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Division Problem of the Perturbed Form of Order One of the Chebyshev Form of Second Kind

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Abstract. We study the regularity of the form (linear functional) satisfying the functional equation $(x^2 - c^2)u = \lambda v$ where $c \in \mathbb{C}, \lambda \in \mathbb{C} - \{0\}$ and v is the perturbed form of order one of the Chebyshev form of second kind. The integral representation of the form u is highlighted. Moreover, some symmetric second degree forms of class two and four are given.

1. Introduction

Several authors have studied the problem of division of a regular form by a polynomial [1,3,4,8,10,11,15]: Let v be a regular form, find all regular forms u fulfilling the following algebraic equation

$$Ru = \lambda v, \ \lambda \in \mathbb{C} - \{0\},$$

where R is a monic polynomial.
In particular, for $R(x) = x$ see [4] and for $R(x) = x^2$ consult [8]. Moreover, the cases $R(x) = x^3$ and $R(x) = x^4$, have been studied in [10] and [11] respectively.
Recently, we have studied the case where $R(x) = x^2 - c^2$, $c \in \mathbb{C} - \{0\}$. Basically, we dealt with $v = \mathcal{U}$ the Chebyshev form of second kind [1]. The present work concerns the study of the case where v is the perturbed form of order one of the Chebyshev form of second kind [7,12,13] and fulfilling the following algebraic equation [1,12]

$$(x^2 - a^2)v = -\frac{\omega}{4(\omega - 1)}\mathcal{U}, \ \omega \in \mathbb{C} - \{0, 1\}, \ a^2 = \frac{\omega^2}{4(\omega - 1)}.$$

Then,

$$(x^2 - c^2)(x^2 - a^2)u = -\frac{\omega}{4(\omega - 1)}\mathcal{U},$$

which give a particular study of the division problem of a regular form by a polynomial with $\deg(R) = 4$ and and v is the Chebyshev form of second kind.
The preliminaries results and notations are thoroughly presented in section two. In the third section, we

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give explicitly the regularity conditions of the form u a detailed study regarding the class of is provided. Also we establish the second degree character of the form u. In the fourth section, we study some particular cases, which allowed us to give some examples of symmetric second degree forms of class four (which is not given in the literature as far as we know) and of class two (that are not dealt with in [14]) . The last section is devoted to establish the integral representations of the forms given above.

2. Preliminaries and fundamental results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle w, f\rangle$ the effect of $w \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(w)_n := \langle w, x^n\rangle$, $n \geq 0$, the moments of w. For any form w, any polynomial g, any $a \in \mathbb{C} - \{0\}$ any $c \in \mathbb{C}$ and $b \in \mathbb{C}$, let w', gw, $h_a w$ and $(x-c)^{-1}w$ be the forms defined by duality

$\langle w', f\rangle := -\langle w, f'\rangle$, $\langle gw, f\rangle := \langle w, gf\rangle$, $\langle (x-c)^{-1}w, f\rangle := \langle w, \theta_c f\rangle$,
where $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$, $f \in \mathcal{P}$.
We also define the right-multiplication of a form by a polynomial with
$(wf)(x) := \langle w, \frac{xf(x)-\xi f(\xi)}{x-\xi}\rangle$, $w \in \mathcal{P}'$, $f \in \mathcal{P}$.
Next, the product of two forms is defined as
$\langle uv, f\rangle := \langle u, vf\rangle$, $u, v \in \mathcal{P}'$, $f \in \mathcal{P}$.
We call polynomial sequence (PS), the sequence of polynomials $\{P_n\}_{n\geq 0}$ when $\deg P_n = n$, $n \geq 0$. Then any polynomial P_n can be supposed monic and the sequence becomes a monic polynomial sequence (MPS). Let $\{w_n\}_{n\geq 0}$ be its dual sequence defined by $\langle w_n, P_m\rangle = \delta_{n,m}$, $n, m \geq 0$.
The MPS $\{P_n\}_{n\geq 0}$ is orthogonal (MOPS) with respect to $w \in \mathcal{P}'$ when the following conditions hold $\langle w, P_m P_n\rangle = r_n \delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$. In this case the form w is said to be regular. The form w is called normalized if $(w)_0 = 1$. In this paper, we suppose that the forms are normalized. Thus, $w = w_0$ and $\{P_n\}_{n\geq 0}$ satisfies the standard recurrence relation

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0^P, \\ P_{n+2}(x) = (x - \beta_{n+1}^P)P_{n+1}(x) - \gamma_{n+1}^P P_n(x),\ n \geq 0;\ \gamma_{n+1}^P \neq 0,\ n \geq 0. \end{cases} \tag{1}$$

The regular form w_0 is said to be symmetric when $(w_0)_{2n+1} = 0$, $n \geq 0$, or equivalently $\beta_n^P = 0$, $n \geq 0$, in (1) [5].
Let us recall some results:

Lemma 2.1. [5] *Let $w \in \mathcal{P}'$, $f, g \in \mathcal{P}$ and a, $b \in \mathbb{C}$. The following formulas hold:*

$$(fw)' = fw' + f'w, \tag{2}$$

$$(x-a)^{-1}\Big((x-a)w\Big) = w - (w)_0\delta_a, \tag{3}$$

$$(x-a)^{-1}\delta_b = \frac{1}{b-a}\Big(\delta_b - \delta_a\Big),\ a \neq b \quad ; \quad (x-a)^{-1}\delta_a = -\delta_a', \tag{4}$$

$$(\theta_a\theta_b f)(x) = (\theta_b\theta_a f)(x) = \frac{1}{a-b}\Big(\theta_a f - \theta_b f\Big)(x),\ a \neq b, \tag{5}$$

where $\langle \delta_a, f\rangle := f(a)$, $f \in \mathcal{P}$.

Definition 2.2. *A MOPS $\{P_n\}_{n\geq 0}$ with respect to w is called semi-classical sequence if w fulfils an equation*

$$(\phi w)' + \psi w = 0, \tag{6}$$

where ϕ monic and $\deg\psi \geq 1$. In this case, the form w is called semi-classical.

