Filomat 36:10 (2022), 3527–3543
https://doi.org/10.2298/FIL2210527K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Solutions to the ∞-point Fractional BVP Posed on Half-Line via a Family of Measure of Noncompactness in the Hölder Space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$

Mahammad Khuddush[a], K. Rajendra Prasad[b], D. Leela[b]

[a]Department of Mathematics, Dr. Lankapalli Bullayya College of Engineering, Resupuvanipalem, Visakhapatnam, 530013, India
[b]Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, India

Abstract. This paper deals with the existence of solutions for the Riemann-Liouville fractional order boundary value problem with infinite-point boundary conditions posed on half-line via the concept of a family of measures of noncompactness in the space of functions $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ satisfying the Hölder condition and a generalized Darbo fixed point theorem.

1. Introduction

Fractional differential equations with boundary conditions have occupied an important role in the fractional calculus domain, since these problems appear in various applications of sciences and engineering: electricity, mechanics, finance, control theory, biology, chemistry, chemistry, economics [12, 15, 19, 38–41]. In recent years, there are certain papers and monographs dealing with the existence, uniqueness and stability analysis of fractional order nonlinear boundary value problems, see ([5, 17, 23–25, 31, 33, 34, 43, 44, 46]) and references therein.

Measures of noncompactness play an important area in fixed point theory and have many applications in various branches of nonlinear analysis, including differential equations, integral and integro-differential equations, optimization, etc. Roughly speaking, a measure of noncompactness is a function defined on the family of all nonempty and bounded subsets of a certain metric space such that it is equal to zero on the whole family of relatively compact sets. The concept of measure of noncompactness was first introduced by Kuratowski [26]. Later, the Italian mathematician Darbo [14] used the Kuratowski measure in order to investigate a class of operators (condensing operators) whose properties can be characterized as being intermediate between those of contraction and compact mappings. Darbo's fixed point theorem is useful in establishing existence results for different classes of operator equations. For details on various measures of noncompactness see [7].

Existence of solutions for differential and integral equations has been investigated by many authors in various types of measure of noncompactness. Benchohra et al. [8] investigated a class of boundary value

2020 *Mathematics Subject Classification*. Primary 47H08, 47H10; Secondary 45E10
Keywords. Fractional derivative, boundary value problem, Family of measure of noncompactness, Darbo fixed point theorem, Hölder space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$
Received: 27 June 2021; Revised: 27 December 2021; Accepted: 30 December 2021
Communicated by Hari M. Srivastava
Email addresses: khuddush89@gmail.com (Mahammad Khuddush), krajendra92@rediffmail.com (K. Rajendra Prasad), leelaravidadi@gmail.com (D. Leela)

problems for fractional differential equations involving nonlinear integral conditions using the technique associated with measures of weak noncompactness. Aghajani et al. [3] proved the solvability of a large class of nonlinear fractional integro-differential equations by establishing some fractional integral inequalities and using the nonlinear alternative of Leray-Schauder type. Li et al. [28] studied on the existence of mild solutions for fractional semilinear differential equations with nonlocal conditions. In [30], Mohiuddine et al. studied the existence of solutions of infinite systems of second-order differential equations in the Banach sequence space ℓ_p by using technique based upon measures of noncompactness in conjunction with a Darbo-type fixed point theorem. In [37], Sriviastava et al. obtained existence results for an infinite system of differential equations of order n with boundary conditions in the Banach spaces c_0 and ℓ_1 with the help of a technique associated with measures of noncompactness. B. Hazarika et al. [21] established existence of solution for infinite system of nonlinear integral equations in the Banach spaces ℓ_p, $p > 1$ with help of measure of noncompactness and generalized Meir-Keeler fixed point theorem later in [22], they studied nonlinear functional integral equation with help of measure of noncompactness, simulation function and generalized Darbo fixed point theorem. Wang et al. [45] applied a new variant flxed point theorem to investigate some fractional differential equations in Banach spaces. Liang et al. [29] studied the solvability for a coupled system of nonlinear fractional difierential equations in a Banach space using the measures of noncompactness and the well known fixed point theorem of Monch type. Borisut et al [10] studied fractional order boundary value problem based on Kransnoselskii's fixed point theorem and Darbo's fixed point theorem together and the idea of the measure of noncompactness.

In [16], Derbazi et al. established the existence of weak solutions for the following fractional boundary value problem

$$\begin{gathered} \mathtt{D}^{\alpha}_{0^+}\mathtt{z}(\mathtt{t}) = \mathtt{g}(\mathtt{t}, x(\mathtt{t})),\ 0 \le \mathtt{t} \le \mathtt{T},\ 1 < \alpha \le 2, \\ a_1\mathtt{z}(0) + b_1\mathtt{z}(\mathtt{T}) = \lambda_1 \mathtt{I}^{\beta_1}\mathtt{z}(\eta),\ \beta_1 > 0 \\ a_2\,\mathtt{D}^{\beta_2}_{0^+}\mathtt{z}(\xi) + b_2\,\mathtt{D}^{\beta_3}_{0^+}\mathtt{z}(\eta) = \lambda_2,\ 0 < \beta_2, \beta_3 \le 1, \end{gathered}$$

by using the Monch's fixed point theorem combined with the technique of measures of weak noncompactness. In [35], Prasad et al. investigated the existence of solutions for infinite systems of regular fractional Sturm-Liouville problems

$$\begin{gathered} \mathtt{D}^{\gamma}_{\mathtt{T}^-}\Big[\mathtt{p}_\mathtt{j}(\mathtt{t})\mathtt{D}^{\gamma}_{0^+}(\mathtt{z}_\mathtt{j}(\mathtt{t}))\Big] = \lambda_\mathtt{j}\mathtt{g}_\mathtt{j}(\mathtt{t}, \mathtt{z}(\mathtt{t})),\ 0 < \mathtt{t} < \mathtt{T},\ \gamma \in (0,1), \\ \mathtt{z}_\mathtt{j}(0) = 0,\ \mathtt{D}^{\gamma}_{0^+}\mathtt{z}_\mathtt{j}(\mathrm{T}) = 0,\ \mathtt{j} \in \mathbb{N}, \end{gathered}$$

by an application of Meir-Keeler fixed point theorem in the tempered sequence spaces. Recently, Salem et al. [36] investigated the existence of solutions for the infinite system

$$\begin{gathered} {}^{\rho_\mathtt{j}}\mathtt{D}^{\lambda_\mathtt{j}}\Big({}^{\rho_\mathtt{j}}\mathtt{D}^{\mu_\mathtt{j}} + \xi\Big)\mathtt{z}_\mathtt{j}(\mathtt{t}) = \mathtt{g}_\mathtt{j}(\mathtt{t}, \mathtt{z}_\mathtt{j}(\mathtt{t}), \varphi({}^{\rho}\mathtt{D}^{\nu}\mathtt{z}(\mathtt{t}))),\ 0 < \mathtt{t} < 1,\ 0 < \rho_\mathtt{j}, \mu_\mathtt{j}, \nu < 1 < \lambda_\mathtt{j} \le 2, \\ \mathtt{z}_\mathtt{j}(0) = {}^{\rho_i}\mathtt{D}^{\mu_i}\mathtt{z}_\mathtt{j}(0) = 0,\ \mathtt{z}_\mathtt{j}(1) = a_\mathtt{j}\mathtt{z}_\mathtt{j}(\eta_i),\ 0 < \eta_\mathtt{j} < 1,\ \mathtt{j} \in \mathbb{N}, \end{gathered}$$

by using the measure of noncompactness technique and applying the Darbo's fixed point theorem in the Banach spaces $\ell_\mathtt{p}$, $\mathtt{p} \ge 1$. Motivated by aforementioned works, in this paper we study fractional order differential equation with infinite-point boundary conditions

$$\left.\begin{aligned} &\mathtt{D}^{\delta}_{0^+}\mathtt{z}(\mathtt{t}) = \mathtt{g}\Big(\mathtt{t}, \mathtt{z}(\mathtt{t})\Big),\ \ell < \delta < \ell + 1,\ \mathtt{t} \in \mathbb{R}_+ := [0, \infty), \\ &\mathtt{z}(0) = \mathtt{z}''(0) = \cdots = \mathtt{z}^{(\ell)}(0) = 0, \\ &\lim_{\mathtt{t}\to+\infty} \mathtt{D}^{\delta-1}_{0^+}\mathtt{z}(\mathtt{t}) + \sum_{\mathtt{j}=1}^{\infty} \mathtt{c}_\mathtt{j}\mathtt{z}\Big(\varphi(\tau_\mathtt{j})\Big) = 0, \end{aligned}\right\} \tag{1}$$

where $\mathtt{c}_\mathtt{j}$ is a positive real number, $\mathtt{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and $\mathtt{D}^{\delta}_{0^+}$ is the Riemann-Liouville fractional derivative of order δ.

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which will be useful in our main results. In Section 3, we study existence of at least one fixed point theorems for class of operators. In Section 4, we study existence of solution for fractional order boundary value problem (1) as an application fixed point theorem. Finally, we provide an example to check feasibility of our results.

