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Hybrid Subgradient Method for Pseudomonotone Equilibrium Problem and Fixed Points of Relatively Nonexpansive Mappings in Banach Spaces

Murtala Haruna Harbau[a], Bashir Ali[b]

[a]*Department of Science and Technology Education, Bayero University, Kano, Nigeria*
[b]*Department of Mathematical Sciences, Bayero University, Kano, Nigeria*

Abstract. In this paper, we propose a new hybrid subgradient algorithm for finding a common point in the set of solutions of pseudomonotone equilibrium problem and the set of fixed points of relatively nonexpansive mapping in a real uniformly convex and uniformly smooth Banach spaces. Weak and strong convergence of the iterative scheme are established. Our results generalizes and improves several recent results in the literature.

1. Introduction

Let E be a real Banach space and E^* be the dual of E. Let K be a nonempty closed and convex subset of E. A bifunction $f : K \times K \to \mathbb{R}$ ia said to be an equilibrium bifunction if it satisfies $f(x, x) = 0$. The equilibrium problem with respect to f and K in the sense of Blum and Oettli [4] is a problem of finding $z \in K$ such that

$$f(z, y) \geq 0 \quad \forall\ y \in K. \tag{1}$$

We denote by $EP(f, K)$ to be the set of solutions of equilibrium problem (1), i.e.

$$EP(f, K) = \{z \in K : f(z, y) \geq 0\ \ \forall\ \ y \in K\}.$$

Many problems in economics, physics, transportation, engineering etc (see for example [4, 9, 28, 30] and the refrences contained therein) can be reformulated as equilibrium problems. the existence of a solutions of equillibrium problem and its characterization can be found in [13].

Let $T : K \to K$ be a map. T is called $L-$Lipchitzian if $\|Tx - Ty\| \leq L\|x - y\| \quad \forall\ x, y \in K$ for some $L > 0$. A point $x \in K$ is said to be a fixed point of T if $x = Tx$. The set of fixed points of T is denoted by $F(T)$, i.e $F(T) = \{x \in K : x = Tx\}$.

The problem of finding a common elements of the set of equilibrium and the set of fixed points of nonlinear mappings have recently and continue to be an attractive subject of researches, and various techniques have

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Email addresses: murtalaharbau@yahoo.com (Murtala Haruna Harbau), bashiralik@yahoo.com (Bashir Ali)

been developed and investigated for solving this problem. (see for example [1, 3, 7, 16, 17, 21, 22, 24, 27, 31] and the references contained therein). However, most of the algorithms for this type of problem are based on the proximal point method which combine Mann iterative procedure for the fixed point of nonexpansive mappings in which the convergence analysis has been considered if the bifunction f is monotone. This is because the proximal point method is not valid if the underlying bifunction f in question is pseudomonone see Wen, [25].
Tran et al, [23] introduced the so called extragradient method for pseudomonotone equilibrium problems in which the computation is expensive because it involved two projection maps defined on the constrained set. Santos and Scheimberg [18] proposed an inexact subgradient algorithm for solving a wide class of equilibrium problems which involved one projection map instead of the two projection maps introduced by Tran et al[23].
Recently, Wen [25] proposed a hybrid subgradient method for pseudomonotone equilibrium problems and multivalued nonexpansive mappings in Hilbert spaces. He studied the following iterative sequence:

$$\begin{cases} x_0 \in K \\ w_n \in \partial_{\epsilon_n} f(x_n, .)x_n \\ u_n = P_K(x_n - \gamma_n w_n), \qquad \gamma_n = \frac{\beta_n}{max\{\sigma_n, \|w_n\|\}} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \end{cases} \tag{2}$$

where $T_n = T_{n(modN)}$, $z_n \in T_n u_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{\sigma_n\}$ are nonnegative sequences. He proved weak and strong convergence of the scheme (2) under some mild condtions on the pseudomonotone bifunction f.

In this paper, motivated by the above results, we extend the subgradient method for pseudomonotone equilibrium problems from real Hilbert spaces to the framework of more general Banach spaces than Hilbert spaces. In fact, we propose an algoritm for finding a common point point in the set of psedomonotone equilibrium problem and the set of fixed points of relatively nonexpansive mapping in real uniformly convex Banach spaces and uniformly smooth Banach spaces.