Let us introduce the integer $s(\phi,\psi)=\max\Big(\deg\phi-2,\deg\psi-1\Big)$. Then $s=\min s(\phi,\psi)$ where the minimum is taken over all the pairs (ϕ,ψ) occurring in (6) is called the class of w. By extension, the integer s is also the class of $\{P_n\}_{n\geq0}$ [5].
We have the following result:

Proposition 2.3. [5] *The form w satisfying (6) is of class $s=s(\phi,\psi)$ if and only if*
$\prod_{c\in Z(\phi)}\Big(|\,\psi(c)+\phi'(c)\,|+|\,\langle w,\theta_c\psi+\theta_c^2\phi\rangle\,|\Big)\neq0,$
where $Z(\phi)=\{c,\ \phi(c)=0\}$.
In the case when there exists $c\in\mathbb{C}$ such that $\phi(c)=0$ and $|\,\psi(c)+\phi'(c)\,|+|\,\langle w,\theta_c\psi+\theta_c^2\phi\rangle\,|=0$, we have

$$(\theta_c\phi w)'+\Big(\theta_c\psi+\theta_c^2\phi\Big)w=0.$$

Let $\{P_n\}_{n\geq0}$ be a MOPS with respect to w and $\{P_n^{(1)}\}_{n\geq0}$ be the first associated polynomial sequence where $P_n^{(1)}(x)=\langle w,\frac{P_{n+1}(x)-P_{n+1}(\xi)}{x-\xi}\rangle$, $n\geq0$. The sequence $\{P_n^{(1)}\}_{n\geq0}$ is orthogonal with respect to $w^{(1)}$ and satisfies (1) with $\beta_n^{(1)}=\beta_{n+1}^P$, $\gamma_{n+1}^{(1)}=\gamma_{n+2}^P$, $n\geq0$.
The monic second kind Chebyshev polynomials are defined by [2,6,7]

$$\begin{cases} U_0(x)=1 \quad , \quad U_1(x)=x,\\ U_{n+2}(x)=xU_{n+1}(x)-\frac{1}{4}U_n(x),\ n\geq0. \end{cases} \tag{7}$$

We denote by v the perturbed Chebyshev form of order one of the Chebyshev form of second kind and $\{S_n\}_{n\geq0}$ its MOPS, we have [12]

$$\begin{cases} S_0(x)=1,\quad S_1(x)=x,\\ S_{n+2}(x)=xS_{n+1}(x)-\sigma_{n+1}S_n(x),\ n\geq0, \end{cases} \tag{8}$$

with

$$\sigma_1=\frac{1}{4}\omega,\ \sigma_{n+1}=\frac{1}{4},\ n\geq1,\ \omega\in\mathbb{C}-\{0,1,2\}, \tag{9}$$

$$(\phi v)'+\psi v=0, \tag{10}$$

$$\phi(x)=(x^2-1)(x^2-a^2),\ \psi(x)=-3x(x^2-a^2),\ a^2=\frac{\omega^2}{4(\omega-1)}. \tag{11}$$

We know that [1]

$$S_{n+2}(x)=U_{n+2}(x)-\frac{\omega-1}{4}U_n(x),\ n\geq0. \tag{12}$$

Now, let us consider the following problem:
find all regular symmetric forms u satisfying

$$(x^2-c^2)u=\lambda v,\quad \lambda\in\mathbb{C}-\{0\},\quad c\in\mathbb{C}-\{0\}, \tag{13}$$

with the constraint $(u)_0=1$.
When u is regular, let $\{Z_n\}_{n\geq0}$ be its corresponding MOPS fulfilling

$$\begin{cases} Z_0(x)=1,\quad Z_1(x)=x,\\ Z_{n+2}(x)=xZ_{n+1}(x)-\gamma_{n+1}Z_n(x),\ n\geq0. \end{cases} \tag{14}$$

From (13), the existence of the sequence $\{Z_n\}_{n\geq0}$ is among all the strictly quasi-orthogonal sequences of order two with respect to λv that is [1,9]

$$Z_{n+2}(x)=S_{n+2}(x)+a_nS_n(x),\ n\geq0, \tag{15}$$

with $a_n\neq0,\ n\geq0$.

Remark 2.4. *Let $\{S_n(.,\mu)\}_{n\geq 0}$, be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$. One may write [2,5]*

$$S_n(x,\mu) = S_n(x) - \mu S_{n-1}^{(1)}(x),\ n \geq 0, \tag{16}$$

where $S_{-1}^{(1)} = 0$.

Let us recall the following results:

Proposition 2.5. [1] *The form u is regular, if and only if, $S_n(c, -\frac{\lambda}{c}) \neq 0$, $n \geq 0$.*
Moreover,

$$a_n = -\frac{S_{n+2}(c,-\frac{\lambda}{c})}{S_n(c,-\frac{\lambda}{c})},\ n \geq 0, \tag{17}$$

$$\gamma_1 = cS_1(c, -\frac{\lambda}{c}), \tag{18}$$

$$\gamma_2 = -\lambda \frac{S_2(c,-\frac{\lambda}{c})}{\lambda + c^2}, \tag{19}$$

$$\gamma_{n+3} = \frac{a_{n+1}}{a_n}\sigma_{n+1},\ n \geq 0. \tag{20}$$

Proposition 2.6. *The form u given by (13) is regular if and only if*

$$\Delta_n \neq 0,\ n \geq 0,$$

with

$$\Delta_n = \frac{c^2+\lambda}{c}U_n(c) - \frac{\omega}{4}U_{n-1}(c) \neq 0,\ n \geq 0,\ U_{-1} = 0. \tag{21}$$

In this case, we have

$$a_0 = -\Big(c^2 + \lambda - \frac{\omega}{4}\Big), \tag{22}$$

$$a_{n+1} = -\frac{\Delta_{n+2}}{\Delta_n},\ n \geq 0, \tag{23}$$

$$\gamma_1 = c^2 + \lambda, \tag{24}$$

$$\gamma_2 = -\frac{\lambda}{4}\frac{4c^2+4\lambda-\omega}{\lambda+c^2}, \tag{25}$$

$$\gamma_3 = \frac{\omega}{4}\frac{a_1}{a_0}, \tag{26}$$

$$\gamma_{n+4} = \frac{1}{4}\frac{a_{n+2}}{a_{n+1}},\ n \geq 0. \tag{27}$$

Proof. We have

$$S_n(c, -\frac{\lambda}{c}) = S_n(c) + \frac{\lambda}{c}S_{n-1}^{(1)}(c),\ n \geq 0.$$