2. Preliminaries

In this section, we present some definitions and lemmas on fractional calculus and the concept of measure of noncompactness. Throughout the paper we assume that X is a Banach space. For a subset E of X, the closure and convex hull of E in X are denoted by $\overline{E}$ and $conv(E)$, respectively. Further, we denote the family of nonempty bounded subsets of X by $\mathfrak{M}_X$ and $\mathfrak{N}_X$ is the subfamily consisting of all relatively compact subsets of X. Let $\mathbb{C}(I)$ be the space of continuous functions defined on I with the norm given by

$$\|\mathtt{z}\| = \sup_{\mathtt{t}\in I} |\mathtt{z}(\mathtt{t})|.$$

In addition, $L_1(I)$ denotes the class of the Lebesgue integrable functions on the interval I with the norm given by

$$\|\mathtt{z}\|_{L_1} = \int_I \mathtt{z}(\mathtt{s})d\mathtt{s}.$$

Definition 2.1. *[25] The Riemann-Liouville fractional integral of order* $\beta > 0$ *for a function* $\mathtt{f} : (0, \infty) \to \mathbb{R}$ *is defined as*

$$\mathtt{I}^{\beta}_{0^+}\mathtt{f}(\mathtt{t}) = \frac{1}{\Gamma(\beta)}\int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\beta-1}\mathtt{f}(\mathtt{s})d\mathtt{s},$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. *[25] The Riemann-Liouville fractional derivative of order* $\beta > 0$ *for a continuous function* $\mathtt{f} : (0, \infty) \to \mathbb{R}$ *is defined as*

$$\mathtt{D}^{\beta}_{0^+}\mathtt{f}(\mathtt{t}) = \frac{1}{\Gamma(m-\beta)}\left(\frac{d}{d\mathtt{t}}\right)^m \int_0^{\mathtt{t}} \frac{\mathtt{f}(\mathtt{s})}{(\mathtt{t}-\mathtt{s})^{\gamma-m-1}}d\mathtt{s},$$

where $m = [\beta] + 1$, *provided that the right side is pointwise defined on* $(0, \infty)$.

Remark 2.3. *([25]) In this work we need the following composition relations:*

(*a*) $\mathtt{D}^{\beta}_{0^+}\mathtt{I}^{\beta}_{0^+}\mathtt{f}(\mathtt{t}) = \mathtt{f}(\mathtt{t}),\ \beta > 0, \mathtt{f}(\mathtt{t}) \in L^1(0, +\infty)$;

(*b*) $\mathtt{D}^{\gamma}_{0^+}\mathtt{I}^{\beta}_{0^+}\mathtt{f}(\mathtt{t}) = \mathtt{I}^{\beta-\gamma}_{0^+}\mathtt{f}(\mathtt{t}),\ \beta > \gamma > 0, \mathtt{f}(\mathtt{t}) \in L^1(0, +\infty)$.

Remark 2.4. *([5]) For* $\eta > -1$, *we have*

$$\mathtt{D}^{\beta}_{0^+}\mathtt{t}^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\beta+1)}\mathtt{t}^{\eta-\beta},$$

giving inparticular $\mathtt{D}^{\beta}_{0^+}\mathtt{t}^{\beta-m} = 0$, $m = 1, 2, \cdots, N$, *where* N *is the smallest integer greater than or equal to* β.

Lemma 2.5. *[25] The general solution to* $\mathtt{D}^{\beta}_{0^+}\mathtt{y}(\mathtt{t}) = 0$ *with* $\beta \in (m-1, m]$ *and* $m > 1$ *is the function*

$$\mathtt{y}(\mathtt{t}) = a_1\mathtt{t}^{\beta-1} + a_2\mathtt{t}^{\beta-2} + \cdots + a_m\mathtt{t}^{\beta-m},\ a_{\mathtt{i}} \in \mathbb{R},\ \mathtt{i} = 1, 2, \cdots, m.$$

Lemma 2.6. *[25] Let $\beta > 0$. Then the following equality holds for $y(t)$:*

$$I_{0^+}^{\beta} D_{0^+}^{\beta} y(t) = y(t) + a_1 t^{\beta-1} + a_2 t^{\beta-2} + \cdots + a_m t^{\beta-m},\ a_i \in \mathbb{R},\ i = 1, 2, \cdots, m$$

and m is the smallest integer greater than or equal to γ.

Lemma 2.7. *Suppose $\sum_{j=1}^{m} c_j \varphi(\tau_j)^{\delta-1}$ converges to $\Gamma(\delta)$ and let $\varpi \in L^1(\mathbb{R}_+)$. Then the fractional order boundary value problem*

$$D_{0^+}^{\delta} z(t) = \varpi(t),\ \ell < \delta < \ell + 1,\ t \in \mathbb{R}_+ \tag{2}$$

satisfying infinite-point boundary conditions

$$z(0) = 0, z'(0) = z''(0) = \cdots = z^{(\ell-1)}(0) = 0, \tag{3}$$

$$\lim_{t\to+\infty} D_{0^+}^{\delta-1} z(t) + \sum_{j=1}^{\infty} c_j z\big(\varphi(\tau_j)\big) = 0, \tag{4}$$

has a unique solution

$$\left.\begin{aligned} z(t) = \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds - \frac{t^{\delta-1}}{2\Gamma(\delta)} \int_0^{\infty} \varpi(s) ds \\ - \frac{t^{\delta-1}}{2\Gamma(\delta)} \sum_{j=1}^{\infty} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds. \end{aligned}\right\} \tag{5}$$

Proof. Let $z(t)$ be a solution of (2). Then, by Lemma 2.6, we have

$$z(t) = a_1 t^{\delta-1} + a_2 t^{\delta-2} + \cdots + a_{\ell+1} t^{\delta-\ell-1} + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds \tag{6}$$

for some $a_m \in \mathbb{R}$, $m = 1, 2, \cdots, \ell + 1$. Using conditions (3), we get $a_2 = a_3 = \cdots = a_{\ell+1} = 0$. So, (6) reduces to

$$z(t) = a_1 t^{\delta-1} + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds. \tag{7}$$

Before applying (4), we first take

$$\lim_{t\to+\infty} D_{0^+}^{\delta-1} z(t) = -\sum_{j=1}^{m} c_j z\big(\varphi(\tau_j)\big)$$

and using Remark 2.4 to get

$$a_1 = \frac{-1}{\sum_{j=1}^{m} c_j \varphi(\tau_j)^{\delta-1} + \Gamma(\delta)} \left[\int_0^{\infty} \varpi(s) ds + \sum_{j=1}^{m} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds\right].$$

Plugging a_2 value into (7), we obtain

$$\left.\begin{aligned} z(t) = & \frac{-t^{\delta-1}}{\sum_{j=1}^{m} c_j \varphi(\tau_j)^{\delta-1} - \Gamma(\delta)} \left[\int_0^{\infty} \varpi(s) ds + \sum_{j=1}^{m} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds\right] \\ & + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \varpi(s) ds \end{aligned}\right\} \tag{8}$$

Since

$$\left|c_j\varpi\Big(\varphi(\tau_j)\Big)\right| \le c_j\|\varpi\|$$

and

$$\begin{aligned}\left|c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds\right| &\le \frac{c_j}{\Gamma(\delta)}\left|\int_0^{\varphi(\tau_j)}u(s)\varpi(s)ds\right|\\ &\le \frac{c_j}{\Gamma(\delta)}\left|\int_0^{\tau_j}u(s)\varpi(s)ds\right|\\ &\le \frac{c_j}{\Gamma(\delta)}\|u\|_{L_1}\|z\|_{L_1},\end{aligned}$$

where $u(s)=(\varphi(\tau_j)-s)^{\delta-1}$. Then, by comparison test, the series in (4) and

$$\sum_{j=1}^{\infty}c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds$$

are convergent. So, by taking the limit as $m\to\infty$ in (8), we obtain (5).

Conversely, it is clear that (3) satisfies (6). Next applying the operator $D_{0^+}^{\delta-1}$ and $D_{0^+}^{\delta}$ to the two sides of (5) respectively and using Remark 2.3 and Remark 2.4, we obtain

$$\lim_{t\to+\infty}D_{0^+}^{\delta-1}z(t)=\frac{1}{2}\int_0^{\infty}\varpi(s)ds-\frac{1}{2}\sum_{j=1}^{\infty}c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds \tag{9}$$

and

$$D_{0^+}^{\delta}z(t)=\varpi(s).$$

Also note that

$$\begin{aligned}\sum_{j=1}^{m}c_jz\Big(\varphi(\tau_j)\Big)&=\sum_{j=1}^{m}c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds-\frac{1}{2\Gamma(\delta)}\sum_{j=1}^{m}c_j\varphi(\tau_j)^{\delta-1}\int_0^{\infty}\varpi(s)ds\\ &\quad-\frac{1}{2\Gamma(\delta)}\sum_{i=1}^{m}c_i\varphi(\tau_i)^{\delta-1}\sum_{j=1}^{\infty}c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds.\end{aligned}$$

Since $\sum_{i=1}^{m}c_i\varphi(\tau_i)^{\delta-1}$ converges to $\Gamma(\delta)$, it follows by taking limit $t\to+\infty$ that

$$\sum_{j=1}^{\infty}c_jz\Big(\varphi(\tau_j)\Big)=-\frac{1}{2}\int_0^{\infty}\varpi(s)ds+\frac{1}{2}\sum_{j=1}^{\infty}c_j\int_0^{\varphi(\tau_j)}\frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)}\varpi(s)ds. \tag{10}$$

By adding (9) and (10), we get (4). Which shows that the solution of the integral equation (5) satisfies the differential equation (2) under infinite-point boundary conditions (3) and (4). □

