2008) 59-66.
[51] N. Yu. Svetova, *The property of convexity of mutual multifractal dimension*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. 17 (2010) 15-24.