On account of (7), (8) and (9), we get $S_n^{(1)} = U_n$, $n \geq 0$, then last relation becomes

$$S_n(c, -\frac{\lambda}{c}) = S_n(c) + \frac{\lambda}{c}U_{n-1}(c),\ n \geq 0. \tag{28}$$

Also, by (7), (8) and (9), we have

$$S_0(c, -\frac{\lambda}{c}) = 1,\ S_1(c, -\frac{\lambda}{c}) = \frac{c^2+\lambda}{c}. \tag{29}$$

We may write

$$S_{n+2}(c,-\frac{\lambda}{c}) = S_{n+2}(c) + \frac{\lambda}{c}U_{n+1}(c),\ n \geq 0,$$

using the formula (12), we obtain

$$S_{n+2}(c,-\frac{\lambda}{c}) = \frac{c^2+\lambda}{c}U_{n+1}(c) - \frac{\omega}{4}U_n(c),\ n \geq 0.$$

Based on (29), we can deduce that

$$S_{n+1}(c,-\frac{\lambda}{c}) = \frac{c^2+\lambda}{c}U_n(c) - \frac{\omega}{4}U_{n-1}(c) = \Delta_n,\ n \geq 0. \tag{30}$$

Taking into account relation (30) and Proposition 2.5, we can deduce that the form u is regular

$$\Delta_n \neq 0,\ n \geq 0.$$

Furthermore, the relations (22)-(27) are an immediate consequence from (29)-(30) and Proposition 2.5. □

3. The class of the form *u* and its second degree character

3.1. The class of the form u

From the relations (10)-(13), we get

$$(\tilde{\phi}u)' + \tilde{\psi}u = 0, \tag{31}$$

with

$$\tilde{\phi}(x) = (x^2-c^2)(x^2-a^2)(x^2-1) \quad , \quad \tilde{\psi}(x) = -3x(x^2-c^2)(x^2-a^2). \tag{32}$$

We see that the class of the form u is at most $s = 4$.
Let us recall the following result:

Proposition 3.1. [1] *The class of the form u depends only on the zeros εc, where $\varepsilon = \pm 1$.*

Proposition 3.2. *When u is a regular form, let*

$$\vartheta_1 = 2a^2(a^2-1) + \lambda(2a^2 - 1 - \frac{\omega}{2}), \tag{33}$$

$$\vartheta_2 = 2(1-a^2) + \lambda(2a^2 - 1 - \frac{\omega}{2}). \tag{34}$$

We have
(a) *The form u is a semi-classical of class $s = 4$ when $c^2 \neq a^2,\ c^2 \neq 1$ or $c^2 = a^2,\ \vartheta_1 \neq 0$ or $c^2 = 1,\ \vartheta_2 \neq 0$ and u satisfies (31)-(32).*
(b) *When $c^2 = a^2,\ \vartheta_1 = 0$, the class of the form u is $s = 2$, and we have*

$$(\tilde{\phi}_1 u)' + \tilde{\psi}_1 u = 0, \tag{35}$$

$$\tilde{\phi}_1(x) = (x^2-a^2)(x^2-1),\ \tilde{\psi}_1(x) = -x^3 + (3a^2-2)x. \tag{36}$$

(c) *If $c^2 = 1,\ \vartheta_2 = 0$, the class of the form u is $s = 2$, and we have*

$$(\tilde{\phi}_2 u)' + \tilde{\psi}_2 u = 0, \tag{37}$$

$$\tilde{\phi}_2(x) = (x^2-a^2)(x^2-1),\ \tilde{\psi}_2(x) = -2x(x^2-a^2) \tag{38}$$

Proof. The following formula is needed [5]

$$\Big(\theta_a(fg)\Big)(x) = f(x)(\theta_a g)(x) + g(a)(\theta_a f)(x), a \in \mathbb{C},\ f,\ g \in \mathcal{P}. \tag{39}$$

From (32), we get

$$\tilde{\psi}(\epsilon c) + \tilde{\phi}'(\varepsilon c) = 2\epsilon c(c^2 - a^2)(c^2 - 1). \tag{40}$$

Also, we have

$$(\theta_{\varepsilon c}\tilde{\psi})(x) = (\theta_{\varepsilon c}((x^2 - c^2)\psi))(x),$$

using the formula (39), it follows that

$$(\theta_{\varepsilon c}\tilde{\psi})(x) = (x^2 - c^2)(\theta_{\varepsilon c}\psi)(x) + (x + \varepsilon c)\psi(\varepsilon c). \tag{41}$$

Similarly, we get

$$(\theta^2_{\varepsilon c}\tilde{\phi})(x) = (x^2 - c^2)(\theta^2_{\varepsilon c}\phi)(x) + (x + \varepsilon c)\phi'(\varepsilon c) + \phi(\varepsilon c). \tag{42}$$

Based on (41) and (42), we can deduce that

$$(\theta_{\varepsilon c}\tilde{\psi})(x) + (\theta^2_{\varepsilon c}\tilde{\phi})(x) = (x^2 - c^2)\Big(\theta_{\varepsilon c}\psi) + \theta^2_{\varepsilon c}\phi\Big)(x) + (x + \varepsilon c)\Big(\psi(\varepsilon c) + \phi'(\varepsilon c)\Big) + \phi(\varepsilon c),$$

but from (11), we obtain successively

$$\Big(\theta_{\varepsilon c}\psi) + \theta^2_{\varepsilon c}\phi\Big)(x) = -2x^2 - \varepsilon cx + 2a^2 - 1,$$

and

$$\psi(\varepsilon c) + \phi'(\varepsilon c) = 2\varepsilon c(2c^2 - a^2 - 1).$$

Which implies that

$$\begin{aligned}(\theta_{\varepsilon c}\tilde{\psi})(x) + (\theta^2_{\varepsilon c}\tilde{\phi})(x) &= (x^2 - c^2)\Big(-2x^2 - \varepsilon cx + 2a^2 - 1\Big)\\ &+2\varepsilon c(2c^2 - a^2 - 1)(x + \varepsilon c) + (c^2 - a^2)(c^2 - 1).\end{aligned} \tag{43}$$