Remark 2.8. *The supposition* $\sum\limits_{j=1}^{m}c_j\varphi(\tau_j)^{\delta-1}$ *converges to* $\Gamma(\delta)$ *in the Lemma 2.7 is valid. For example: Let* $\delta=\frac{9}{2}$, $c_j=\frac{99225}{16\pi^{11/2}j^4}$, $\tau_j=\frac{1}{j}$ *and* $\varphi(t)=t^{4/7}$. *Then*

$$\sum_{j=1}^{m}c_j\varphi(\tau_j)^{\delta-1}=\sum_{j=1}^{m}\frac{99225}{16\pi^{11/2}j^6}\to\frac{105}{16}\sqrt{\pi}=\Gamma(\delta)\ \text{ as }\ m\to+\infty.$$

For $\ell \in \mathbb{N}$, $\mathbb{C}^{\ell}(\mathbb{R}_+)$ denotes the space of all real functions which are ℓ-times continuously differentiable on $\mathbb{R}_+$ with the family of seminorms

$$|\mathtt{z}|^{\mathtt{T}} = \sup\{\mathtt{z}^{(i)}(\mathtt{t}) : 0 \le i \le \ell,\ \mathtt{t} \in [0, \mathtt{T}]\}$$

for all $\mathtt{T} \ge 1$. Then the space $\mathbb{C}^{\ell}(\mathbb{R}_+)$ is a Fréchet space with respect to the distance

$$d(u, v) = \sup\left\{\frac{1}{2^{\mathtt{T}}} \min\left\{1, |u - v|^{\mathtt{T}}_{\mathbb{C}^{\ell}}\right\} : \mathtt{T} \in \mathbb{N}\right\}.$$

For $\alpha \in (0, 1]$, the space α-Hölder continuous functions $\mathcal{H}_{\alpha}(\mathbb{R}_+)$ is the family of all continuous functions $\mathtt{z} = \mathtt{z}(\mathtt{t})$ on $\mathbb{R}_+$ such that, for every $\mathtt{T} > 0$, there exists $N_{\mathtt{T}} > 0$,

$$\sup\left\{\frac{|\mathtt{z}(\mathtt{t}_1) - \mathtt{z}(\mathtt{t}_2)|}{|\mathtt{t}_1 - \mathtt{t}_2|^{\alpha}} : \mathtt{t}_1, \mathtt{t}_2 \in [0, \mathtt{T}], \mathtt{t}_1 \ne \mathtt{t}_2\right\} < N_{\mathtt{T}}.$$

The space $\mathcal{H}_{\alpha}(\mathbb{R}_+)$ is equipped with the family of seminorms

$$|\mathtt{z}|^{\mathtt{T}}_{\mathcal{H}_{\alpha}} = |\mathtt{z}(0)| + \sup\left\{\frac{|\mathtt{z}(\mathtt{t}_1) - \mathtt{z}(\mathtt{t}_2)|}{|\mathtt{t}_1 - \mathtt{t}_2|^{\alpha}} : \mathtt{t}_1, \mathtt{t}_2 \in [0, \mathtt{T}], \mathtt{t}_1 \ne \mathtt{t}_2\right\}.$$

Then, $\mathcal{H}_{\alpha}(\mathbb{R}_+)$ is a Fréchet space with respect to the distance

$$d(u, v) = \sup\left\{\frac{1}{2^{\mathtt{T}}} \min\left\{1, |u - v|^{\mathtt{T}}_{\mathcal{H}_{\alpha}}\right\} : \mathtt{T} \in \mathbb{N}\right\}.$$

For fixed $\alpha \in (0, 1]$, the space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ denotes the space of all functions $\mathtt{z} \in \mathbb{C}^{\ell}(\mathbb{R}_+)$ whose ℓth-derivative is Hölder continuous with exponent α. Then $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ equipped with the family of seminorms

$$|\mathtt{z}|^{\mathtt{T}}_{\alpha} = |\mathtt{z}|^{\mathtt{T}}_{\mathbb{C}^{k}} + |\mathtt{z}^{(\ell)}|^{\mathtt{T}}_{\mathcal{H}_{\alpha}}$$

for all $\mathtt{T} \in \mathbb{N}$. Then the space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ is a Fréchet space with respect to the distance

$$d(u, v) = \sup\left\{\frac{1}{2^{\mathtt{T}}} \min\left\{1, |u - v|^{\mathtt{T}}_{\alpha}\right\} : \mathtt{T} \in \mathbb{N}\right\}.$$

It can be seen that $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ is a linear subspace of $\mathbb{C}^{\ell}(\mathbb{R}_+)$. Further, let $\mathfrak{M}_{\mathbb{C}^{\ell,\alpha}}$ be the family of all nonempty and bounded subsets of $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$, and let $\mathfrak{N}_{\mathbb{C}^{\ell,\alpha}}$ be the family of all nonempty and relatively compact subsets of $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$,

Definition 2.9. *[20] A family of mapping $\left\{\aleph^{\mathtt{T}}_{\alpha,\ell}\right\}_{\mathtt{T}\in\mathbb{N}}$, where $\alpha \in (0, 1]$ and $\aleph^{\mathtt{T}}_{\alpha,\ell} : \mathfrak{M}_{\mathbb{C}^{\ell,\alpha}} \to \mathbb{R}_+$ is said to be a measure of noncompactness in $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ if it satisfies the following conditions,*

($\mathtt{C}_1$) *The family Ker $\left\{\aleph^{\mathtt{T}}_{\alpha,\ell}\right\} = \{\mathtt{S} \in \mathfrak{M}_{\mathbb{C}^{\ell,\alpha}} : \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S}) = 0\ \forall\ \mathtt{T} \in \mathbb{N}\} \ne \emptyset$ and Ker $\left\{\aleph^{\mathtt{T}}_{\alpha,\ell}\right\} \subseteq \mathfrak{N}_{\mathbb{C}^{\ell,\alpha}}$.*

($\mathtt{C}_2$) $\mathtt{S} \subseteq \mathcal{T} \implies \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S}) \le \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{T})$ *for* $\mathtt{T} \in \mathbb{N}$.

($\mathtt{C}_3$) $\aleph^{\mathtt{T}}_{\alpha,\ell}(\overline{\mathtt{S}}) = \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})$ *for* $\mathtt{T} \in \mathbb{N}$.

($\mathtt{C}_4$) $\aleph^{\mathtt{T}}_{\alpha,\ell}(conv(\mathtt{S})) = \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})$ *for* $\mathtt{T} \in \mathbb{N}$.

($\mathtt{C}_5$) $\aleph^{\mathtt{T}}_{\alpha,\ell}(c\mathtt{S} + (1 - c)\mathcal{T}) \le c\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S}) + (1 - c)\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{T})$ *for* $c \in [0, 1]$ *and* $\mathtt{T} \in \mathbb{N}$.

($\mathtt{C}_6$) *If $\{\mathtt{S}_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of closed chains from $\mathfrak{M}_{\mathbb{C}^{\ell,\alpha}}$ such that $\lim_{n\to\infty} \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S}_n) = 0$ for each $\mathtt{T} \in \mathbb{N}$, then $\mathtt{S}_{\infty} = \bigcap_{n=1}^{\infty} \mathtt{S}_n \ne \emptyset$.*

Suppose $\mathtt{T} \in \mathbb{N}$, *and* $\mathtt{S}$ *is a bounded subset of the space* $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$. *For* $\mathtt{z} \in \mathtt{S}$ *and* $\varepsilon > 0$ *we denote*

$$\begin{aligned}
\mu^{\mathtt{T}}(\mathtt{z},\varepsilon) &= \sup\left\{|\mathtt{z}^{(l)}(\mathtt{t}_1) - \mathtt{z}^{(l)}(\mathtt{t}_2)| : \mathtt{t}_1, \mathtt{t}_2 \in [0,\mathtt{T}], |\mathtt{t}_1 - \mathtt{t}_2| \le \varepsilon, l = 1, 2, \cdots, k\right\},\\
\mu^{\mathtt{T}}_{\alpha}(\mathtt{z},\varepsilon) &= \sup\left\{\frac{|\mathtt{z}^{(\ell)}(\mathtt{t}_1) - \mathtt{z}^{(\ell)}(\mathtt{t}_2)|}{|\mathtt{t}_1 - \mathtt{t}_2|^{\alpha}} : \mathtt{t}_1, \mathtt{t}_2 \in [0,\mathtt{T}], \mathtt{t}_1 \ne \mathtt{t}_2, |\mathtt{t}_1 - \mathtt{t}_2| \le \varepsilon\right\},\\
\mu^{\mathtt{T}}(\mathtt{S},\varepsilon) &= \sup_{\mathtt{z}\in\mathtt{S}} \mu^{\mathtt{T}}(\mathtt{z},\varepsilon),\\
\mu^{\mathtt{T}}_{\alpha}(\mathtt{S},\varepsilon) &= \sup_{\mathtt{z}\in\mathtt{S}} \mu^{\mathtt{T}}_{\alpha}(\mathtt{z},\varepsilon),\\
\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S},\varepsilon) &= \mu^{\mathtt{T}}(\mathtt{S},\varepsilon) + \mu^{\mathtt{T}}_{\alpha}(\mathtt{S},\varepsilon).
\end{aligned}$$

Theorem 2.10. *(Darbo [20]) Let* $\mathtt{C}$ *be a nonempty, closed and convex subset of the Fréchet space* $\mathbb{C}^{\alpha,\ell}(\mathbb{R}_+)$ *and* $\mathcal{F} : \mathtt{C} \to \mathtt{C}$ *be a continuous operator such that, for each* $\mathtt{T} \in \mathbb{N}$, *there exists* $\mathcal{L}_{\mathtt{T}} \in [0,1)$ *so that*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}(\mathtt{S})) \le \mathcal{L}_{\mathtt{T}} \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})$$

for each $\mathtt{S} \in \mathtt{C}$. *Then* $\mathcal{F}$ *has at least one fixed point in the set* $\mathtt{C}$.

