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A Note on Some New Inequalities of Fusion Frames in Hilbert C^*-Modules

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Abstract. In this note, we give a new one-sided inequality for fusion frames in Hilbert C^*-modules, which corrects one corresponding result. We also present some double inequalities for fusion frames in Hilbert C^*-modules, which, compared to previous ones on this topic, possess different structures.

1. Introduction

In 1952, Duffin and Schaeffer [11] formally defined the concept of frames, which have become to be a research hotspot since 1986, when Daubechies et al. [10] discovered the close relationship between frame theory and wavelet theory. Now frames have played an important role in dozens of fields such as acoustics [5], quantum mechanics [17], signal processing [7], and sampling theory [12, 24, 26]. As a generalization of frames, the notion of fusion frames (also known as frames of subspaces) was introduced independently in [8] and [13] to deal with some large systems. Noting that some new characteristics for fusion frames do arise, although most properties of them can be induced from those for frames. For applications of fusion frames see, for example, the references [6, 9].

On the other hand, some researchers have analogized the concepts of frames and fusion frames to the case of Hilbert C^*-modules [14, 19], which provides us a new direction to examine frame theory. We remark, however, that because of the complexity of the C^*-algebra and some essential differences between Hilbert C^*-modules and Hilbert spaces, the problems of frames and fusion frames in Hilbert C^*-modules are more complicated than those in Hilbert spaces. Frames and fusion frames in Hilbert C^*-modules have attracted many researchers' attention, for more details see [1–3, 18, 20, 23].

To continue with this introductory section, we need to collect some notations and definitions.

Throughout this paper, the notations $\mathscr{A}$ and $\mathbb{J}$ denote, respectively, a unital C^*-algebra with identity $1_{\mathscr{A}}$ and a finite or countable index set. For two Hilbert C^*-modules $\mathscr{H}$ and $\mathscr{K}$ over $\mathscr{A}$, we denote by $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ the collection of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and, if $\mathscr{K}=\mathscr{H}$, then $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ is abbreviated to $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. We set $|f|^2=\langle f,f\rangle$ for each $f\in\mathscr{H}$. A closed submodule $\mathscr{M}$ of $\mathscr{H}$ is said to be

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orthogonally complemented if $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$ and in this case, $\pi_{\mathscr{M}}$, the orthogonal projection onto $\mathscr{M}$, is an adjointable operator on $\mathscr{H}$. Let $\ell^{\infty}(\mathscr{A})$ be the set

$$\ell^{\infty}(\mathscr{A}) = \left\{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathscr{A} : \sup_{j\in\mathbb{J}} \|a_j\| < \infty\right\}.$$

Let $\{\omega_j\}_{j\in\mathbb{J}}$ be a sequence of weights, i.e., each ω_j is a positive invertible element from the center of $\mathscr{A}$. For any $j \in \mathbb{J}$, let W_j be a closed submodule of $\mathscr{H}$ which is orthogonally complemented. One calls that $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ is a *fusion frame* for $\mathscr{H}$ with fusion frame bounds C and D, if there are real numbers $0 < C \leq D < \infty$ such that

$$C\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \leq D\langle f, f\rangle. \tag{1}$$

If only D in (1) is required to exist, then W is said to be a *Bessel fusion sequence* with bound D.

Let $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$. Then, there is always a self-adjoint, positive and invertible operator related to W, called the *fusion frame operator* of W and is defined by

$$S_W : \mathscr{H} \to \mathscr{H}, \quad S_W f = \sum_{j\in\mathbb{J}} \omega_j^2 \pi_{W_j}(f), \quad \forall f \in \mathscr{H}.$$

Recall that a Bessel fusion sequence $V = \{(V_j, v_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ is said to be an *alternate dual* of W, if for each $f \in \mathscr{H}$, we have

$$f = \sum_{j\in\mathbb{J}} v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f).$$