Then,

$$\begin{aligned}\langle u, \theta_{\varepsilon c}\tilde{\psi} + \theta^2_{\varepsilon c}\tilde{\phi}\rangle &= \langle u, (x^2 - c^2)\Big(-2x^2 - \varepsilon cx + 2a^2 - 1)\Big)\rangle\\ &+\langle u, 2\varepsilon c(2c^2 - a^2 - 1)(x + \varepsilon c) + (c^2 - a^2)(c^2 - 1)\rangle\end{aligned}$$

Equivalently,

$$\begin{aligned}\langle u, \theta_{\varepsilon c}\tilde{\psi} + \theta^2_{\varepsilon c}\tilde{\phi}\rangle &= \langle (x^2 - c^2)u, -2x^2 - \varepsilon cx + 2a^2 - 1\rangle\\ &+\langle u, 2\varepsilon c(2c^2 - a^2 - 1)(x + \varepsilon c) + (c^2 - a^2)(c^2 - 1)\rangle,\end{aligned}$$

by (13) and the fact that u is a symmetric form, the last relation can be written as

$$\begin{aligned}\langle u, \theta_{\varepsilon c}\tilde{\psi} + \theta^2_{\varepsilon c}\tilde{\phi}\rangle &= \lambda\langle v, -2x^2 - \varepsilon cx + 2a^2 - 1\rangle\\ &+2c^2(2c^2 - a^2 - 1) + (c^2 - a^2)(c^2 - 1).\end{aligned}$$

But $\langle v, -2x^2 - \varepsilon cx + 2a^2 - 1\rangle = 2a^2 - 1 - \frac{\omega}{2}$, then

$$\langle u, \theta_{\varepsilon c}\tilde{\psi} + \theta^2_{\varepsilon c}\tilde{\phi}\rangle = \lambda\Big(2a^2 - 1 - \frac{\omega}{2}\Big) + 2c^2(2c^2 - a^2 - 1) + (c^2 - a^2)(c^2 - 1). \tag{44}$$

(a) If $c^2 = a^2$, relation (44) becomes

$$\langle u, \theta_{\varepsilon a}\tilde{\psi} + \theta^2_{\varepsilon a}\tilde{\phi}\rangle = \lambda\Big(2a^2 - 1 - \frac{\omega}{2}\Big) + 2a^2(a^2 - 1) = \vartheta_1. \tag{45}$$

When $c^2 = 1$, by (44), we get

$$\langle u, \theta_\varepsilon \tilde{\psi} + \theta_\varepsilon^2 \tilde{\phi} \rangle = \lambda\Big(2a^2 - 1 - \frac{\omega}{2}\Big) + 2(1 - a^2) = \vartheta_2. \tag{46}$$

In the case where $c^2 \neq a^2$, $c^2 \neq 1$, based on (40), (45), (46) and Proposition 2.3, we can not simplify the equation (31). According to Proposition 3.1, we can deduce the desired results.
(b) In this case, from (40) and (45), we can simplify equation (31) by the factor $x^2 - a^2$, and we obtain (35) with

$$\tilde{\phi}_1 = \theta_a \theta_{-a} \tilde{\phi},\ \tilde{\psi}_1 = \theta_a \theta_{-a} \tilde{\psi} + (\theta_a \theta_{-a}^2 + \theta_a^2 \theta_{-a}) \tilde{\phi}.$$

Since $\tilde{\phi}(x) = (x^2 - a^2)^2(x^2 - 1)$, $\tilde{\psi}(x) = -3x(x^2 - a^2)^2$, we obtain (36).
Moreover, by (36), we have $\tilde{\phi}_1'(\varepsilon a) + \tilde{\psi}(\varepsilon a) = 4\varepsilon a(a^2 - 1) \neq 0$, then we cannot simplify equation (35) and the class of the form u is $s = 2$.
(c) When $c^2 = 1$, $\vartheta_2 = 0$, by (40) and (46), equation (35) can be is simplified by the factor $x^2 - 1$ and we obtain (37) with

$$\tilde{\phi}_2 = \theta_1 \theta_{-1} \tilde{\phi},\ \tilde{\psi}_2 = \theta_1 \theta_{-1} \tilde{\psi} + (\theta_1 \theta_{-1}^2 + \theta_1^2 \theta_{-1}) \tilde{\phi}.$$

Since $\tilde{\phi}(x) = (x^2 - 1)^2(x^2 - a^2)$, $\tilde{\psi}(x) = -3x(x^2 - 1)(x^2 - a^2)$, we obtain (38).
Furthermore, by (38), we have $\tilde{\phi}_2'(\varepsilon) + \tilde{\psi}_2(\varepsilon) = -2\varepsilon(1 - a^2) \neq 0$, then we cannot simplify the equation (37) and the class of the form u is $s = 2$.
□

3.2. The second degree character of the form u

Let us introduce the formal Stieltjes function $S(.; w)$ of the form w:

$$S(z, w) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}. \tag{47}$$

Definition 3.3. *A regular form w is said to be a second degree form if there exist two polynomials $B(z, w)$ (monic) and $C(z, w)$ such that*

$$B(z, w)S^2(z, w) + C(z, w)S(z, w) + D(z, w) = 0, \tag{48}$$

where

$$D(z, w) = (w\theta_0 C(., w))(z) - (w^2 \theta_0^2 B(., w))(z). \tag{49}$$

The regularity of w means that $B(., w) \neq 0$, $C^2((., w) - 4B(., w)D(., w) \neq 0$ and $D(., w) \neq 0$ [6].

A second-degree form w whose corresponding formal Stieltjes function $S(z, w)$ satisfies (48) is necessarily semi-classical and fulfils [6]

$$(\phi(x, w)w)' + \psi(x, w)w = 0, \tag{50}$$

with

$$\alpha\phi(x, w) = B(x, w)(C^2(x, w) - 4B(x, w)D(x, w)),$$
$$\alpha\psi(x, w) = -\frac{3}{2}B(x, w)(C^2(x, w) - 4B(x, w)D(x, w))', \tag{51}$$

where α represents a normalization constant.