2. Preliminaries

Let E be a real Banach space and E^* be the dual of E. Let K be a nonempty closed and convex subset of E. We denote by $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle ., . \rangle$ denotes the duality pairing between the element of E and that of E^*. It is well known that $J(x)$ is nonempty for each $x \in E$, see [22]. We denote weak and strong convergence by $\rightharpoonup$ and $\to$ respectively.

Let $S(E)$ be a unit sphere centered at the origin. A Banach space is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x,\ y \in S(E)$ and $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(t) = \inf\Big\{1 - \frac{1}{2}\|x + y\| : \|x\| = 1 = \|y\|,\ \|x - y\| \geq \epsilon\Big\},\ \forall\ t \in [0, 2].$$

E is called uniformly convex if $\delta_E(t) \geq 0\ \forall\ t \in [0, 2]$ and p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(t) \geq c_p t^p\ \forall\ t \in [0, 2]$. Note that every p-uniformly convex Banach space is uniformly convex and every uniformly convex is strictly convex and reflexive. The modulus of smoothness $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = \sup\Big\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\Big\}.$$

E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$ and E is $q-$uniformly smooth if there exists $d_q > 0$ such that $\rho_E(\tau) \leq d_q \tau^q$. It is well known that if E is $q-$uniformly smooth, then $q \leq 2$ and E uniformly smooth.

We know that (See for example [8], if E is smooth, strictly convex and reflexive, then J is single-valued, one-to-one and onto respectively and J^{-1} is also single-valued, one-to-one, onto and it is the duality mapping from E^* into E. In addition if E is uniformly smooth, then the norm on E is fr$\acute{e}$chet differentiable and J is uniformly norm-to-norm continuous on bounded subsets of E and E is uniformly smooth if and only if E^* is uniformly convex. Furthermore if E has a uniformly G$\hat{a}$teaux differentiable norm, then J is norm-to-weak* uniformly continuous on bounded subsets of E.

Let E be a smooth Banach space and K be a closed convex subset of E. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \ \forall\ x, y \in E, \tag{3}$$

is called Lyapunov bifunction introduced by Alber [2], where J is the normalized duality mapping. Observe from the definition of ϕ in (3) above, we have that,

$$\phi(x, y) = \phi(z, y) - \phi(z, x) + 2\langle x - z, Jx - Jy\rangle, \quad \forall x,\ y,\ z \in E, \ \ and \tag{4}$$

$$\Big(||x|| - ||y||\Big)^2 \leq \phi(x, y) \leq \Big(||x|| + ||y||\Big)^2, \ \ \forall x,\ y \in E \tag{5}$$

Follwing Alber [2], the generalized projection $\Pi_K : E \to K$ is a mapping defined by

$$\Pi_K(x) = arg\ \min_{y\in K}\phi(y, x) \ \ \forall x \in E.$$

Remark 2.1. *(1) If E is a Hilbert space, then $\phi(y, x) = ||y - x||^2$, and the generalized projection reduces to metric projection P_K of E onto K.*

(2) If E is smooth and strictly convex, then $\phi(x, y) = 0$ if and only if $x = y\ \forall x, y \in E$, see for example[22]

Following Alber [2], the function $G : E \times E^* \to [0, +\infty)$ is defined by

$$G(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \ \ \forall x \in E \text{ and } x^* \in E^*.$$

Then $G(x, x^*) = \phi(x, J^{-1}x^*)\ \forall x \in E$ and $x^* \in E^*$. It is well known that if E is reflexive, strictly convex and smooth Banach space with its dual E^*, then

$$G(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq G(x, x^* + y^*), \ \forall x \in E \ \ and\ x^*,\ y^* \in E^*. \tag{6}$$

Following Takahashi and Zembayashi [22], A fixed point $z \in K$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in K such that $x_n \rightharpoonup z$ and $\lim_{n\to\infty}||x_n - Tx_n|| = 0$. The set of all asymptotic fixed points of T is denoted by $\tilde{F}(T)$. T is relatively nonexpansive if $F(T) \neq \emptyset$, $F(T) = \tilde{F}(T)$ and $\phi(z, Tx) \leq \phi(z, x)\ \forall x \in K,\ z \in F(T)$. It is well known that (see for example [14]) if K is nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and $T : K \to K$ is relatively nonexpansive mapping, then $F(T)$ is closed and convex.