For any $\mathbb{I} \subset \mathbb{J}$, let $\mathbb{I}^c = \mathbb{J}\backslash\mathbb{I}$, and define an operator in $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ associated with $\mathbb{I}$ and W in the following form:

$$S_W^{\mathbb{I}} : \mathscr{H} \to \mathscr{H}, \quad S_W^{\mathbb{I}} f = \sum_{j\in\mathbb{I}} \omega_j^2 \pi_{W_j}(f), \quad \forall f \in \mathscr{H}.$$

Since

$$\begin{aligned}\langle S_W^{\mathbb{I}} f, f\rangle &= \left\langle \sum_{j\in\mathbb{I}} \omega_j^2 \pi_{W_j}(f), f \right\rangle = \sum_{j\in\mathbb{I}} \langle \omega_j^2 \pi_{W_j}(f), f\rangle \\ &= \sum_{j\in\mathbb{I}} \omega_j^2 \langle \pi_{W_j}(f), f\rangle = \sum_{j\in\mathbb{I}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \\ &\geq 0\end{aligned}$$

for any $f \in \mathscr{H}$, meaning that $S_W^{\mathbb{I}}$ is positive.

Balan et al. [4] showed us a surprising inequality for Parseval frames when they further investigated the remarkable Parseval frames identity arising from their study on efficient algorithms for signals reconstructions, which was later extended to canonical dual frames and alternate dual frames [16]. After above works, much attention has been paid to the generalization of those inequalities [15, 21, 25, 27, 28].

Recently, the authors in [22] presented some inequalities for fusion frames in Hilbert C^*-modules with a scalar in a finite interval, borrowing the ideas from [21, 25]. The purpose of this paper is to establish some new inequalities for fusion frames in Hilbert C^*-modules with a scalar in $\mathbb{R}$, the set of real numbers, and the motivation is derived from an observation of Theorem 4 in [22], which we list as follows.

Theorem 1.1. *Let $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ and $V = \{(V_j, v_j)\}_{j\in\mathbb{J}}$ be an alternate dual fusion frame of W. Then for any $\lambda \in [0,1]$, for any bounded sequence $\{a_j\}_{j\in\mathbb{J}}$ and any $f \in \mathscr{H}$, we have*

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{J}} a_j v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle + \Big|\sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j)v_j\omega_j \pi_{V_j} S_W^{-1}\pi_{W_j}(f)\Big|^2 \\
=&\mathrm{Re}\sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle + \Big|\sum_{j\in\mathbb{J}} a_j v_j\omega_j \pi_{V_j} S_W^{-1}\pi_{W_j}(f)\Big|^2 \\
\geq&(2\lambda - \lambda^2)\mathrm{Re}\sum_{j\in\mathbb{J}} a_j v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&+ (1-\lambda^2)\mathrm{Re}\sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle.
\end{aligned}$$

The representation of above inequality is wrong since, the sums involved take values in the C^*-algebra $\mathscr{A}$ rather than the field of complex numbers, one can not take the real parts on them. We provide a correction to Theorem 1.1 in next section, and we also give some new double inequalities for fusion frames in Hilbert C^*-modules, which differ in structure from previous ones on this topic.

2. The Main Results and Their Proofs

We begin with a simple result on adjointable operators, which shows that the condition "self-adjoint operator" in Lemma 2 of [22] can be deleted, and that the scalar λ can take values in $\mathbb{R}$, not merely in the interval $[0,1]$.

Lemma 2.1. *Suppose that $U, V \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ and that $U + V = \mathrm{Id}_{\mathscr{H}}$. Then for any $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned}
U^*U + \lambda(V^* + V) &= V^*V + (1-\lambda)(U^* + U) + (2\lambda - 1)\mathrm{Id}_{\mathscr{H}} \\
&\geq (2\lambda - \lambda^2)\mathrm{Id}_{\mathscr{H}}.
\end{aligned}$$

Proof. The proof is similar to Proposition 3.6 in [25], we omit the details. □

Theorem 1.1 can be corrected as follows.