Remark 3.4. *Keeping the same notation in [12], we have $v = \hat{u}(a, -\frac{\omega}{4(\omega-1)})$, then we have the following results:*

Proposition 3.5. [12] *On has*

$$B(z, v)S^2(z, v) + C(z, v)S(z, v) + D(z, v) = 0, \tag{52}$$

with

$$B(z, v) = (z^2 - a^2),\ C(z, v) = \frac{\omega - 2}{\omega - 1}z,\ D(z, v) = -\frac{1}{\omega - 1}, \tag{53}$$

$$S(z,v)=\frac{1}{z^2-a^2}\Big\{\frac{\omega}{2(\omega-1)}\frac{1}{z+\sqrt{z^2-1}}-z\Big\},\ z\in\mathbb{C}-\{-a,+a\}, \tag{54}$$

where the principal square root of a complex number z is $\sqrt{z}=\sqrt{|z|}\exp(i\frac{\arg(z)}{2}),\ -\pi<\arg(z)\le\pi.$

Proposition 3.6. *When the form u is regular it is a second degree form and we have*

$$B(z,u)S^2(z,u)+C(z,u)S(z,u)+D(z,u)=0, \tag{55}$$

with

$$\begin{cases} B(z,u)=(z^2-c^2)^2(z^2-a^2),\\ C(z,u)=(z^2-c^2)\Big(2z(z^2-a^2)+\lambda\frac{\omega-2}{\omega-1}z\Big),\\ D(z,u)=z^2(z^2-a^2)+\lambda\frac{\omega-2}{\omega-1}z^2-\frac{\lambda^2}{\omega-1}, \end{cases} \tag{56}$$

$$S(z,u)=\frac{\lambda}{(z^2-c^2)^2(z^2-a^2)}\Big(\frac{\omega}{2(\omega-1)}\frac{1}{z+\sqrt{z^2-1}}-z\Big)-\frac{z}{z^2-c^2}. \tag{57}$$

Proof. We know that the Stieltjes function $S(z,u)$ of the form u defined by (13) is given by [1]

$$S(z,u)=\frac{1}{z^2-c^2}\Big(\lambda S(z,v)-z\Big). \tag{58}$$

Equivalently,

$$S(z,v)=\frac{1}{\lambda}(z^2-c^2)S(z,u)+\frac{1}{\lambda}z.$$

Using the equation (52), we can deduce that

$$B(z,u)S^2(z,u)+C(z,u)S(z,u)+D(z,u)=0, \tag{59}$$

with

$$\begin{cases} B(z,u)=(z^2-c^2)^2B(z,v),\\ C(z,u)=(z^2-c^2)\Big(2zB(z,v)+\lambda C(z,v)\Big),\\ D(z,u)=z^2B(z,v)+\lambda zC(z,v)+\lambda^2D(z,v). \end{cases} \tag{60}$$

Taking into account (53), we obtain (56).
The result in (57) is a consequence of equations (58) and (54). □

4. Some particular cases

Remark 4.1. *In this section, we denote* $a(\omega):=a,\ u(a(\omega),c,\lambda):=u,\ \Delta_n(a(\omega),c,\lambda)=\Delta_n,\ a_n(a(\omega),c,\lambda):=a_n,\ \gamma_{n+1}(a(\omega),c,\lambda)=\gamma_{n+1},\ n\ge0.$

4.1. The case where $c=\frac{1}{2}$

Proposition 4.2. *The form* $u(a(\omega),\frac{1}{2},\lambda)$, *is regular if and only if,* $4\lambda+1\ne0,\ 4\lambda-\omega+1\ne0.$
In this case, we have

$$\Delta_{3n}(a(\omega),\frac{1}{2},\lambda)=\frac{(-1)^n}{2^{3n+1}}(4\lambda+1),\ \Delta_{3n+1}(a(\omega),\frac{1}{2},\lambda)=\frac{(-1)^n}{2^{3n+2}}(4\lambda-\omega+1),$$

$$\Delta_{3n+2}(a(\omega),\frac{1}{2},\lambda)=-\omega\frac{(-1)^n}{2^{3n+3}},\ n\ge0. \tag{61}$$

$$a_0(a(\omega),\frac{1}{2},\lambda)=-\Big(\lambda+\frac{1-\omega}{4}\Big),\ a_{3n+1}(a(\omega),\frac{1}{2},\lambda)=\frac{\omega}{4(4\lambda+1)},$$

$$a_{3n+2}(a(\omega),\frac{1}{2},\lambda)=\frac{4\lambda+1}{4(4\lambda-\omega+1)},$$

$$a_{3n+3}(a(\omega),\frac{1}{2},\lambda)=-\frac{4\lambda-\omega+1}{4\omega},\ n\geq 0. \tag{62}$$

$$\gamma_1(a(\omega),\frac{1}{2},\lambda)=\frac{4\lambda+1}{4},\ \gamma_2(a(\omega),\frac{1}{2},\lambda)=-\lambda\frac{4\lambda-\omega+1}{4\lambda+1},$$

$$\gamma_3(a(\omega),\frac{1}{2},\lambda)=-\frac{\omega^2}{4}\frac{1}{(4\lambda+1)(4\lambda-\omega+1)},$$

$$\gamma_{3n+4}(a(\omega),\frac{1}{2},\lambda)=\frac{1}{4\omega}\frac{(4\lambda+1)^2}{4\lambda-\omega+1},\ n\geq 0,$$

$$\gamma_{3n+5}(a(\omega),\frac{1}{2},\lambda)=-\frac{(4\lambda-\omega+1)^2}{4\omega(4\lambda+1},\ n\geq 0,$$

$$\gamma_{3n+6}(a(\omega),\frac{1}{2},\lambda)=-\frac{\omega^2}{4}\frac{1}{(4\lambda+1)(4\lambda-\omega+1)},\ n\geq 0. \tag{63}$$