3. Main Results

In this section, we present an extension of Darbo's fixed point theorem 2.10 and consequence results.

Definition 3.1. *[18] Let* $\mathcal{R}$ *denotes the class of functions* $\pi : \mathbb{R}_+ \to [0,1)$ *such that*

$$\pi(\xi_n) \to 1 \implies \xi_n \to 0.$$

Theorem 3.2. *[1] Let* $\mathcal{E}$ *be a Hausdorff locally convex linear topological space,* $\mathtt{C}$ *be a nonempty convex subset of* $\mathcal{E}$ *and* $\mathcal{F} : \mathtt{C} \to \mathcal{E}$ *be a continuous mapping such that*

$$\mathcal{F}(\mathtt{C}) \subset \mathcal{A} \subset \mathtt{C},$$

with $\mathcal{A}$ *compact. Then* $\mathcal{F}$ *has at least one fixed point.*

Theorem 3.3. *Let* $\mathtt{C}$ *be a nonempty, bounded, closed and convex subset of the Fréchet space* $\mathtt{C}^{\alpha,\ell}(\mathbb{R}_+)$ *and* $\mathcal{F} : \mathtt{C} \to \mathtt{C}$ *be a continuous operator such that, for each* $\mathtt{T} \in \mathbb{N}$,

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}(\mathtt{S})) \le \pi\left(\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})\right)\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S}) \tag{11}$$

for any subset $\mathtt{S} \in \mathtt{C}$ *and* $\pi \in \mathcal{R}$. *Then* $\mathcal{F}$ *has at least one fixed point in* $\mathtt{C}$.

Proof. By induction, we define a sequence $\{\mathtt{C}_{\mathtt{p}}\}$ by taking $\mathtt{C}_0 = \mathtt{C}$ and $\mathtt{C}_{\mathtt{p}} = conv(\mathcal{F}\mathtt{C}_{\mathtt{p}-1})$, $p \ge 1$. We have $\mathtt{C}_1 = conv(\mathcal{F}\mathtt{C}_0) \subseteq \mathtt{C}_0$, so continuing this process we get

$$\mathtt{C}_0 \supseteq \mathtt{C}_1 \supseteq \mathtt{C}_2 \supseteq \cdots.$$

If $\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_N) = 0$ for some positive integer N and for all $\mathtt{T}$, then $\mathtt{C}_N$ is relatively compact. Thus, Theorem 3.2 gives that $\mathcal{F}$ has a fixed point. Otherwise, let $\mathtt{T} \ge 0$ be a number such that $\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}}) \ne 0$ for any $\mathtt{p} \ge 0$. From (11), we have

$$\begin{aligned}
\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}+1}) &= \aleph^{\mathtt{T}}_{\alpha,\ell}(conv(\mathcal{F}\mathtt{C}_{\mathtt{p}}))\\
&= \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}\mathtt{C}_{\mathtt{p}}) \le \pi\left(\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}})\right)\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}}) \le \aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}}),
\end{aligned} \tag{12}$$

which implies that $\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p})$ is a positive decreasing sequence of real numbers, so, there exists a $\zeta \geq 0$ such that $\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p}) \to \zeta$ as $\mathtt{p} \to \infty$. Now we show that $\zeta = 0$. Suppose by contrary $\zeta \neq 0$. Then, from (12), we have

$$\frac{\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_{\mathtt{p}+1})}{\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p})} \leq \pi\Big(\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p})\Big) < 1 \implies \pi\Big(\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p})\Big) \to 1 \text{ as } n \to \infty.$$

Since $\pi \in \mathcal{R}$, we obtain $\zeta = 0$ and hence $\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{C}_\mathtt{p}) \to 0$ as $\mathtt{p} \to \infty$. In view of $(\mathtt{C}_6)$, $\mathtt{C}_\infty = \bigcap_{\mathtt{p}=1}^{\infty} \mathtt{C}_\mathtt{p}$ is nonempty, closed, convex and $\mathtt{C}_\infty \subset \mathtt{C}$. Also, the set $\mathtt{C}_\infty$ is invariant under the operator $\mathcal{F}$ and $\mathtt{C}_\infty \in Ker\left\{\aleph^{\mathtt{T}}_{\alpha,\ell}\right\}$. Therefore, by Theorem 3.2 the operator $\mathcal{F}$ has a fixed point. □

Corollary 3.4. *Let $\mathtt{C}$ be a nonempty, bounded, closed and convex subset of the Fréchet space $\mathtt{C}^{\alpha,\ell}(\mathbb{R}_+)$ and $\mathcal{F} : \mathtt{C} \to \mathtt{C}$ be a continuous operator such that, for each $\mathtt{T} \in \mathbb{N}$,*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}(\mathtt{S})) \leq \phi\left(\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})\right)$$

for any subset $\mathtt{S} \in \mathtt{C}$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and upper semicontinuous function such that $\phi(\mathtt{t}) < \mathtt{t}$ for all $\mathtt{t} > 0$. Then $\mathcal{F}$ has at least one fixed point.

Proof. The proof is similar to the proof of Corollary 2.2. in [2]. □

Corollary 3.5 (Darbo [20]). *Let $\mathtt{C}$ be a nonempty, bounded, closed and convex subset of the Fréchet space $\mathtt{C}^{\alpha,\ell}(\mathbb{R}_+)$ and $\mathcal{F} : \mathtt{C} \to \mathtt{C}$ be a continuous operator such that, for each $\mathtt{T} \in \mathbb{N}$, there exists $\mathfrak{L}_\mathtt{T} \in [0, 1)$,*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}(\mathtt{S})) \leq \mathfrak{L}_\mathtt{T}\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathtt{S})$$

for any subset $\mathtt{S} \in \mathtt{C}$. Then $\mathcal{F}$ has at least one fixed point in $\mathtt{C}$.

Proof. Put $\pi(\mathtt{t}) = \mathfrak{L}_\mathtt{T}$ in Theorem 3.3, we get desired result. □

Theorem 3.6. *Suppose $\left\{\aleph^{\mathtt{T}}_{\alpha,\ell,1}\right\}_{\mathtt{T}\in\mathbb{N}}, \left\{\aleph^{\mathtt{T}}_{\alpha,\ell,2}\right\}_{\mathtt{T}\in\mathbb{N}}, \cdots, \left\{\aleph^{\mathtt{T}}_{\alpha,\ell,n}\right\}_{\mathtt{T}\in\mathbb{N}}$, are n families of measures of noncompactness in Fréchet spaces $\mathbb{C}^{\ell,\alpha}_1(\mathbb{R}_+)$, $\mathbb{C}^{\ell,\alpha}_2(\mathbb{R}_+)$, $\cdots$, $\mathbb{C}^{\ell,\alpha}_n(\mathbb{R}_+)$ respectively. Moreover, assume that the function $\mathcal{F} : [0,\infty)^n \to [0,\infty)$ is convex and $\mathcal{F}(\varpi_1, \varpi_2, \cdots, \varpi_n) = 0$ if and only if $\varpi_i = 0$ for $i = 1, 2, \cdots, n$. Then*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(E) = \mathcal{F}\Big(\aleph^{\mathtt{T}}_{\alpha,\ell,1}(E_1), \aleph^{\mathtt{T}}_{\alpha,\ell,2}(E_2), \cdots, \aleph^{\mathtt{T}}_{\alpha,\ell,n}(E_n)\Big)$$

defines a measure of noncompactness in $\mathbb{C}^{\ell,\alpha}_1(\mathbb{R}_+) \times \mathbb{C}^{\ell,\alpha}_2(\mathbb{R}_+) \times \cdots \times \mathbb{C}^{\ell,\alpha}_n(\mathbb{R}_+)$ where E_i denotes the natural projection of E into $\mathbb{C}^{\ell,\alpha}_i(\mathbb{R}_+)$ for $i = 1, 2, \cdots, n$.

Proof. The proof is similar to the proof of Theorem 3.3.1. in [7]. □

Definition 3.7. *[13] An element $(\mathtt{z}_1, \mathtt{z}_2) \in E \times E$ is called a coupled fixed point of a mapping $\mathcal{F} : E \times E \to E$ if*

$$\mathcal{F}(\mathtt{z}_1, \mathtt{z}_2) = \mathtt{z}_1, \ \mathcal{F}(\mathtt{z}_2, \mathtt{z}_1) = \mathtt{z}_2.$$

Theorem 3.8. *Let Λ be a nonempty, bounded, closed and convex subset of a Fréchet space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function such that $\psi(\mathtt{t}) < \mathtt{t}$ for all $\mathtt{t} > 0$. Suppose that $\mathcal{F} : \Lambda \times \Lambda \to \Lambda$ is a continuous operator satisfying*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2)\right) \leq \psi\left[\frac{1}{2}\Big(\aleph^{\mathtt{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathtt{T}}_{\alpha,\ell}(E_2)\Big)\right] \tag{13}$$

for all $E_1, E_2 \subset \Lambda$. Then $\mathcal{F}\mathcal{G}$ has at least a coupled fixed point.