Theorem 2.2. *Let $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ and $V = \{(V_j, v_j)\}_{j\in\mathbb{J}}$ be an alternate dual fusion frame of W. Then for each $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathscr{A})$, for any $\lambda \in \mathbb{R}$ and any $f \in \mathscr{H}$, we have*

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j)v_j\omega_j \pi_{V_j} S_W^{-1}\pi_{W_j}(f)\Big|^2 + \sum_{j\in\mathbb{J}}\langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle (a_j v_j \omega_j)^* \\
=&\Big|\sum_{j\in\mathbb{J}} a_j v_j\omega_j \pi_{V_j} S_W^{-1}\pi_{W_j}(f)\Big|^2 + \sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
\geq&(2\lambda - \lambda^2)\sum_{j\in\mathbb{J}} a_j v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&+ (1 + \lambda - \lambda^2)\sum_{j\in\mathbb{J}}(1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&- \lambda\sum_{j\in\mathbb{J}}\langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle((1_{\mathscr{A}} - a_j)v_j\omega_j)^*. \qquad (2)
\end{aligned}$$

Proof. We define two adjointable operators $L_1, L_2 : \mathscr{H} \to \mathscr{H}$ as follows.

$$L_1 f = \sum_{j\in\mathbb{J}} a_j v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f), \quad L_2 f = \sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j) v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f). \tag{3}$$

Then, clearly, $L_1 + L_2 = \mathrm{Id}_{\mathscr{H}}$. By Lemma 2.1,

$$|L_1 f|^2 + \lambda(\langle L_2 f, f\rangle + \langle f, L_2 f\rangle) = |L_2 f|^2 + (1-\lambda)(\langle L_1 f, f\rangle + \langle f, L_1 f\rangle) + (2\lambda - 1)|f|^2$$

for any $f \in \mathscr{H}$ and any $\lambda \in \mathbb{R}$. Therefore

$$\begin{aligned}
|L_1 f|^2 &= |L_2 f|^2 + (1-\lambda)(\langle L_1 f, f\rangle + \langle f, L_1 f\rangle) \\
&\quad + (2\lambda - 1)|f|^2 - \lambda(\langle L_2 f, f\rangle + \langle f, L_2 f\rangle) \\
&= |L_2 f|^2 + \langle L_1 f, f\rangle + \langle f, L_1 f\rangle - \lambda(\langle L_1 f, f\rangle + \langle L_2 f, f\rangle) \\
&\quad - \lambda(\langle f, L_1 f\rangle + \langle f, L_2 f\rangle) + (2\lambda - 1)|f|^2 \\
&= |L_2 f|^2 + \langle L_1 f, f\rangle + \langle f, L_1 f\rangle - 2\lambda|f|^2 + (2\lambda - 1)|f|^2 \\
&= |L_2 f|^2 + \langle L_1 f, f\rangle + \langle f, L_1 f\rangle - \langle L_1 f, f\rangle - \langle L_2 f, f\rangle.
\end{aligned}$$

It follows that

$$|L_2 f|^2 + \langle f, L_1 f\rangle = |L_1 f|^2 + \langle L_2 f, f\rangle,$$

which gives

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j) v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\right|^2 + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle (a_j v_j \omega_j)^* \\
=&\left|\sum_{j\in\mathbb{J}} a_j v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\right|^2 + \sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle.
\end{aligned}$$

For the inequality in (2), we apply Lemma 2.1 again,

$$\begin{aligned}
|L_1 f|^2 &\geq (2\lambda - \lambda^2)|f|^2 - \lambda(\langle L_2 f, f\rangle + \langle f, L_2 f\rangle) \\
&= (2\lambda - \lambda^2)\langle L_1 f, f\rangle + (2\lambda - \lambda^2)\langle L_2 f, f\rangle - \lambda\langle L_2 f, f\rangle - \lambda\langle f, L_2 f\rangle \\
&= (2\lambda - \lambda^2)\langle L_1 f, f\rangle + (\lambda - \lambda^2)\langle L_2 f, f\rangle - \lambda\langle f, L_2 f\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{J}} a_j v_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\right|^2 + \sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
\geq&(2\lambda - \lambda^2)\langle L_1 f, f\rangle + (1 + \lambda - \lambda^2)\langle L_2 f, f\rangle - \lambda\langle f, L_2 f\rangle \\
=&(2\lambda - \lambda^2)\sum_{j\in\mathbb{J}} a_j v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&+ (1 + \lambda - \lambda^2)\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j) v_j \omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&- \lambda\sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle ((1_{\mathscr{A}} - a_j) v_j \omega_j)^*,
\end{aligned}$$