Proof. By virtue of [13], we have $U_{3n}(\frac{1}{2})=\frac{(-1)^n}{2^{3n}}$, $U_{3n+1}(\frac{1}{2})=\frac{(-1)^n}{2^{3n+1}}$, $U_{3n+2}(\frac{1}{2})=0$, $n\geq 0$. This implies (61).
On one hand, based on (4.1) and Proposition 2.6, we can deduce that the form $u(a(\omega),\frac{1}{2},\lambda)$ is regular if and only if $4\lambda+1\neq 0,\ 4\lambda-\omega+1\neq 0$.
On the other hand the relations (62)-(63) are a consequence from the relation (61) and Proposition 2.6. □

Proposition 4.3. *When $u(a(\omega),\frac{1}{2},\lambda)$ is regular, one has*
(a) *$u(a(\omega),\frac{1}{2},\lambda)$ it is a semi-classical of class $s=4$, when $\omega\neq\frac{1-i\sqrt{3}}{2}$, $\omega\neq\frac{1+i\sqrt{3}}{2}$ and we have*

$$\Big(\tilde{\phi}(a(\omega),\frac{1}{2},\lambda)u(a,\frac{1}{2},\lambda)\Big)'+\tilde{\psi}(a(\omega),\frac{1}{2},\lambda)u(a(\omega),\frac{1}{2},\lambda)=0, \tag{64}$$

with

$$\tilde{\phi}(a(\omega),\frac{1}{2},\lambda)(x)=(x^2-\frac{1}{4})(x^2-a(\omega)^2)(x^2-1),$$

$$\tilde{\psi}(a(\omega),\frac{1}{2},\lambda)(x)=-3x(x^2-\frac{1}{4})(x^2-a(\omega)^2). \tag{65}$$

(b) *When $\omega=\omega_k$, $k=1,2$, with $\omega_1=\frac{1-i\sqrt{3}}{2}$ and $\omega_2=\frac{1+i\sqrt{3}}{2}$, the form $u(a(\omega_k),\frac{1}{2},\lambda)$ is of class two and satisfy the following equation*

$$\Big(\tilde{\phi}(a(\omega_k),\frac{1}{2},\lambda)u(a(\omega_k),\frac{1}{2},\lambda)\Big)'+\tilde{\psi}(a(\omega),\frac{1}{2},\lambda)u(a(\omega_k),\frac{1}{2},\lambda)=0, \tag{66}$$

with

$$\tilde{\phi}(a(\omega_k),\frac{1}{2},\lambda)(x)=(x^2-\frac{1}{4})(x^2-1),$$

$$\tilde{\psi}(a(\omega_k),\frac{1}{2},\lambda)(x)=-3x^2-\frac{5}{4}x. \tag{67}$$

Proof. **(a)** We have $a^2(\omega)=\frac{1}{4}$ if and only if $\omega=\omega_k$, $k=1,2$.
From the affirmation (a) in Proposition 3.2, the form $u(a(\omega),\frac{1}{2},\lambda)$ is of class $s=4$ and satisfies (61)-(62) since $c=\frac{1}{2}\neq a^2$, $c=\frac{1}{2}\neq 1$.
(b) When $\omega=\omega_k$, $k=1,2$, by virtue of the statement (b) in Proposition 3.2, the form $u(a(\omega_k),\frac{1}{2},\lambda)$ is of class two and fulfilled (66) with (67) since $a(\omega_k)^2=\frac{1}{4}$. □

4.2. The case where $c = 1$

Proposition 4.4. *The form $u(a(\omega), 1, \lambda)$ is regular if and only if $\big(2(\lambda+1)-\omega\big)n + 2(\lambda+1) \neq 0$, $n \geq 0$. In this case, one has*

$$\Delta_n(a(\omega), 1, \lambda) = \frac{\big(2(\lambda+1)-\omega\big)n + 2(\lambda+1)}{2^{n+1}}, \; , \; n \geq 0. \tag{68}$$

$$a_0(a(\omega), 1, \lambda) = \frac{4(\lambda+1)-\omega}{4},$$

$$a_{n+1}(a(\omega), 1, \lambda) = -\frac{1}{4}\frac{\big(2(\lambda+1)-\omega\big)(n+2) + 2(\lambda+1)}{\big(2(\lambda+1)-\omega\big)n + 2(\lambda+1)}, \; n \geq 0. \tag{69}$$

$$\gamma_1(a(\omega), 1, \lambda) = \lambda + 1, \;\; \gamma_2(a(\omega), 1, \lambda) = -\frac{\lambda}{4}\frac{4(\lambda+1)-\omega}{\lambda+1},$$

$$\gamma_3(a(\omega), 1, \lambda) = -\frac{\omega}{4}\frac{5(\lambda+1)-3\omega}{\big(4(\lambda+1)-\omega\big)^2},$$

$$\gamma_{n+4}(a(\omega), 1, \lambda) = \frac{1}{4}\frac{\big(2(\lambda+1)-\omega\big)(n+3) + 2(\lambda+1)}{\big(2(\lambda+1)-\omega\big)(n+2) + 2(\lambda+1)} \times$$
$$\frac{\big(2(\lambda+1)-\omega\big)n + 2(\lambda+1)}{\big(2(\lambda+1)-\omega\big)(n+1) + 2(\lambda+1)}, \; n \geq 0. \tag{70}$$

Proof. We know that [2]

$$U_n(x) = \frac{1}{2^n}\frac{\sin((n+1)\theta)}{\sin(\theta)}, \; n \geq 0, \;\; x = \cos(\theta). \tag{71}$$

It follows that , $U_n(1) = \frac{n+1}{2^n}$, $n \geq 0$. From the previous relation, we obtain (68).
By Proposition 2.6 and (68), we see that $u(a(\omega), 1, \lambda)$ is regular when $\big(2(\lambda+1)-\omega\big)n + 2(\lambda+1) \neq 0$, $n \geq 0$.
The relations (69)-(70) can be deduced from (68) and Proposition 2.6. □