Definition 2.2. *A map $T : K \to K$ is said to be semi-compact if for any bounded sequence $\{x_n\} \subset K$, with $\lim_{n\to\infty}||x_n - Tx_n|| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges in K*

Definition 2.3. *(see [6, 11]) Let $\psi : K \to \mathbb{R}\cup\{\infty\}$ be a proper convex function. For a given $\epsilon > 0$, the ϵ-subdifferential of ψ at $x_0 \in D(\psi)$ is given by*

$$\partial_\epsilon\psi(x_0) = \{x \in K : \psi(y) - \psi(x_0) \geq \langle x, y - x_0\rangle - \epsilon \ \ \forall y \in K\}.$$

Remark 2.4. *It is known that if the function ψ is proper, lower semicontinuous and convex, then for each $x \in D(\psi)$ the ϵ-subdifferential $\partial_\epsilon\psi(x_0)$ is a nonempty closed convex set.*

Definition 2.5. *(Browder, [5]) The duality mapping J is said to be weakly sequentially continuous if for any sequence $\{x_n\}$ in E such that $x_n \rightharpoonup x$, implies $J(x_n) \overset{*}{\rightharpoonup} J(x)$, where $\overset{*}{\rightharpoonup}$ means weak* convergence.*

Definition 2.6. *A bifunction $f : K \times K \to \mathbb{R}$ is said to be;*

1. *γ-strongly monotone on K if there exists $\gamma > 0$ such that*
$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2 \ \ \forall \ \ x, y \in K,$$
2. *Monotone on K if*
$$f(x, y) + f(y, x) \leq 0 \ \ \forall \ \ x, y \in K,$$
3. *Pseudomonotone on K with respect to $x \in K$ if*
$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq 0 \ \ \forall \ \ y \in K,$$
4. *Pseudomonotone on K with respect to $B \subseteq K$, if it is pseudomonotone on K with respect to every $x \in B$ (See [18]).*

From Definition 2.6, we obviously have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). But in the following example, it is shown that (3) $\not\Rightarrow$ (2), i.e., the class of monotone bifunctions is a proper subclass of pseudomonotone bifunctions.

Example 2.7. *Let $E = \mathbb{R}$, $K = [\frac{1}{2}, 1]$. Define $f : K \times K \to \mathbb{R}$ by*
$$f(x, y) = x(x - y) \ \ \forall \ x, y \in K.$$

To show that f is pseudomonotone, we proceed as follows: let $f(x, y) \geq 0$. Since $x \geq \frac{1}{2}$, it follows that $(x - y) \geq 0$. As $y \geq \frac{1}{2}$ we have $f(y, x) = y(y - x) \leq 0$, showing that f is pseudomonotone. But f is not monotone since for $x \neq y$, $f(x, y) + f(y, x) = (x - y)^2 > 0$.
To study equilibrium problem (1), we make the following assumptions on the bifunction f:

(A1) $f(x, x) = 0$ for every $x \in K$ and $f(x, .)$ is convex and lower semicontinuous on K,

(A2) $f(., y)$ is weakly upper semicontinuous for every $y \in K$,

(A3) f is pseudomonotone on K with respect to $EP(f, K)$ and satisfies the strict paramonotonicity property i.e $f(y, x) = 0$ for $x \in EP(f, K)$ and $y \in K$ implies $y \in EP(f, K)$,

(A4) If $\{x_n\} \subseteq K$ is bounded and $\epsilon_n \to 0$ as $n \to \infty$, then the sequence $\{w_n\}$, $w_n \in \partial_{\epsilon_n} f(x_n, .)x_n$ is bounded.