and the proof is completed. □

We can immediately arrive at the following result, if in Theorem 2.2 we take $\mathbb{I} \subset \mathbb{J}$ and

$$a_j = \begin{cases} 1_{\mathscr{A}}, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases}$$

Corollary 2.3. *Let $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ and $V = \{(V_j, \upsilon_j)\}_{j\in\mathbb{J}}$ be an alternate dual fusion frame of W. Then for any $\mathbb{I} \subset \mathbb{J}$, for any $\lambda \in \mathbb{R}$ and any $f \in \mathscr{H}$, we have*

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{I}^c} \upsilon_j\omega_j\pi_{V_j}S_W^{-1}\pi_{W_j}(f)\right|^2 + \sum_{j\in\mathbb{I}}\langle\pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle(\upsilon_j\omega_j)^* \\
=&\left|\sum_{j\in\mathbb{I}} \upsilon_j\omega_j\pi_{V_j}S_W^{-1}\pi_{W_j}(f)\right|^2 + \sum_{j\in\mathbb{I}^c}\upsilon_j\omega_j\langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
\geq&(2\lambda - \lambda^2)\sum_{j\in\mathbb{I}}\upsilon_j\omega_j\langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&+ (1 + \lambda - \lambda^2)\sum_{j\in\mathbb{I}^c}\upsilon_j\omega_j\langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&- \lambda\sum_{j\in\mathbb{I}^c}\langle\pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle(\upsilon_j\omega_j)^*.
\end{aligned}$$

Remark 2.4. *Corollary 2.3 is a correction of Corollary 4 in [22].*

The following three double inequalities for fusion frames in Hilbert C^*-modules admit new structures, compared with those in Theorems 3, 5, and 6 of [22].

Theorem 2.5. *Let $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$. Then for each $\lambda \in \mathbb{R}$, for any $\mathbb{I} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, we have*

$$\begin{aligned}
&\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(f), \pi_{W_j}(f)\rangle - 2\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f), \pi_{W_j}(f)\rangle \\
\leq&\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f), \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f)\rangle - 2\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f), \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle \\
\leq&(1 + \lambda^2)\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(f), \pi_{W_j}(f)\rangle + (\lambda^2 - 2\lambda - 1)\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f), \pi_{W_j}(f)\rangle.
\end{aligned}\tag{4}$$

Proof. Since $S_W^{\mathbb{I}} + S_W^{\mathbb{I}^c} = S_W$, we conclude that $S_W^{-\frac{1}{2}}S_W^{\mathbb{I}}S_W^{-\frac{1}{2}} + S_W^{-\frac{1}{2}}S_W^{\mathbb{I}^c}S_W^{-\frac{1}{2}} = \mathrm{Id}_{\mathscr{H}}$. It is obvious that $U = S_W^{-\frac{1}{2}}S_W^{\mathbb{I}}S_W^{-\frac{1}{2}}$ and $V = S_W^{-\frac{1}{2}}S_W^{\mathbb{I}^c}S_W^{-\frac{1}{2}}$ are positive and commutable. Hence

$$0 \leq UV = U - U^2 = S_W^{-\frac{1}{2}}(S_W^{\mathbb{I}} - S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}})S_W^{-\frac{1}{2}}.$$