Proposition 4.5. *If the form $u(a(\omega), 1, \lambda)$ is regular, we have*
(a) *The class of $u(a(\omega), 1, \lambda)$ is $s = 4$ if $\vartheta_2 \neq 0$ and satisfies the following functional equation*

$$\Big(\tilde{\phi}(a(\omega), 1, \lambda)u(a, 1, \lambda)\Big)' + \tilde{\psi}(a(\omega), 1, \lambda)u(a(\omega), 1, \lambda) = 0, \tag{72}$$

with

$$\tilde{\phi}(a(\omega), 1, \lambda)(x) = (x^2 - a^2(\omega))(x^2-1)^2,$$
$$\tilde{\psi}(a(\omega), 1, \lambda)(x) = -3x(x^2-1)(x^2-a^2(\omega)). \tag{73}$$

(b) *The class of $u(a(\omega), 1, \lambda)$ is $s = 2$ if $\vartheta_2 = 0$ and the following functional equation holds*

$$\Big(\tilde{\phi}_1(a(\omega), 1, \lambda)u(a, 1, \lambda)\Big)' + \tilde{\psi}_1(a(\omega), 1, \lambda)u(a(\omega), 1, \lambda) = 0, \tag{74}$$

where

$$\tilde{\phi}_1(a(\omega), 1, \lambda)(x) = (x^2 - a^2(\omega))(x^2-1),$$
$$\tilde{\psi}_1(a(\omega), 1, \lambda)(x) = -3x^3 + (3 - 2a^2(\omega))x. \tag{75}$$

Proof. **(a)** Taking into account the statement (a) in the Proposition 3.2, we get the desired results.
(b) From the the statement (b) in the Proposition 3.2, we can deduce that the functional equation (72) can be simplified by the factor $x^2 - 1$, and we obtain (74) with (75). □

Remark 4.6. *The cases (b) in Proposition 4.3 and the Proposition 4.5 are not studied in [14].*

5. Integral representation of the form u

In order to determine the integral representation of the form u, we need the following results:

Proposition 5.1. [12] *The form v possess the following integral representation*

$$\langle v, f\rangle = b(f(-a)+f(a)) + \rho\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{x^2-a^2}f(x)dx,\ f\in\mathcal{P},\ a\in\mathbb{C}-]-1,+1[, \tag{76}$$

with

$$b = \frac{1}{2}+\frac{\omega}{4(1-\omega)}\Big(1-\sqrt{(1-\frac{2}{\omega})^2}\Big),\ \rho = \frac{\omega}{2\pi(1-\omega)}. \tag{77}$$

Lemma 5.2. *let $a, c \in \mathbb{C}$. The following formulas hold*

$$(x+c)^{-1}(x-c)^{-1}\Big(\delta_{-a}+\delta_a\Big) = \frac{1}{a^2-c^2}\Big(\delta_{-a}+\delta_a-\delta_{-c}-\delta_c\Big),\ a^2\neq c^2, \tag{78}$$

$$(x+a)^{-1}(x-a)^{-1}\Big(\delta_{-a}+\delta_{\ a}\Big) = \frac{1}{2a}\Big(\delta'_{-a}-\delta'_a\Big). \tag{79}$$

Proof. We need the following formula easy to prove from the definition:

$$(x-b)^{-1}\delta'_a = \frac{1}{a-b}\delta'_a + \frac{1}{(a-b)^2}(\delta_a-\delta_b),\ a\neq b. \tag{80}$$

Let $\varepsilon = \pm 1$, by (4), we get $(x-c)^{-1}\delta_{\varepsilon a} = \frac{1}{\varepsilon a - c}\Big(\delta_{\varepsilon a}-\delta_c\Big)$. Therefore

$$(x+c)^{-1}(x-c)^{-1}\delta_{\varepsilon a} = \frac{1}{\varepsilon a-c}\Big((x+c)^{-1}\delta_{\varepsilon a}-(x+c)^{-1}\delta_c\Big),$$

and by the formula (4), it follows that

$$(x+c)^{-1}(x-c)^{-1}\delta_{\varepsilon a} = \frac{1}{\varepsilon a-c}\Big(\frac{1}{\varepsilon a+c}\Big(\delta_{\varepsilon a}-\delta_{-c}\Big)-\frac{1}{2c}\Big(\delta_c-\delta_{-c}\Big)\Big).$$

Thus,

$$(x+c)^{-1}(x-c)^{-1}\delta_{\varepsilon a} = \frac{1}{a^2-c^2}\delta_{\varepsilon a}+\frac{1}{2c(\varepsilon a+c)}\delta_{-c}-\frac{1}{2c(\varepsilon a-c)}\delta_c.$$

Based on last relation, we can deduce (78).
From (4), we have $(x-a)^{-1}\delta_a = -\delta'_a$. Then

$$(x+a)^{-1}(x-a)^{-1}\delta_a = -(x+a)^{-1}\delta'_a,$$

using the formula (80), it follows that

$$(x+a)^{-1}(x-a)^{-1}\delta_a = -\frac{1}{2a}\delta'_a-\frac{1}{4a^2}(\delta_a-\delta_{-a}). \tag{81}$$

By virtue of (81), we get

$$(x+a)^{-1}(x-a)^{-1}\delta_{-a} = \frac{1}{2a}\delta'_{-a}-\frac{1}{4a^2}(\delta_{-a}-\delta_a). \tag{82}$$

The relation (79) can be deduced by addition both sides of the relations (81) and (82). □

Lemma 5.3. *Let w be the form defined by*

$$\langle w, f\rangle = \int_{-1}^{+1} \frac{\sqrt{1-x^2}}{x^2-a^2} f(x)dx,\ \ f \in \mathcal{P},\ \ a \in \mathbb{C}-]-1,+1[. \tag{83}$$

We have

$$\begin{aligned}\langle (x+c)^{-1}(x-c)^{-1}w, f\rangle &= \int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x^2-c^2)} f(x)dx \\ &-\frac{1}{2c}\Big(\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x-c)}dx\Big)f(c) \\ &+\frac{1}{2c}\Big(\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x+c)}dx\Big)f(-c),\ \ f \in \mathcal{P}, c \in \mathbb{C}-]-1,+1[,\end{aligned} \tag{84}$$

$$\begin{aligned}\langle (x+c)^{-1}(x-c)^{-1}w, f\rangle &= P\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x^2-c^2)} f(x)dx \\ &-\frac{1}{2c}\Big(P\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x-c)}dx\Big)f(c) \\ &+\frac{1}{2c}\Big(P\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x+c)}dx\Big)f(-c),\ \ f \in \mathcal{P}, c \in]-1,+1[,\end{aligned} \tag{85}$$

where $P\int_{-\infty}^{+\infty} \frac{V(x)}{x+\epsilon c}dx$ *and* $P\int_{-\infty}^{+\infty} \frac{V(x)}{x^2-c^2}f(x)dx$ *means the Cauchy principal value of the integral at* ϵc *.*