In the sequel we will need the following lemmas:

Lemma 2.8. *[2] Let E be a strictly convex, smooth and reflexive Banach space and let K be a nonempty closed and convex subset of E. Let $x \in E$, then*
$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x) \quad \forall y \in K.$$

Lemma 2.9. *[29] Let E be a uniformly convex Banach space and $r > 0$, then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, +\infty)$ such that $g(0) = 0$ and*
$$\Big\|\sum_{i=1}^{N} \alpha_i x_i\Big\|^2 \leq \sum_{i=1}^{N} \alpha_i\|x_i\|^2 - \alpha_i\alpha_j g(\|x_i - x_j\|),$$
where $\alpha_i \in (0, 1)$, $\sum_{i=1}^{N} \alpha_i = 1$ and $x_i \in B_r(0)$, $\forall i \in \{1, 2, \ldots, N\}$,

Lemma 2.10. *[12] Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.11. *[20] Let E be a Banach space with a fréchet differentiable norm. For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) = \sup\left\{\left|\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, jx\rangle\right| : \|y\| = 1\right\}. \quad (7)$$

Then $\lim_{t\to 0}\beta^*(t) = 0$ and

$$\|x+h\|^2 \le \|x\|^2 + 2\langle h, jx\rangle + \|h\|\beta^*(\|h\|)\ \forall\ h \in E - \{0\}.$$

Remark 2.12. *In Hilbert space H and L_p, for $2 \le p < \infty$, β^* in (7) is estimated by $\beta^*(t) = t$ and $\beta^*(t) = (p-1)t$ for $t > 0$ respectively, see for example Shehu [19] for details. In our more general setting, in this paper we will assume $\beta^*(t) \le ct$ for $t > 0$ and $c > 1$.*

Lemma 2.13. *[26] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$a_{n+1} \le a_n + b_n,\ \ n \ge 0$. If $\sum_{n=0}^{\infty} b_n < \infty$. Then the $\lim_{n\to\infty} a_n$ exists.

Lemma 2.14. *Let E be a smooth and strictly convex Banach space such that the duality mapping J on E is weakly sequentially continuous and let $\{x_n\}$ be a sequence in E such that $x_n \rightharpoonup x$. Then*

$$\limsup_{n\to\infty}\phi(x, x_n) < \limsup_{n\to\infty}\phi(y, x_n)\ \ \forall x \neq y.$$

Proof. Let $x \neq y$, then from (4) we have

$$\phi(x, x_n) - \phi(y, x_n) = -\phi(y, x) + 2\langle x - y, Jx - Jx_n\rangle.$$

Since $x_n \rightharpoonup x$ and J is weakly sequentially continuous, we obtain

$$\lim_{n\to\infty}(\phi(x, x_n) - \phi(y, x_n)) = -\phi(y, x).$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty}\phi(x, x_n) &= \limsup_{n\to\infty}\Big(\phi(x, x_n) - \phi(y, x_n) + \phi(y, x_n)\Big)\\ &\le \limsup_{n\to\infty}(\phi(x, x_n) - \phi(y, x_n)) + \limsup_{n\to\infty}\phi(y, x_n)\\ &= -\phi(y, x) + \limsup_{n\to\infty}\phi(y, x_n).\end{aligned}$$

Since E is strictly convex, we have

$$\limsup_{n\to\infty}\phi(x, x_n) < \limsup_{n\to\infty}\phi(y, x_n).$$

This completes the proof.
□

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space E with dual E^*. Let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let $T_i : K \to K,\ i = 1, 2, 3, \ldots, N$ be a finite family of relatively nonexpansive mappings such that $\Gamma = \bigcap_{i=1}^{N} F(T_i) \cap EP(f, K) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be* *iteratively defined by*

$$\begin{cases} x_1 \in K \\ w_n \in \partial_{\epsilon_n} f(x_n, .)x_n \\ u_n = \Pi_K J^{-1}(Jx_n - \gamma_n w_n), & \gamma_n = \begin{cases} \frac{\beta_n}{\|w_n\|}, & w_n \neq 0 \\ 0, & Otherwise \end{cases} \\ x_{n+1} = \Pi_K J^{-1}(\alpha_{n,0} Ju_n + \sum_{i=1}^N \alpha_{n,i} JT_i u_n), \end{cases} \tag{8}$$

where $\alpha_{n,i} \in (\eta, 1-\eta)$, $\forall n \geq 1$ *for some* $\eta \in (0,1)$ *and* $\sum_{i=0}^N \alpha_{n,i} = 1$ *and* $\{\beta_n\}$, $\{\epsilon_n\}$ *are nonnegative sequences satisfying* $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty \beta_n^2 < \infty$ *and* $\sum_{n=1}^\infty \beta_n \epsilon_n < \infty$. *If* T_i, $i = 1, 2, \ldots, N$ *is* L_i-*Lipschitzian and the duality mapping* J *is weakly sequentially continuous, then*

1. *The sequence generated by* (8) *converges weakly to some* $\overline{x} \in \Gamma$;
2. *In addition if at least one of* T_i, $i = 1, 2, 3, \ldots, N$ *is semi-compact, then the sequence converges strongly to some* $\overline{x} \in \Gamma$.