For each $f \in \mathscr{H}$, there is $g \in \mathscr{H}$ such that $f = S_W^{-\frac{1}{2}}g$. Since

$$\begin{aligned}
\langle(S_W^{\mathbb{I}} - S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}})f, f\rangle &= \langle(S_W^{\mathbb{I}} - S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}})S_W^{-\frac{1}{2}}g, S_W^{-\frac{1}{2}}g\rangle \\
&= \langle S_W^{-\frac{1}{2}}(S_W^{\mathbb{I}} - S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}})S_W^{-\frac{1}{2}}g, g\rangle \\
&\geq 0,
\end{aligned}$$

it follows that $S_W^{\mathbb{I}} - S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}} \geq 0$, that is, $S_W^{\mathbb{I}} \geq S_W^{\mathbb{I}}S_W^{-1}S_W^{\mathbb{I}}$.

Thus

$$\begin{aligned}
&\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f),\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f)\rangle-2\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f),\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle\\
=&\langle S_WS_W^{-1}S_W^{\mathbb{I}^c}f,S_W^{-1}S_W^{\mathbb{I}^c}f\rangle-2\langle S_WS_W^{-1}S_W^{\mathbb{I}}f,S_W^{-1}S_W^{\mathbb{I}}f\rangle\\
=&\langle S_W^{-1}S_W^{\mathbb{I}^c}f,S_W^{\mathbb{I}^c}f\rangle-2\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle\\
=&\langle S_W^{-1}(S_W-S_W^{\mathbb{I}})f,(S_W-S_W^{\mathbb{I}})f\rangle-2\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle\\
=&\langle S_Wf,f\rangle-2\langle S_W^{\mathbb{I}}f,f\rangle+\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle-2\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle\\
=&\langle S_W^{\mathbb{I}^c}f,f\rangle-2\langle S_W^{\mathbb{I}}f,f\rangle+[\langle S_W^{\mathbb{I}}f,f\rangle-\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle]\\
\geq&\langle S_W^{\mathbb{I}^c}f,f\rangle-2\langle S_W^{\mathbb{I}}f,f\rangle=\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle-2\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle
\end{aligned}\tag{5}$$

for each $f\in\mathscr{H}$. Now replacing U,V and f by $S_W^{-\frac12}S_W^{\mathbb{I}}S_W^{-\frac12}$, $S_W^{-\frac12}S_W^{\mathbb{I}^c}S_W^{-\frac12}$ and $S_W^{\frac12}f$ respectively in Lemma 2.1 yields

$$\begin{aligned}
\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle&=\langle S_W^{-\frac12}S_W^{\mathbb{I}}S_W^{-\frac12}S_W^{\frac12}f,S_W^{-\frac12}S_W^{\mathbb{I}}S_W^{-\frac12}S_W^{\frac12}f\rangle\\
&\geq(2\lambda-\lambda^2)\langle S_W^{\frac12}f,S_W^{\frac12}f\rangle-\lambda(\langle S_W^{-\frac12}S_W^{\mathbb{I}^c}S_W^{-\frac12}S_W^{\frac12}f,S_W^{\frac12}f\rangle\\
&\quad+\langle S_W^{\frac12}f,S_W^{-\frac12}S_W^{\mathbb{I}^c}S_W^{-\frac12}S_W^{\frac12}f\rangle)\\
&=(2\lambda-\lambda^2)\langle S_Wf,f\rangle-2\lambda\langle S_W^{\mathbb{I}^c}f,f\rangle\\
&=(2\lambda-\lambda^2)\langle S_W^{\mathbb{I}}f,f\rangle-\lambda^2\langle S_W^{\mathbb{I}^c}f,f\rangle,
\end{aligned}\tag{6}$$

and consequently,

$$\begin{aligned}
&\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f),\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}^c}f)\rangle-2\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f),\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle\\
=&\langle S_W^{\mathbb{I}^c}f,f\rangle-\langle S_W^{\mathbb{I}}f,f\rangle-\langle S_W^{-1}S_W^{\mathbb{I}}f,S_W^{\mathbb{I}}f\rangle\\
\leq&\langle S_W^{\mathbb{I}^c}f,f\rangle-\langle S_W^{\mathbb{I}}f,f\rangle-(2\lambda-\lambda^2)\langle S_W^{\mathbb{I}}f,f\rangle+\lambda^2\langle S_W^{\mathbb{I}^c}f,f\rangle\\
=&(1+\lambda^2)\langle S_W^{\mathbb{I}^c}f,f\rangle+(\lambda^2-2\lambda-1)\langle S_W^{\mathbb{I}}f,f\rangle\\
=&(1+\lambda^2)\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle+(\lambda^2-2\lambda-1)\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle.
\end{aligned}$$