Proof. From Theorem 3.6, it can be seen that $\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}(E) = \aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2)$ is a measure of noncompactness in the space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+) \times \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$, where E_1, E_2 denote the natural projections of E. Next, consider the map $\widetilde{\mathcal{F}} : \Lambda \times \Lambda \to \Lambda \times \Lambda$ defined by

$$\widetilde{\mathcal{F}}(\varpi_1, \varpi_2) = (\mathcal{F}(\varpi_1, \varpi_2), \mathcal{F}(\varpi_2, \varpi_1)) .$$

Since $\mathcal{F}$ is continuous on $\Lambda \times \Lambda$, $\mathcal{F}$ is continuous on $\Lambda \times \Lambda$. Let $E \subset \Lambda \times \Lambda$ be a nonempty subset. Then, from C_2 and (13), we get

$$\begin{aligned}
\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) &\le \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2) \times \mathcal{F}(E_2 \times E_1)\right) \\
&\le \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2)\right) + \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_2 \times E_1)\right) \\
&\le \psi\left[\frac{1}{2}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2)\right)\right] + \psi\left[\frac{1}{2}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_1)\right)\right] \\
&\le 2\psi\left[\frac{1}{2}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2)\right)\right].
\end{aligned}$$

Taking $\vartheta^{\mathrm{T}}_{\alpha,\ell} = \frac{1}{2}\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}$, we obtain

$$\vartheta^{\mathrm{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) \le \psi(\vartheta^{\mathrm{T}}_{\alpha,\ell}(E)).$$

Since $\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}$ is a measure of noncompactness, $\vartheta^{\mathrm{T}}_{\alpha,\ell}$ is too. Therefore, all the conditions of Corollary 3.4 are satisfied and hence, $\mathcal{F}$ has a coupled fixed point. □

Definition 3.9. *[9] Denote $E^3 := E \times E \times E$. Then an element $(\mathtt{z}_1, \mathtt{z}_2, \mathtt{z}_3) \in E^3$ is called a tripled fixed point of a mapping $\mathcal{F} : E^3 \to E$ if*

$$\mathcal{F}(\mathtt{z}_1, \mathtt{z}_2, \mathtt{z}_3) = \mathtt{z}_1,\ \mathcal{F}(\mathtt{z}_2, \mathtt{z}_1, \mathtt{z}_3) = \mathtt{z}_2,\ \mathcal{F}(\mathtt{z}_3, \mathtt{z}_2, \mathtt{z}_1) = \mathtt{z}_3.$$

Theorem 3.10. *Let Λ be a nonempty, bounded, closed and convex subset of a Fréchet space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function such that $\psi(\mathtt{t}) < \mathtt{t}$ for all $\mathtt{t} > 0$. Suppose that $\mathcal{F} : \Lambda \times \Lambda \times \Lambda \to \Lambda$ is a continuous operator satisfying*

$$\aleph^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2 \times E_3)\right) \le \psi\left[\frac{1}{3}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_3)\right)\right] \tag{14}$$

for all $E_1, E_2, E_3 \subset \Lambda$. Then $\mathcal{F}\mathcal{G}$ has a tripled fixed point.

Proof. From Theorem 3.6, it can be seen that $\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}(E) = \aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_3)$ is a measure of noncompactness in the space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+) \times \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+) \times \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$, where E_1, E_2, E_3 denote the natural projections of E. Next, consider the map $\widetilde{\mathcal{F}} : \Lambda \times \Lambda \times \Lambda \to \Lambda \times \Lambda \times \Lambda$ defined by

$$\widetilde{\mathcal{F}}(\varpi_1, \varpi_2, \varpi_3) = (\mathcal{F}(\varpi_1, \varpi_2, \varpi_3), \mathcal{F}(\varpi_2, \varpi_1, \varpi_3), \mathcal{F}(\varpi_3, \varpi_2, \varpi_1)) .$$

Since $\mathcal{F}$ is continuous on $\Lambda \times \Lambda \times \Lambda$, $\mathcal{F}$ is continuous on $\Lambda \times \Lambda \times \Lambda$. Let $E \subset \Lambda \times \Lambda \times \Lambda$ be a nonempty subset. Then, from C_2 and (14), we get

$$\begin{aligned}
\widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) &\le \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2 \times E_3) \times \mathcal{F}(E_2 \times E_1 \times E_3) \times \mathcal{F}(E_3 \times E_2 \times E_1)\right) \\
&\le \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_1 \times E_2 \times E_3)\right) + \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_2 \times E_1 \times E_3)\right) + \widetilde{\aleph}^{\mathrm{T}}_{\alpha,\ell}\left(\mathcal{F}(E_3 \times E_2 \times E_1)\right) \\
&\le \psi\left[\frac{1}{3}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_3)\right)\right] + \psi\left[\frac{1}{3}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_3)\right)\right] \\
&\qquad + \psi\left[\frac{1}{3}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_3) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_1)\right)\right] \\
&\le 3\psi\left[\frac{1}{3}\left(\aleph^{\mathrm{T}}_{\alpha,\ell}(E_1) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_2) + \aleph^{\mathrm{T}}_{\alpha,\ell}(E_3)\right)\right].
\end{aligned}$$

Taking $\vartheta^{\mathtt{T}}_{\alpha,\ell} = \frac{1}{3}\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}$, we obtain

$$\vartheta^{\mathtt{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) \leq \psi(\vartheta^{\mathtt{T}}_{\alpha,\ell}(E)).$$

Since $\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}$ is a measure of noncompactness, $\vartheta^{\mathtt{T}}_{\alpha,\ell}$ is too. Therefore, all the conditions of Corollary 3.4 are satisfied and hence, $\mathcal{F}$ has a tripled fixed point. □

Definition 3.11. *[32] Denote $E^n := \prod_{\mathtt{j}=1}^{n} E_{\mathtt{j}}$. Then an element $(\mathtt{z}_1, \mathtt{z}_2, \cdots, \mathtt{z}_{n-1}, \mathtt{z}_n) \in E^n$ is called an n-fixed point of a mapping $\mathcal{F} : E^n \to E$ if*

$$\begin{aligned}
&\mathcal{F}(\mathtt{z}_1, \mathtt{z}_2, \mathtt{z}_3, \cdots, \mathtt{z}_{n-1}, \mathtt{z}_n) = \mathtt{z}_1,\\
&\mathcal{F}(\mathtt{z}_2, \mathtt{z}_1, \mathtt{z}_3, \cdots, \mathtt{z}_{n-1}, \mathtt{z}_n) = \mathtt{z}_2,\\
&\mathcal{F}(\mathtt{z}_3, \mathtt{z}_2, \mathtt{z}_1, \cdots, \mathtt{z}_{n-1}, \mathtt{z}_n) = \mathtt{z}_3,\\
&\qquad\vdots\\
&\mathcal{F}(\mathtt{z}_{n-1}, \mathtt{z}_2, \mathtt{z}_3, \cdots, \mathtt{z}_1, \mathtt{z}_n) = \mathtt{z}_{n-1}\\
&\mathcal{F}(\mathtt{z}_n, \mathtt{z}_2, \mathtt{z}_3, \cdots, \mathtt{z}_{n-1}, \mathtt{z}_1) = \mathtt{z}_n
\end{aligned}$$

Theorem 3.12. *Let Λ be a nonempty, bounded, closed and convex subset of a Fréchet space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function such that $\psi(\mathtt{t}) < \mathtt{t}$ for all $\mathtt{t} > 0$. Suppose that $\mathcal{F} : \Lambda^n \to \Lambda$ is a continuous operator satisfying*

$$\aleph^{\mathtt{T}}_{\alpha,\ell}(\mathcal{F}(E^n)) \leq \psi\left[\frac{1}{n}\sum_{\mathtt{j}=1}^{n} \aleph^{\mathtt{T}}_{\alpha,\ell}(E_{\mathtt{j}})\right] \tag{15}$$

for all $E_{\mathtt{j}} \subset \Lambda$, $\mathtt{j} = 1, 2, \cdots, n$. Then $\mathcal{F}\mathcal{G}$ has an n-fixed point.

Proof. From Theorem 3.6, it can be seen that $\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}(E) = \sum_{\mathtt{j}=1}^{n} \aleph^{\mathtt{T}}_{\alpha,\ell}(E_{\mathtt{j}})$ is a measure of noncompactness in the space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)^n$, where $E_{\mathtt{j}}$ $(\mathtt{j} = 1, 2, \cdots, n)$ denote the natural projections of E. Next, consider the map $\widetilde{\mathcal{F}} : \Lambda^n \to \Lambda^n$ defined by

$$\widetilde{\mathcal{F}}(\varpi_1, \varpi_2, \cdots, \varpi_{n-1}, \varpi_n) = \begin{bmatrix} \mathcal{F}(\varpi_1, \varpi_2, \varpi_3, \cdots, \varpi_{n-1}, \varpi_n) \\ \mathcal{F}(\varpi_2, \varpi_1, \varpi_3, \cdots, \varpi_{n-1}, \varpi_n) \\ \mathcal{F}(\varpi_3, \varpi_2, \varpi_1, \cdots, \varpi_{n-1}, \varpi_n) \\ \vdots \\ \mathcal{F}(\varpi_{n-1}, \varpi_2, \varpi_3, \cdots, \varpi_1, \varpi_n) \\ \mathcal{F}(\varpi_n, \varpi_2, \varpi_1, \cdots, \varpi_{n-1}, \varpi_1) \end{bmatrix}$$

Since $\mathcal{F}$ is continuous on Λ^n, $\mathcal{F}$ is continuous on Λ^n. Let $E \subset \Lambda^n$ be a nonempty subset. Then, from $\mathtt{C}_2$, (15) and proceeding similar to the Theorem 3.10, we arrive

$$\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) \leq n\psi\left[\frac{1}{n}\sum_{\mathtt{j}=1}^{n} \aleph^{\mathtt{T}}_{\alpha,\ell}(E_{\mathtt{j}})\right].$$

Taking $\vartheta^{\mathtt{T}}_{\alpha,\ell} = \frac{1}{n}\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}$, we obtain

$$\vartheta^{\mathtt{T}}_{\alpha,\ell}(\widetilde{\mathcal{F}}(E)) \leq \psi(\vartheta^{\mathtt{T}}_{\alpha,\ell}(E)).$$

Since $\widetilde{\aleph}^{\mathtt{T}}_{\alpha,\ell}$ is a measure of noncompactness, $\vartheta^{\mathtt{T}}_{\alpha,\ell}$ is too. Therefore, all the conditions of Corollary 3.4 are satisfied and hence, $\mathcal{F}$ has an n-fixed point. □

4. Existence of Solutions for Fractional Order Boundary Value Problem

In this section we present an existence result for the nonlinear fractional order infinite point boundary value problem (1) in the Frećhet space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ where $\alpha \in (0,1]$ and $\delta - \ell - \alpha > 0$. We also provide en example to check feasibility of our results.