Proof. Let $x^* \in \Gamma$. Then by Lemma 2.8, we obtain

$$\begin{aligned} \phi(x^*, u_n) &= \phi(x^*, \Pi_K J^{-1}(Jx_n - \gamma_n w_n)) \\ &\leq \phi(x^*, J^{-1}(Jx_n - \gamma_n w_n)) \\ &= \|x^*\|^2 - 2\langle x^*, Jx_n - \gamma_n w_n\rangle + \|Jx_n - \gamma_n w_n\|^2 \\ &= \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + 2\langle x^*, \gamma_n w_n\rangle + \|Jx_n - \gamma_n w_n\|^2. \end{aligned}$$

Since E is uniformly convex, E^* is uniformly smooth and taking $\beta^*(t) \leq ct$, $c > 1$ in Lemma 2.11, we have

$$\begin{aligned} \phi(x^*, u_n) &\leq \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + 2\langle x^*, \gamma_n w_n\rangle + \|x_n\|^2 \\ &- 2\langle x_n, \gamma_n w_n\rangle + \|\gamma_n w_n\|\beta^*(\|\gamma_n w_n\|) \\ &\leq \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + 2\langle x^*, \gamma_n w_n\rangle + \|x_n\|^2 \\ &- 2\langle x_n, \gamma_n w_n\rangle + c\gamma_n^2\|w_n\|^2 \\ &= \phi(x^*, x_n) + 2\gamma_n\langle x^* - x_n, w_n\rangle + c\gamma_n^2\|w_n\|^2. \end{aligned}$$

Since $w_n \in \partial_{\epsilon_n} f(x_n, .)x_n$, we have

$$\begin{aligned} \phi(x^*, u_n) &\leq \phi(x^*, x_n) + 2\gamma_n f(x_n, x^*) - 2\gamma_n f(x_n, x_n) + 2\gamma_n \epsilon_n + c\gamma_n^2\|w_n\|^2 \\ &= \phi(x^*, x_n) + 2\gamma_n f(x_n, x^*) + 2\gamma_n \epsilon_n + c\gamma_n^2\|w_n\|^2. \end{aligned} \tag{9}$$

On the other hand, from (8) and Lemma 2.8, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, \Pi_K J^{-1}(\alpha_{n,0} Ju_n + \sum_{i=1}^N \alpha_{n,i} JT_i u_n)) \\ &\leq \phi(x^*, J^{-1}(\alpha_{n,0} Ju_n + \sum_{i=1}^N \alpha_{n,i} JT_i u_n)) \\ &= \|x^*\|^2 - 2\langle x^*, \alpha_{n,0} Ju_n + \sum_{i=1}^N \alpha_{n,i} JT_i u_n\rangle \\ &\quad + \|\alpha_{n,0} Ju_n + \sum_{i=1}^N \alpha_{n,i} JT_i u_n\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|x^*\|^2 - 2\alpha_{n,0}\langle x^*, Ju_n\rangle + \sum_{i=1}^{N} \alpha_{n,i}\langle x^*, JT_iu_n\rangle \\ &\quad + \alpha_{n,0}\|u_n\|^2 - 2\sum_{i=1}^{N} \alpha_{n,i}\|T_iu_n\|^2 \\ &= \alpha_{n,0}\phi(x^*, u_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(x^*, T_iu_n). \end{aligned}$$

Since T_i is relatively nonexpansive for each $i = 1, 2, 3, \ldots, N$ we obtain

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \alpha_{n,0}\phi(x^*, u_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(x^*, u_n) \\ &= \phi(x^*, u_n). \end{aligned} \tag{10}$$