This result and (5) conclude (4). □

Theorem 2.6. *Let $W=\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$. Then for each $\lambda\in\mathbb{R}$, for any $\mathbb{I}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$\begin{aligned}
&(2\lambda+1)\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle-(1+\lambda^2)\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle\\
\leq&\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f),\pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle-\sum_{j\in\mathbb{I}^c}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle\\
\leq&(\lambda^2-1)\sum_{j\in\mathbb{J}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle+(3-2\lambda)\sum_{j\in\mathbb{I}}\omega_j^2\langle\pi_{W_j}(f),\pi_{W_j}(f)\rangle.
\end{aligned}$$

Proof. For each $f \in \mathscr{H}$, by (6) we obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{J}} \omega_j^2 \langle \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f), \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle - \sum_{j\in\mathbb{I}^c} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \\
=&\langle S_W^{-1}S_W^{\mathbb{I}}f, S_W^{\mathbb{I}}f\rangle - \langle S_W^{\mathbb{I}^c}f, f\rangle \\
\geq&(2\lambda - \lambda^2)\langle S_W^{\mathbb{I}}f, f\rangle - \lambda^2\langle S_W^{\mathbb{I}^c}f, f\rangle - \langle S_W^{\mathbb{I}^c}f, f\rangle \\
=&(2\lambda + 1)\langle S_W^{\mathbb{I}}f, f\rangle - (1+\lambda^2)\langle S_W f, f\rangle \\
=&(2\lambda + 1)\sum_{j\in\mathbb{I}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \\
&- (1+\lambda^2)\sum_{j\in\mathbb{J}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle.
\end{aligned}$$

Taking $S_W^{-\frac{1}{2}}S_W^{\mathbb{I}^c}S_W^{-\frac{1}{2}}$, $S_W^{-\frac{1}{2}}S_W^{\mathbb{I}}S_W^{-\frac{1}{2}}$ and $S_W^{\frac{1}{2}}f$ instead of V, U and f respectively in Lemma 2.1 leads to

$$\begin{aligned}
\langle S_W^{-1}S_W^{\mathbb{I}^c}f, S_W^{\mathbb{I}^c}f\rangle &= \langle S_W^{-\frac{1}{2}}S_W^{\mathbb{I}^c}S_W^{-\frac{1}{2}}S_W^{\frac{1}{2}}f, S_W^{-\frac{1}{2}}S_W^{\mathbb{I}^c}S_W^{-\frac{1}{2}}S_W^{\frac{1}{2}}f\rangle \\
&\geq [(2\lambda - \lambda^2) - (2\lambda - 1)]\langle S_W^{\frac{1}{2}}f, S_W^{\frac{1}{2}}f\rangle \\
&\quad - (1-\lambda)(\langle S_W^{-\frac{1}{2}}S_W^{\mathbb{I}}S_W^{-\frac{1}{2}}S_W^{\frac{1}{2}}f, S_W^{\frac{1}{2}}f\rangle \\
&\quad\quad + \langle S_W^{\frac{1}{2}}f, S_W^{-\frac{1}{2}}S_W^{\mathbb{I}}S_W^{-\frac{1}{2}}S_W^{\frac{1}{2}}f\rangle) \\
&= (1-\lambda^2)\langle S_W f, f\rangle - (1-\lambda)(\langle S_W^{\mathbb{I}}f, f\rangle + \langle f, S_W^{\mathbb{I}}f\rangle) \\
&= (1-\lambda^2)\langle S_W f, f\rangle - 2(1-\lambda)\langle S_W^{\mathbb{I}}f, f\rangle \\
&= (1-\lambda^2)\langle (S_W^{\mathbb{I}^c} + S_W^{\mathbb{I}})f, f\rangle - 2(1-\lambda)\langle S_W^{\mathbb{I}}f, f\rangle \\
&= (2\lambda - \lambda^2 - 1)\langle S_W^{\mathbb{I}}f, f\rangle + (1-\lambda^2)\langle S_W^{\mathbb{I}^c}f, f\rangle.
\end{aligned}$$