We assume the following conditions are true in the sequel:

(H_1) $\mathtt{g} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and there exist increasing functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{t\to 0} a(\mathtt{t}) = \lim_{t\to 0} b(\mathtt{t}) = 0, a \in L^1(\mathbb{R}_+)$ and there exits $\lambda > 0$ such that

$$(i)\ |\mathtt{g}(\mathtt{s}, \mathtt{z}_1) - \mathtt{g}(\mathtt{s}, \mathtt{z}_2)| \leq a(|\mathtt{z}_1 - \mathtt{z}_2|),$$

$$(ii)\ \int_0^\infty |\mathtt{g}(\mathtt{s}, \mathtt{z}_1) - \mathtt{g}(\mathtt{s}, \mathtt{z}_2)|\mathtt{ds} \leq \lambda b(|\mathtt{z}_1 - \mathtt{z}_2|)$$

for all $\mathtt{s} \in \mathbb{R}_+$ and $\mathtt{z}_1, \mathtt{z}_2 \in \mathbb{R}$. Also assume that

$$\mathtt{g}^{\mathtt{S}} = \sup\{|\mathtt{g}(\mathtt{s}, 0)| : \mathtt{s} \in \mathbb{R}_+\} < \infty, \quad \mathtt{g}^{\mathtt{I}} = \int_0^\infty |\mathtt{g}(\mathtt{s}, 0)|\mathtt{ds} < \infty.$$

(H_2) for each $\mathtt{T} \in \mathbb{N}$, the exits a positive number $\mathtt{R}_\mathtt{T}$ such that

$$\Big[a(\mathtt{R}_\mathtt{T}) + \mathtt{g}^{\mathtt{S}}\Big]\Big[\mathtt{M}_\mathtt{T} + \mathtt{N}_\mathtt{T} \sum_{\mathtt{j}=1}^{\infty} \mathtt{c}_\mathtt{j} \varphi(\tau_\mathtt{j})^\delta + \frac{2\mathtt{T}^{\delta-\ell-\alpha}}{\Gamma(\delta - \ell + 1)}\Big] + \Big[\lambda b(\mathtt{R}_\mathtt{T}) + \mathtt{g}^{\mathtt{I}}\Big]\mathtt{N}_\mathtt{T} \leq \mathtt{R}_\mathtt{T},$$

where

$$M_\mathtt{T} = \sup_{0\leq \mathtt{p}\leq \ell} \left\{\frac{\mathtt{T}^{\delta-\mathtt{p}}}{\Gamma(\delta - \mathtt{p} + 1)}\right\}, \ N_\mathtt{T} = \sup_{0\leq \mathtt{p}\leq \ell} \left\{\frac{\mathtt{T}^{\delta-\mathtt{p}-1}}{2\Gamma(\delta - \mathtt{p})}\right\}.$$

Theorem 4.1. *Assume that the conditions* (H_1)-(H_2) *are satisfied. The the fractional order infinite point boundary value problem* (1) *has at least one solution in the Fréchet space* $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$.

Proof. Define the operator $\mathcal{F} : \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+) \to \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ by

$$\begin{aligned}\mathcal{F}(\mathtt{z})(\mathtt{t}) = \int_0^\mathtt{t} \frac{(\mathtt{t}-\mathtt{s})^{\delta-1}}{\Gamma(\delta)}\varpi(\mathtt{s})\mathtt{ds} - \frac{\mathtt{t}^{\delta-1}}{2\Gamma(\delta)}\int_0^\infty \varpi(\mathtt{s})\mathtt{ds} \\ - \frac{\mathtt{t}^{\delta-1}}{2\Gamma(\delta)}\sum_{\mathtt{j}=1}^{\infty} \mathtt{c}_\mathtt{j} \int_0^{\varphi(\tau_\mathtt{j})} \frac{(\varphi(\tau_\mathtt{j})-\mathtt{s})^{\delta-1}}{\Gamma(\delta)}\varpi(\mathtt{s})\mathtt{ds}.\end{aligned}$$

Then Lemma 2.7 shows that the fixed points of the operator $\mathcal{F}$ coincides with the solutions of the problem

(1). Let $z \in \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$, then by (H_1)–(H_2), we have

$$\begin{aligned}
|\mathcal{F}(z)(t)| =& \Big| \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} g(s,z(s))ds - \frac{t^{\delta-1}}{2\Gamma(\delta)} \int_0^\infty g(s,z(s))ds \\
&\quad - \frac{t^{\delta-1}}{2\Gamma(\delta)} \sum_{j=1}^{\infty} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} g(s,z(s))ds \Big| \\
\leq& \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \Big[|g(s,z(s)) - g(s,0)| + |g(s,0)|\Big]ds \\
&+ \frac{t^{\delta-1}}{2\Gamma(\delta)} \int_0^\infty \Big[|g(s,z(s)) - g(s,0)| + |g(s,0)|\Big]ds \\
&+ \frac{t^{\delta-1}}{2\Gamma(\delta)} \sum_{j=1}^{\infty} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} \Big[|g(s,z(s)) - g(s,0)| + |g(s,0)|\Big]ds \\
\leq& \Big[a(|z|_\alpha^T) + g^S\Big] \Bigg[\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} ds + \frac{t^{\delta-1}}{2\Gamma(\delta)} \sum_{j=1}^{\infty} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} ds \Bigg] \\
&+ \Big[\lambda b(|z|_\alpha^T) + g^I\Big] \frac{t^{\delta-1}}{2\Gamma(\delta)}.
\end{aligned}$$

Thus,

$$|\mathcal{F}(z)(t)| \leq \Big[\frac{a(|z|_\alpha^T) + g^S}{\Gamma(\delta+1)} \Big] \Big[T^\delta + \frac{T^{\delta-1}}{2\Gamma(\delta)} \sum_{j=1}^{\infty} c_j \varphi(\tau_j)^\delta \Big] + \Big[\lambda b(|z|_\alpha^T) + g^I\Big] \frac{T^{\delta-1}}{2\Gamma(\delta)}. \tag{16}$$

The similar argument gives that

$$|\mathcal{F}'(z)(t)| \leq \Big[a(|z|_\alpha^T) + g^S\Big] \Big[\frac{T^{\delta-1}}{\Gamma(\delta)} + \frac{T^{\delta-2}}{2\Gamma(\delta-1)} \sum_{j=1}^{\infty} c_j \varphi(\tau_j)^\delta \Big] + \Big[\lambda b(|z|_\alpha^T) + g^I\Big] \frac{T^{\delta-2}}{2\Gamma(\delta-1)}. \tag{17}$$

and

$$\begin{aligned}
|\mathcal{F}^{(p)}(z)(t)| =& \Big| \int_0^t \frac{(t-s)^{\delta-p-1}}{\Gamma(\delta-p)} g(s,z(s))ds - \frac{t^{\delta-p-1}}{2\Gamma(\delta-p)} \int_0^\infty g(s,z(s))ds \\
&\quad - \frac{t^{\delta-p-1}}{2\Gamma(\delta-p)} \sum_{j=1}^{\infty} c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j)-s)^{\delta-1}}{\Gamma(\delta)} g(s,z(s))ds \Big| \\
\leq& \Big[a(|z|_\alpha^T) + g^S\Big] \Big[\frac{T^{\delta-p}}{\Gamma(\delta-p+1)} + \frac{T^{\delta-p-1}}{2\Gamma(\delta-p)} \sum_{j=1}^{\infty} c_j \varphi(\tau_j)^\delta \Big] + \Big[\lambda b(|z|_\alpha^T) + g^I\Big] \frac{T^{\delta-p-1}}{2\Gamma(\delta-p)},
\end{aligned} \tag{18}$$

for any $t \in \mathbb{R}_+$ and $p = 2, 3, \cdots, \ell - 1$. From (16)-(18), we get

$$\left.\begin{aligned}
|\mathcal{F}^{(p)}(z)(t)| \leq& \Big[a(|z|_\alpha^T) + g^S\Big] \Big[\sup_{0\leq p\leq \ell} \left\{ \frac{T^{\delta-p}}{\Gamma(\delta-p+1)} \right\} + \sup_{0\leq p\leq \ell} \left\{ \frac{T^{\delta-p-1}}{2\Gamma(\delta-p)} \right\} \sum_{j=1}^{\infty} c_j \varphi(\tau_j)^\delta \Big] \\
&+ \Big[\lambda b(|z|_\alpha^T) + g^I\Big] \sup_{0\leq p\leq \ell} \left\{ \frac{T^{\delta-p-1}}{2\Gamma(\delta-p)} \right\}.
\end{aligned}\right\} \tag{19}$$