Putting (9) in (10) and using our assumption on γ_n, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) + 2\gamma_n f(x_n, x^*) + 2\gamma_n\epsilon_n + c\gamma_n^2\|w_n\|^2 \\ &\leq \phi(x^*, x_n) + 2\gamma_n f(x_n, x^*) + 2\gamma_n\epsilon_n + c\beta_n^2. \end{aligned} \tag{11}$$

Since $x^* \in \Gamma$, it implies $f(x^*, x_n) \geq 0\ \forall n \in \mathbb{N}$. By pseudomonotone property of f we get $f(x_n, x^*) \leq 0\ \forall n \in \mathbb{N}$. Hence

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) + 2\gamma_n\epsilon_n + c\beta_n^2. \tag{12}$$

From (12), taking $b_n = 2\gamma_n\epsilon_n + c\beta_n^2$ and using Lemma 2.13, we have $\lim_{n\to\infty} \phi(x^*, x_n)$ exists.

We note that from the scheme (8), Lemma 2.8, and Lemma 2.11

$$\begin{aligned} \phi(x_n, u_n) &= \phi(x_n, \Pi_K J^{-1}(Jx_n - \gamma_n w_n)) \\ &\leq \phi(x_n, J^{-1}(Jx_n - \gamma_n w_n)) \\ &= \|x_n\|^2 - 2\langle x_n, Jx_n - \gamma_n w_n\rangle + \|Jx_n - \gamma_n w_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jx_n\rangle + 2\langle x_n, \gamma_n w_n\rangle + \|Jx_n - \gamma_n w_n\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|x_n\|^2 - 2\langle x_n, Jx_n\rangle + 2\gamma_n\langle x_n, w_n\rangle + \|x_n\|^2 \\ &\quad - 2\gamma_n\langle x_n, w_n\rangle + c\gamma_n^2\|w_n\|^2 \\ &= c\gamma_n^2\|w_n\|^2 = c\beta_n^2. \end{aligned}$$

By our assumption $\sum_{n=1}^{\infty} \beta_n^2 < \infty$, it follows that

$$\lim_{n\to\infty} \phi(x_n, u_n) = 0. \tag{13}$$

Using Lemma 2.10, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{14}$$

Since $\{\phi(x^*, x_n)\}$ converges, it is bounded. Therefore, it follows that $\{x_n\}$ is bounded. Also from (14) and the fact that $T_i,\ i = 1, 2, 3, \ldots, N$ is relatively nonexpansive, we have $\{u_n\}$, $\{T_iu_n\}$ and $\{T_ix_n\}$ are all bounded.

Let $r = \max_{1\le i\le N}\sup_{n\ge 1}\{\|u_n\|,\ \|T_i u_n\|\}$. Since E is uniformly smooth Banach space, then E^* is uniformly convex Banach space, therefore from (8), Lemma 2.8 and Lemma 2.9, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, \Pi_K J^{-1}(\alpha_{n,0}Ju_n + \sum_{i=1}^{N}\alpha_{n,i}JT_iu_n)) \\
&\le \phi(x^*, J^{-1}(\alpha_{n,0}Ju_n + \sum_{i=1}^{N}\alpha_{n,i}JT_iu_n)) \\
&= \|x^*\|^2 - 2\langle x^*, \alpha_{n,0}Ju_n + \sum_{i=1}^{N}\alpha_{n,i}JT_iu_n\rangle \\
&\quad + \|\alpha_{n,0}Ju_n + \sum_{i=1}^{N}\alpha_{n,i}JT_iu_n\|^2
\end{aligned}$$

$$\begin{aligned}
&\le \|x^*\|^2 - 2\alpha_{n,0}\langle x^*, Ju_n\rangle + \sum_{i=1}^{N}\alpha_{n,i}\langle x^*, JT_iu_n\rangle \\
&\quad + \alpha_{n,0}\|u_n\|^2 - 2\sum_{i=1}^{N}\alpha_{n,i}\|T_iu_n\|^2 - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big) \\
&= \alpha_{n,0}\phi(x^*, u_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi(x^*, T_iu_n) - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big)
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_{n,0}\phi(x^*, u_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi(x^*, u_n) - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big) \\
&= \phi(x^*, u_n) - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big)
\end{aligned}$$

$$\begin{aligned}
&\le \phi(x^*, x_n) + 2\gamma_n\epsilon_n + c\gamma_n^2\|w_n\|^2 - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big) \\
&\le \phi(x^*, x_n) + 2\gamma_n\epsilon_n + c\beta_n^2 - \alpha_{n,0}\alpha_{n,i}g\Big(\|Ju_n - JT_iu_n\|\Big). \qquad (15)
\end{aligned}$$