From this and taking into account the fact that $S_W^{\mathbb{I}^c}S_W^{-1}S_W^{\mathbb{I}^c} \leq S_W^{\mathbb{I}^c}$, we get

$$\begin{aligned}
&\sum_{j\in\mathbb{J}} \omega_j^2 \langle \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f), \pi_{W_j}(S_W^{-1}S_W^{\mathbb{I}}f)\rangle - \sum_{j\in\mathbb{I}^c} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \\
=&\langle S_W^{-1}S_W^{\mathbb{I}}f, S_W^{\mathbb{I}}f\rangle - \langle S_W^{\mathbb{I}^c}f, f\rangle \\
\leq&\langle S_W^{\mathbb{I}}f, f\rangle - \langle S_W^{-1}S_W^{\mathbb{I}^c}f, S_W^{\mathbb{I}^c}f\rangle \\
\leq&\langle S_W^{\mathbb{I}}f, f\rangle - (2\lambda - \lambda^2 - 1)\langle S_W^{\mathbb{I}}f, f\rangle - (1-\lambda^2)\langle S_W^{\mathbb{I}^c}f, f\rangle \\
=&(\lambda^2 - 1)\langle S_W f, f\rangle + (3 - 2\lambda)\langle S_W^{\mathbb{I}}f, f\rangle \\
=&(\lambda^2 - 1)\sum_{j\in\mathbb{J}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle \\
&+ (3 - 2\lambda)\sum_{j\in\mathbb{I}} \omega_j^2 \langle \pi_{W_j}(f), \pi_{W_j}(f)\rangle.
\end{aligned}$$

This completes the proof. □

Theorem 2.7. *Let* $W = \{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ *be a fusion frame for* $\mathscr{H}$ *and* $V = \{(V_j, \upsilon_j)\}_{j\in\mathbb{J}}$ *be an alternate dual fusion frame* *of W. Then for each* $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathscr{A})$*, for any* $\lambda \in \mathbb{R}$ *and any* $f \in \mathscr{H}$*, we have*

$$\begin{aligned}\frac{3}{4}|f|^2 &\le \Big|\sum_{j\in\mathbb{J}} a_j \upsilon_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\Big|^2 + \frac{1}{2}\Big(\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j)\upsilon_j\omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle ((1_{\mathscr{A}} - a_j)\upsilon_j\omega_j)^*\Big) \\ &= \Big|\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j)\upsilon_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\Big|^2 + \frac{1}{2}\Big(\sum_{j\in\mathbb{J}} a_j\upsilon_j\omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle (a_j\upsilon_j\omega_j)^*\Big) \\ &\le \frac{3 + \|L_1 - L_2\|^2}{4}|f|^2,\end{aligned}$$