Proof. Let $f \in \mathcal{P}$, we have

$$\langle (x+c)^{-1}(x-c)^{-1}w, f\rangle = \langle w, (\theta_{-c}\theta_c)f\rangle,$$

and by the formula (5), we obtain

$$\langle (x+c)^{-1}(x-c)^{-1}w, f\rangle = \frac{1}{2c}\langle w, \theta_c f - \theta_{-c} f\rangle.$$

Taking into account (83), we get

$$\langle (x+c)^{-1}(x-c)^{-1}w, f\rangle = \frac{1}{2c}\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{x^2-a^2}\Big\{\frac{f(x)-f(c)}{x-c} - \frac{f(x)-f(-c)}{x+c}\Big\}dx. \tag{86}$$

The relation (86) implies (84) and (85). □

Proposition 5.4. *Let $f \in \mathcal{P}$, we have*
(a) *If $c \in \mathbb{C}-]-1,+1[$, $c^2 \neq a^2$, the form u is represented by*

$$\begin{aligned}\langle u, f\rangle &= \frac{\lambda b}{a^2-c^2}\Big(f(-a)+f(a)\Big) \\ &+\Big\{\frac{1}{2} - \frac{\lambda b}{a^2-c^2} + \frac{\lambda\rho}{2c}\Big(\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x+c)}dx\Big)\Big\}f(-c) \\ &+\Big\{\frac{1}{2} - \frac{\lambda b}{a^2-c^2} - \frac{\lambda\rho}{2c}\Big(\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x-c)}dx\Big)\Big\}f(c) \\ &+\lambda\rho\int_{-1}^{+1} \frac{\sqrt{1-x^2}}{(x^2-a^2)(x^2-c)}f(x)dx.\end{aligned} \tag{87}$$

(b) *When* $c \in]-1,+1[$*, one has*

$$\begin{aligned}\langle u,f\rangle &= \frac{\lambda b}{a^2-c^2}\Big(f(-a)+f(a)\Big)\\ &+\Big\{\frac{1}{2}-\frac{\lambda b}{a^2-c^2}+\frac{\lambda\rho}{2c}\Big(P\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)(x+c)}dx\Big)\Big\}f(-c)\\ &+\Big\{\frac{1}{2}-\frac{\lambda b}{a^2-c^2}-\frac{\lambda\rho}{2c}\Big(P\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)(x-c)}dx\Big)\Big\}f(c)\\ &+\lambda\rho P\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)(x^2-c)}f(x)dx. \qquad (88)\end{aligned}$$

(c) *In the case when* $a^2=c^2$*, the form* u *possesses the following integral representation*

$$\begin{aligned}\langle u,f\rangle &= \Big\{\frac{1}{2}+\frac{\lambda\rho}{2a}\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)(x+a)}dx\Big\}f(-a)\\ &+\Big\{\frac{1}{2}-\frac{\lambda\rho}{2a}\Big(P\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)(x+a)}dx\Big)\Big\}f(a)\\ &+\frac{\lambda b}{2a}\Big(f'(a)-f'(-a)\Big)+\lambda\rho\int_{-1}^{+1}\frac{\sqrt{1-x^2}}{(x^2-a^2)^2}f(x)dx. \qquad (89)\end{aligned}$$

Proof. Based on (77) and (83), we may write

$$v = b(\delta_{-a}+\delta_a)+\rho w, \qquad (90)$$

where w is the form defined by (83).
We know that [1]

$$u = \frac{1}{2}(\delta_{-c}+\delta_c)+\lambda(x+c)^{-1}(x-c)^{-1}v. \qquad (91)$$

Taking into account (90), we get

$$\begin{aligned}u &= \frac{1}{2}(\delta_{-c}+\delta_c)+\lambda b(x+c)^{-1}(x-c)^{-1}(\delta_{-a}+\delta_a)\\ &+\lambda\rho(x+c)^{-1}(x-c)^{-1}w,\end{aligned}$$

and by Lemma 5.2, we obtain the following formulas

$$\begin{aligned}u &= \frac{\lambda b}{a^2-c^2}\Big(\delta_{-a}+\delta_a\Big)+\Big(\frac{1}{2}-\frac{\lambda b}{a^2-c^2}\Big)\Big(\delta_{-c}+\delta_c\Big)\\ &+\lambda\rho(x+c)^{-1}(x-c)^{-1}w,\ c^2\neq a^2, \qquad (92)\end{aligned}$$

and

$$u = \frac{1}{2}\Big(\delta_{-a}+\delta_a\Big)+\frac{\lambda b}{2a}\Big(\delta'_{-a}-\delta'_a\Big)+\lambda\rho(x+a)^{-1}(x-a)^{-1}w. \qquad (93)$$

(a) In this case, by (92), we have

$$\begin{aligned}\langle u,f\rangle &= \frac{\lambda b}{a^2-c^2}\Big(f(-a)+f(a)\Big)+\Big(\frac{1}{2}-\frac{\lambda b}{a^2-c^2}\Big)\Big(f(-c)+f(c)\Big)\\ &+\lambda\rho\langle(x+c)^{-1}(x-c)^{-1}w,f\rangle \qquad (94)\end{aligned}$$

By virtue of (84), we obtain (87).
(b) When $c \in]-1,+1[$. The relation (88) is a consequence from (94) and (85).
(c) When $c^2=a^2$, by (93), we obtain

$$\langle u,f\rangle = \frac{1}{2}\Big(f(-a)+f(a)\Big)+\frac{\lambda b}{2a}\Big(-f'(-a)+f'(a)\Big)+\lambda\rho\langle(x+a)^{-1}(x-a)^{-1}w,f\rangle.$$

Using (84) since $a \in \mathbb{C}-]-1,+1[$, we can deduce (89). □

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