Further, let $t_1, t_2 \in [0, T]$ with $t_2 > t_1$. Similar to the above process, we arrive

$$\begin{aligned}\frac{|\mathcal{F}^{(\ell)}(z)(t_2) - \mathcal{F}^{(\ell)}(z)(t_1)|}{|t_2 - t_1|^\alpha} &\le \frac{1}{|t_2 - t_1|^\alpha}\bigg|\int_0^{t_2} \frac{(t_2 - s)^{\delta-\ell-1}}{\Gamma(\delta - \ell)} g(s, z(s))ds \\ &\quad - \int_0^{t_1} \frac{(t_1 - s)^{\delta-\ell-1}}{\Gamma(\delta - \ell)} g(s, z(s))ds - \frac{[t_2^{\delta-\ell-1} - t_1^{\delta-\ell-1}]}{2\Gamma(\delta - \ell)}\int_0^\infty g(s, z(s))ds \\ &\quad - \frac{[t_2^{\delta-\ell-1} - t_1^{\delta-\ell-1}]}{2\Gamma(\delta - \ell)}\sum_{j=1}^\infty c_j \int_0^{\varphi(\tau_j)} \frac{(\varphi(\tau_j) - s)^{\delta-1}}{\Gamma(\delta)} g(s, z(s))ds\bigg| \\ &\le \frac{1}{|t_2 - t_1|^\alpha}\left[\frac{a(|z|_\alpha^T) + g^S}{\Gamma(\delta - \ell)}\right]\bigg[\frac{2(t_2 - t_1)^{\delta-\ell}}{\delta - \ell} + \Big(\frac{t_1^{\delta-\ell}}{\delta - \ell} - \frac{t_2^{\delta-\ell}}{\delta - \ell}\Big) \\ &\quad + \frac{1}{2}\Big(\frac{t_1^{\delta-\ell-1}}{\Gamma(\delta - \ell)} - \frac{t_2^{\delta-\ell-1}}{\Gamma(\delta - \ell)}\Big)\sum_{j=1}^\infty c_j\varphi(\tau_j)^\delta\bigg] + \frac{1}{2}\Big[\frac{\lambda b(|z|_\alpha^T) + g^I}{|t_2 - t_1|^\alpha}\Big]\Big[\frac{t_1^{\delta-\ell-1}}{\Gamma(\delta - \ell)} - \frac{t_2^{\delta-\ell-1}}{\Gamma(\delta - \ell)}\Big].\end{aligned}$$

Since $t_2 > t_1$, $\dfrac{t_1^{\delta-\ell}}{\delta - \ell} - \dfrac{t_2^{\delta-\ell}}{\delta - \ell} \le 0$, $\dfrac{t_1^{\delta-\ell-1}}{\Gamma(\delta - \ell)} - \dfrac{t_2^{\delta-\ell-1}}{\Gamma(\delta - \ell)} \le 0$ and so we get

$$\begin{aligned}\frac{|\mathcal{F}^{(\ell)}(z)(t_2) - \mathcal{F}^{(\ell)}(z)(t_1)|}{|t_2 - t_1|^\alpha} &\le \frac{2\Big[a(|z|_\alpha^T) + g^S\Big]|t_2 - t_1|^{\delta-\ell}}{(\delta - \ell)\Gamma(\delta - \ell)|t_2 - t_1|^\alpha} \\ &\le \frac{2\Big[a(|z|_\alpha^T) + g^S\Big]}{\Gamma(\delta - \ell + 1)} T^{\delta-\ell-\alpha}.\end{aligned} \tag{20}$$

From (19) and the above inequality, we deduce that

$$|\mathcal{F}z|_\alpha^T \le \Big[a(|z|_\alpha^T) + g^S\Big]\Big[M_T + N_T\sum_{j=1}^\infty c_j\varphi(\tau_j)^\delta + \frac{2T^{\delta-\ell-\alpha}}{\Gamma(\delta - \ell + 1)}\Big] + \Big[\lambda b(|z|_\alpha^T) + g^I\Big]N_T.$$

Therefore, $\mathcal{F}(z) \in \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$. Next, let

$$z = \Big\{z \in \mathbb{C}^{\ell,\alpha}(\mathbb{R}_+) : |z|_\alpha^T \le R_T \text{ for } T > 0\Big\}.$$

Then z is nonempty, bounded, closed and convex subset of $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ and from (H_2) it is clear that $\mathcal{F}$ is self mapping on z. Now, we show that $\mathcal{F}$ is continuous on z. for this, let $z_1, z_2 \in z$ and ε be any positive number such that $|z_1 - z_2|_\alpha^T \le \varepsilon$. Let $t \in [0, T]$. Then

$$|(\mathcal{F}z_1)(t) - (\mathcal{F}z_2)(t)| \le \frac{a\Big(|z_1 - z_2|_\alpha^T\Big)}{\Gamma(\delta + 1)}\left[t^\delta + \frac{t^{\delta-1}}{2\Gamma(\delta)}\sum_{j=1}^\infty c_j\varphi(\tau_j)^\delta\right] + \lambda b\Big(|z_1 - z_2|_\alpha^T\Big)\frac{t^{\delta-1}}{2\Gamma(\delta)}. \tag{21}$$

Similar to the above argument, we have

$$\begin{aligned}|(\mathcal{F}z_1)'(t) - (\mathcal{F}z_2)'(t)| &\le a\Big(|z_1 - z_2|_\alpha^T\Big)\left[\frac{t^{\delta-1}}{\Gamma(\delta)} + \frac{T^{\delta-2}}{2\Gamma(\delta - 1)}\sum_{j=1}^\infty c_j\varphi(\tau_j)^\delta\right] \\ &\quad + \lambda b\Big(|z_1 - z_2|_\alpha^T\Big)\frac{T^{\delta-2}}{2\Gamma(\delta - 1)}\end{aligned} \tag{22}$$

and

$$\begin{aligned}|(\mathcal{F}z_1)^{(p)}(t) - (\mathcal{F}z_2)^{(p)}(t)| &\le a\Big(|z_1 - z_2|_\alpha^T\Big)\left[\frac{t^{\delta-p}}{\Gamma(\delta - p + 1)} + \frac{T^{\delta-p-1}}{2\Gamma(\delta - p)}\sum_{j=1}^\infty c_j\varphi(\tau_j)^\delta\right] \\ &\quad + \lambda b\Big(|z_1 - z_2|_\alpha^T\Big)\frac{T^{\delta-p-1}}{2\Gamma(\delta - p)}\end{aligned} \tag{23}$$

for $p = 2, 3 \cdots, \ell$. From (21)-(23), we obtain

$$\begin{aligned}
&|(\mathcal{F}z_1)^{(p)}(t) - (\mathcal{F}z_2)^{(p)}(t)| \\
&\quad \leq a\Big(|z_1 - z_2|_\alpha^T\Big)\left[\sup_{0\leq p\leq\ell}\left\{\frac{T^{\delta-p}}{\Gamma(\delta-p+1)}\right\} + \sup_{0\leq p\leq\ell}\left\{\frac{T^{\delta-p-1}}{2\Gamma(\delta-p)}\right\}\sum_{j=1}^{\infty} c_j\varphi(\tau_j)^\delta\right] \\
&\qquad + \lambda b\Big(|z_1 - z_2|_\alpha^T\Big)\sup_{0\leq p\leq\ell}\left\{\frac{T^{\delta-p-1}}{2\Gamma(\delta-p)}\right\}.
\end{aligned} \tag{24}$$

for $p = 0, 1, \cdots, \ell$. Similar to the above process, for $t_1, t_2 \in [0, T]$ with $t_2 > t_1$, we obtain

$$\frac{\Big|[(\mathcal{F}z_1)^{(\ell)}(t_2) - (\mathcal{F}z_2)^{(\ell)}(t_2)] - [(\mathcal{F}z_1)^{(\ell)}(t_1) - (\mathcal{F}z_2)^{(\ell)}(t_1)]\Big|}{|t_2 - t_1|^\alpha} \leq \frac{2T^{\delta-\ell-\alpha}}{\Gamma(\delta-\ell+1)} a\Big(|z_1 - z_2|_\alpha^T\Big).$$

Combining above inequality and (24), we get

$$|\mathcal{F}z_1 - \mathcal{F}z_2|_\alpha^T \leq a\Big(|z_1 - z_2|_\alpha^T\Big)\Big[M_T + N_T\sum_{j=1}^{\infty} c_j\varphi(\tau_j)^\delta + \frac{2T^{\delta-\ell-\alpha}}{\Gamma(\delta-\ell+1)}\Big] + \lambda b\Big(|z_1 - z_2|_\alpha^T\Big)N_T.$$

From which we conclude that $\mathcal{F}$ is continuous on z. Finally, we verify the condition (11). Let B be any bounded subset of $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$, ε be any positive number and $T \in \mathbb{N}$. Let us select $z \in B$ and $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leq \varepsilon$, we have

$$\begin{aligned}
&|(\mathcal{F}z)(t_2) - (\mathcal{F}z)(t_1)| \\
&\quad \leq \left[\frac{a(z(s)) + g^S}{\Gamma(\delta)}\right]\left[\frac{2(t_2 - t_1)^\delta}{\delta} + \frac{t_1^\delta}{\delta} - \frac{t_2^\delta}{\delta} + \frac{1}{2}\left(\frac{t_1^{\delta-1}}{\Gamma(\delta)} - \frac{t_2^{\delta-1}}{\Gamma(\delta)}\right)\sum_{j=1}^{\infty} c_j\varphi(\tau_j)^\delta\right] \\
&\qquad + \frac{1}{2}\Big[\lambda b\Big(|z(s)|\Big) + g^I\Big]\left(\frac{t_1^{\delta-1}}{\Gamma(\delta)} - \frac{t_2^{\delta-1}}{\Gamma(\delta)}\right).
\end{aligned}$$