where L_1 *and* L_2 *are given in (3).*

Proof. Since $L_1 + L_2 = \mathrm{Id}_{\mathscr{H}}$, we have

$$\begin{aligned}&\Big|\sum_{j\in\mathbb{J}} a_j \upsilon_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\Big|^2 + \frac{1}{2}\Big(\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j)\upsilon_j\omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle ((1_{\mathscr{A}} - a_j)\upsilon_j\omega_j)^*\Big) \\ &= |L_1 f|^2 + \frac{1}{2}(\langle L_2 f, f\rangle + \langle f, L_2 f\rangle) \\ &= \langle f - L_2 f, f - L_2 f\rangle + \frac{1}{2}\langle L_2 f, f\rangle + \frac{1}{2}\langle f, L_2 f\rangle \\ &= |L_2 f|^2 + \frac{1}{2}(\langle f, f\rangle - \langle L_2 f, f\rangle) + \frac{1}{2}(\langle f, f\rangle - \langle f, L_2 f\rangle) \\ &= |L_2 f|^2 + \frac{1}{2}(\langle L_1 f, f\rangle + \langle f, L_1 f\rangle) \\ &= \Big|\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j)\upsilon_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\Big|^2 + \frac{1}{2}\Big(\sum_{j\in\mathbb{J}} a_j\upsilon_j\omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle (a_j\upsilon_j\omega_j)^*\Big)\end{aligned}$$

for each $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathscr{A})$, for any $\lambda \in \mathbb{R}$ and any $f \in \mathscr{H}$. Now by Lemma 2.1,

$$\begin{aligned}&\Big|\sum_{j\in\mathbb{J}} a_j \upsilon_j \omega_j \pi_{V_j} S_W^{-1} \pi_{W_j}(f)\Big|^2 + \frac{1}{2}\Big(\sum_{j\in\mathbb{J}} (1_{\mathscr{A}} - a_j)\upsilon_j\omega_j \langle S_W^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad + \sum_{j\in\mathbb{J}} \langle \pi_{V_j}(f), S_W^{-1}\pi_{W_j}(f)\rangle ((1_{\mathscr{A}} - a_j)\upsilon_j\omega_j)^*\Big) \\ &= |L_1 f|^2 + \frac{1}{2}(\langle L_2 f, f\rangle + \langle f, L_2 f\rangle) \\ &\ge \frac{3}{4}|f|^2.\end{aligned}$$

The opposite inequality follows from the following calculation:

$$
\begin{aligned}
&\Big|\sum_{j\in\mathbb{J}} a_j\upsilon_j\omega_j\pi_{V_j}S_W^{-1}\pi_{W_j}(f)\Big|^2+\frac{1}{2}\Big(\sum_{j\in\mathbb{J}}(1_{\mathscr{A}}-a_j)\upsilon_j\omega_j\langle S_W^{-1}\pi_{W_j}(f),\pi_{V_j}(f)\rangle\\
&+\sum_{j\in\mathbb{J}}\langle\pi_{V_j}(f),S_W^{-1}\pi_{W_j}(f)\rangle((1_{\mathscr{A}}-a_j)\upsilon_j\omega_j)^*\Big)\\
=&\langle L_1f,L_1f\rangle+\frac{1}{2}\langle L_2f,f\rangle+\frac{1}{2}\langle f,L_2f\rangle\\
=&\langle L_1f,L_1f\rangle+\frac{1}{2}\langle f,f\rangle-\frac{1}{2}\langle L_1f,f\rangle+\frac{1}{2}\langle f,f\rangle-\frac{1}{2}\langle f,L_1f\rangle\\
=&\langle f,f\rangle-\frac{1}{2}(\langle L_1f,f\rangle-\langle L_1f,L_1f\rangle)-\frac{1}{2}(\langle f,L_1f\rangle-\langle L_1f,L_1f\rangle)\\
=&\langle f,f\rangle-\frac{1}{2}\langle L_1f,L_2f\rangle-\frac{1}{2}\langle L_2f,L_1f\rangle\\
=&\frac{3}{4}\langle f,f\rangle+\frac{1}{4}\langle(L_1+L_2)f,(L_1+L_2)f\rangle-\frac{1}{2}\langle L_1f,L_2f\rangle-\frac{1}{2}\langle L_2f,L_1f\rangle\\
=&\frac{3}{4}\langle f,f\rangle+\frac{1}{4}\langle(L_1-L_2)f,(L_1-L_2)f\rangle\\
\leq&\frac{3}{4}|f|^2+\frac{1}{4}\|L_1-L_2\|^2|f|^2\\
=&\frac{3+\|L_1-L_2\|^2}{4}|f|^2.
\end{aligned}
$$

□

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