From (15) and assumption that $\alpha_{n,0}, \alpha_{n,i} \in (\eta, 1-\eta)$, for some $\eta \in (0,1)$, we obtain

$$\eta^2 g\Big(\|Ju_n - JT_iu_n\|\Big) \le \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + 2\gamma_n\epsilon_n + c\beta_n^2.$$

Hence

$$0 \le \sum_{n=1}^{\infty} g\Big(\|Ju_n - JT_iu_n\|\Big) \le \phi(x^*, x_1) + 2\sum_{n=1}^{\infty}\gamma_n\epsilon_n + c\sum_{n=1}^{\infty}\beta_n^2. \qquad (16)$$

From (16), it follows that $\lim_{n\to\infty} g\Big(\|Ju_n - JT_iu_n\|\Big) = 0$ for each $i = 1,2,3,\dots N$. And using the property of g we have $\lim_{n\to\infty}\|Ju_n - JT_iu_n\| = 0$ for each $i = 1,2,3,\dots N$ Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\|u_n - T_iu_n\| = 0, \text{ for each } i = 1,2,3,\dots N. \qquad (17)$$

From (14) and (17), we obtain

$$\lim_{n\to\infty} \|x_n - T_i u_n\| = 0, \text{ for each } i = 1, 2, 3, \dots N. \tag{18}$$

Observe

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - T_i u_n\| + \|T_i u_n - T_i x_n\| \\ &\leq \|x_n - T_i u_n\| + L_i \|u_n - x_n\|. \end{aligned}$$

Using (13) and (17), we have

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \text{ for each } i = 1, 2, 3, \dots N. \tag{19}$$

Next we show $\limsup\limits_{n\to\infty} f(x_n, x^*) = 0$.

Since $\sum_{n=1}^{\infty} \beta_n \epsilon_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n = 0$, we get $\lim\limits_{n\to\infty} \epsilon_n = 0$. This together with boundedness of $\{x_n\}$ and (A4), we have $\{w_n\}$ is bounded. Therefore there exists $M > 0$ such that $\|w_n\| \leq M \quad \forall\, n \in \mathbb{N}$. From (11), we have

$$2\gamma_n \left[-f(x_n, x^*)\right] \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + 2\gamma_n \epsilon_n + c\beta_n^2.$$

Using the assumption on γ_n, we have

$$0 \leq \frac{2}{M} \sum_{n=1}^{\infty} \beta_n \left[-f(x_n, x^*)\right] \leq \phi(x^*, x_1) + 2 \sum_{n=1}^{\infty} \gamma_n \epsilon_n + c \sum_{n=1}^{\infty} \beta_n^2 < \infty.$$

By pseudomonotone property of f, $-f(x_n, x^*) \geq 0$. Since by our assumption $\sum_{n=1}^{\infty} \beta_n = \infty$, we obtain

$$\limsup_{n\to\infty} f(x_n, x^*) = 0. \tag{20}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\limsup\limits_{n\to\infty} f(x_n, x^*) = \lim\limits_{k\to\infty} f(x_{n_k}, x^*)$. Without loss of generality, assume $x_{n_k} \rightharpoonup q$ for some $q \in E$. The fact that K is closed and convex we have $q \in K$. Also by the assumption that $f(., x^*)$ is weakly upper semicontinuous, for $x^* \in \Gamma$, together with (20), we obtain

$$\begin{aligned} f(q, x^*) &\geq \limsup_{k\to\infty} f(x_{n_k}, x^*) \\ &= \lim_{k\to\infty} f(x_{n_k}, x^*) \\ &= \limsup_{n\to\infty} f(x_n, x^*) = 0. \end{aligned}$$

That is

$$f(q, x^*) \geq 0 \quad \forall\, x^* \in \Gamma.$$

By pseudomonotonicity of f, we have $f(q, x^*) \leq 0$. Hence $f(q, x^*) = 0$. Using (A3) it follows that $q \in EP(f, K)$.

On the other hand since T_i is relatively nonexpansive mapping, $x_{n_k} \rightharpoonup q$ and $\lim\limits_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$, for each $i = 1, 2, 3, \dots N$, we have $q \in \bigcap_{i=1}^{N} F(T_i)$.
Therefore, from the above discussions we obtain $q \in \bigcap_{i=1}^{N} F(T_i) \cap EP(f, K)$.

Finally, we show $x_n \rightharpoonup q$. Let w be another weak limit of $\{x_n\}$ and $q \neq w$. Then we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ as $j \to \infty$. Since $\lim\limits_{n\to\infty} \phi(x^*, x_n)$ exists for $x^* \in \Gamma$, we obtain from Lemma 2.14 that

$$\begin{aligned} \lim_{n\to\infty} \phi(q, x_n) &= \limsup_{k\to\infty} \phi(q, x_{n_k}) < \limsup_{k\to\infty} \phi(w, x_{n_k}) \\ &= \lim_{n\to\infty} \phi(w, x_n) = \limsup_{j\to\infty} \phi(w, x_{n_j}) \\ &< \limsup_{j\to\infty} \phi(q, x_{n_j}) = \lim_{n\to\infty} \phi(q, x_n) \end{aligned}$$

which is a contradiction, thus $q = w$. This completes the proof of conclusion 1.

To prove conclusion 2, assume T_h is semi-compact for some $h \in \{1, 2, 3, \ldots, N\}$. Since $\{x_n\}$ is bounded and $\|x_n - T_h x_n\| = 0$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \overline{u} \in K$. Since $x_n \rightharpoonup q$ we have $q = \overline{u}$. On the other hand, as $x_{n_j} \to q$ and J is norm-to-weak* uniformly continuous on bounded subsets of E, we get $\lim_{j\to\infty} \phi(q, x_{n_j}) = 0$. Since the limit of $\phi(q, x_n)$ exists, it follows that $\lim_{n\to\infty} \phi(q, x_n) = 0$ and consequently by Lemma 2.10 we obtain $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof.
□

Definition 3.2. *A mapping $T : K \to K$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset K$ such that $x_n \rightharpoonup x$ and $Tx_n \to 0$, implies $Tx = 0$.*

Remark 3.3. *It is well known if T is nonexpasive on H, then $I - T$ is demiclosed at 0 (see [10]).*

We know (see for example [15]) that every Hilbert space H satisfies Opial condition. If E is a real Hilbert space H together with Remark 3.3, Theorem 3.1 reduces to the following:

Corollary 3.4. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and let $T_i : K \to K$, $i = 1, 2, 3, \ldots, N$ be a finite family of nonexpansive mappings such that $\Gamma = \bigcap_{i=1}^N F(T_i) \cap EP(f, K) \neq \emptyset$. Let $\{x_n\}_{n=1}^\infty$ be iteratively defined by*

$$\begin{cases} x_1 \in K \\ w_n \in \partial_{\epsilon_n} f(x_n, .)x_n \\ u_n = P_K(x_n - \gamma_n w_n), \qquad \gamma_n = \begin{cases} \frac{\beta_n}{\|w_n\|}, & w_n \neq 0 \\ 0, & Otherwise \end{cases} \\ x_{n+1} = \alpha_{n,0} u_n + \sum_{i=1}^N \alpha_{n,i} T_i u_n, \end{cases} \tag{21}$$

where $\alpha_{n,i} \in (\eta, 1 - \eta)$, $\forall n \geq 1$ for some $\eta \in (0, 1)$ and $\sum_{i=0}^N \alpha_{n,i} = 1$ and $\{\beta_n\}$, $\{\epsilon_n\}$ are nonnegative sequences satisfying $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty \beta_n^2 < \infty$ and $\sum_{n=1}^\infty \beta_n \epsilon_n < \infty$, then

1. *The sequence generated by (21) converges weakly to some $\overline{x} \in \Gamma$;*
2. *In addition if at least one of T_i, $i = 1, 2, 3, \ldots, N$ is semi-compact, then the sequence converges strongly to some $\overline{x} \in \Gamma$.*

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