Since $\dfrac{t_1^\delta}{\delta} - \dfrac{t_2^\delta}{\delta} \leq 0$, $\dfrac{t_1^{\delta-1}}{\Gamma(\delta)} - \dfrac{t_2^{\delta-1}}{\Gamma(\delta)} \leq 0$, $|t_2 - t_1| \leq \varepsilon$ and $\varepsilon > 0$ was arbitrary, we have

$$|(\mathcal{F}z)(t_2) - (\mathcal{F}z)(t_1)| \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{25}$$

Similar to the above arguments, it can be shown that

$$|(\mathcal{F}z)'(t_2) - (\mathcal{F}z)'(t_1)| \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{26}$$

$$|(\mathcal{F}z)^{(p)}(t_2) - (\mathcal{F}z)^{(p)}(t_1)| \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{27}$$

From (25)-(27), we obtain

$$\begin{aligned}
\mu^T(\mathcal{F}z, \varepsilon) &= \sup\Big\{|z^{(p)}(t_2) - z^{(p)}(t_1)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon, p = 0, 1, \cdots, \ell\Big\} \\
&\to 0 \ \text{ as } \ \varepsilon \to 0.
\end{aligned}$$

$$\mu^T(\mathcal{F}B, \varepsilon) = \sup_{z\in B}\mu^T(\mathcal{F}z, \varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{28}$$

In view of (20), we deduce that

$$\begin{aligned}
\mu_\alpha^T(\mathcal{F}z, \varepsilon) &= \sup\left\{\frac{|z^{(\ell)}(t_1) - z^{(\ell)}(t_2)|}{|t_1 - t_2|^\alpha} : t_1, t_2 \in [0, T], t_1 \neq t_2, |t_1 - t_2| \leq \varepsilon\right\} \\
&\leq \frac{2\Big[a(|z|_\alpha^T) + g^S\Big]}{\Gamma(\delta-\ell+1)}\varepsilon^{\delta-\ell-\alpha} \\
&\to 0 \ \text{ as } \ \varepsilon \to 0.
\end{aligned}$$

Which gives that

$$\mu_{\alpha}^{\mathtt{T}}(\mathcal{F}B,\varepsilon)=\sup_{\mathtt{z}\in B}\mu_{\alpha}^{\mathtt{T}}(\mathcal{F}\mathtt{z},\varepsilon)\to 0\ \text{ as }\ \varepsilon\to 0. \tag{29}$$

From (28)-(29), we get

$$\aleph_{\alpha,\ell}^{\mathtt{T}}(\mathcal{F}B)=\lim_{\varepsilon\to 0}\aleph_{\alpha,\ell}^{\mathtt{T}}(\mathcal{F}B,\varepsilon)=\lim_{\varepsilon\to 0}\left[\mu^{\mathtt{T}}(\mathcal{F}B,\varepsilon)+\mu_{\alpha}^{\mathtt{T}}(\mathcal{F}B,\varepsilon)\right].$$

Thus,

$$\aleph_{\alpha,\ell}^{\mathtt{T}}(\mathcal{F}B)\le\pi\left(\aleph_{\alpha,\ell}^{\mathtt{T}}(B)\right)\aleph_{\alpha,\ell}^{\mathtt{T}}(B),$$

where $\pi\left(\aleph_{\alpha,\ell}^{\mathtt{T}}(B)\right)=0$. Therefore, from Theorem 3.3, $\mathcal{F}$ has a fixed point z in the Fréchet space $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$ which belongs to the set z and hence the fractional order infinite point boundary value problem (1) has at least one solution in $\mathbb{C}^{\ell,\alpha}(\mathbb{R}_+)$. This completes the proof. □

Example 4.2. *Consider the following fractional order infinite-point boundary value problem*

$$\left.\begin{aligned}&\mathtt{D}_{0^+}^{\frac{5}{2}}\mathtt{z}(\mathtt{t})=\frac{e^{-2t}\sin(\mathtt{z}(\mathtt{t})+1)}{\sqrt{\mathtt{t}+3}},\ \mathtt{t}\in\mathbb{R}_+,\\&\mathtt{z}(0)=\mathtt{z}'(0)=\mathtt{z}''(0)=0,\ \lim_{t\to+\infty}\mathtt{D}_{0^+}^{\frac{3}{2}}\mathtt{z}(\mathtt{t})=\sum_{\mathtt{j}=1}^{\infty}\frac{135}{2\pi^{7/2}\mathtt{j}}\mathtt{z}\left(\frac{1}{\mathtt{j}^2}\right).\end{aligned}\right\} \tag{30}$$

Comparing (30) with (1), we have $\delta=\frac{5}{2},\ell=2,\mathtt{c}_{\mathtt{j}}=\frac{135}{2\pi^{7/2}\mathtt{j}},\tau_{\mathtt{j}}=\frac{1}{\mathtt{j}},\varphi(\tau_{\mathtt{j}})=\tau_{\mathtt{j}}^2$ and

$$\mathtt{g}(\mathtt{t},\mathtt{z})=\frac{e^{-2\mathtt{t}}\sin(\mathtt{z}(\mathtt{t})+1)}{\sqrt{\mathtt{t}+3}}.$$

Let $\alpha=\frac{1}{3}$. Then, $\delta-\ell-\alpha=\frac{1}{6}>0$. Taking $a(\mathtt{t})=b(\mathtt{t})=\mathtt{t}$ and $\lambda=\frac{1}{2\sqrt{3}}$. Now, we check the conditions of Theorem 4.1. For this, let $\mathtt{s}\in\mathbb{R}_+$ and $\mathtt{z}_1,\mathtt{z}_2\in\mathbb{R}$. Then

$$\begin{aligned}|\mathtt{g}(\mathtt{s},\mathtt{z}_1)-\mathtt{g}(\mathtt{s},\mathtt{z}_2)|&=\left|\frac{e^{-2\mathtt{s}}}{\sqrt{\mathtt{s}+3}}\Big[\sin(\mathtt{z}_1(\mathtt{s})+1)-\sin(\mathtt{z}_2(\mathtt{s})+1)\Big]\right|\\&\le\frac{1}{\sqrt{3}}|(\mathtt{z}_1+1)-(\mathtt{z}_2+1)|\\&\le|\mathtt{z}_1-\mathtt{z}_2|,\end{aligned}$$

$$\begin{aligned}\int_0^{\infty}|\mathtt{g}(\mathtt{s},\mathtt{z}_1)-\mathtt{g}(\mathtt{s},\mathtt{z}_2)|d\mathtt{s}&=\int_0^{\infty}\left|\frac{e^{-2\mathtt{s}}}{\sqrt{s+3}}\Big[\sin(\mathtt{z}_1(\mathtt{s})+1)-\sin(\mathtt{z}_2(\mathtt{s})+1)\Big]\right|d\mathtt{s}\\&\le|(\mathtt{z}_1+1)-(\mathtt{z}_2+1)|\int_0^{\infty}\frac{e^{-2\mathtt{s}}}{\sqrt{3}}d\mathtt{s}\\&\le\frac{1}{2\sqrt{3}}|\mathtt{z}_1-\mathtt{z}_2|=\lambda b\Big(|\mathtt{z}_1-\mathtt{z}_2|\Big),\end{aligned}$$

$$\mathtt{g}^{\mathtt{S}}=\sup\left\{\left|\frac{e^{-2\mathtt{s}}}{\sqrt{s+3}}\sin(1)\right|:\mathtt{s}\in\mathbb{R}_+\right\}=\frac{1}{\sqrt{3}}\quad\text{and}\quad\mathtt{g}^{\mathtt{I}}=\int_0^{\infty}\frac{e^{-2\mathtt{s}}}{\sqrt{\mathtt{s}+3}}\sin(1)d\mathtt{s}\le\frac{1}{\sqrt{3}}.$$

Thus, (H_1) satisfied. Also,

$$\sum_{\mathtt{j}=1}^{m}\mathtt{c}_{\mathtt{j}}\varphi(\tau_{\mathtt{j}})^{\delta-1}=\sum_{\mathtt{j}=1}^{m}\frac{135}{2\pi^{7/2}\mathtt{j}^4}\to\frac{3}{4}\sqrt{\pi}=\Gamma(\delta)\ \text{ as }\ m\to+\infty,$$

$$\sum_{j=1}^{\infty} c_j \varphi(\tau_j)^{\delta} = \sum_{j=1}^{m} \frac{135}{2\pi^{7/2} j^6} \to \frac{1}{14}\pi^{5/2} \text{ as } m \to +\infty,$$

and the relation

$$\Big[a(R_T) + g^S\Big]\Big[M_T + N_T \sum_{j=1}^{\infty} c_j \varphi(\tau_j)^{\delta} + \frac{2T^{\delta-\ell-\alpha}}{\Gamma(\delta-\ell+1)}\Big] + \Big[\lambda b(R_T) + g^I\Big] N_T \le R_T$$

gives

$$\Big[R_T + \frac{1}{\sqrt{3}}\Big]\Big[M_T + N_T \frac{\pi^{5/2}}{14} + \frac{2T^{1/6}}{\sqrt{\pi}}\Big] + \Big[\frac{1}{2\sqrt{3}} R_T + g^I\Big] N_T \le R_T,$$

which is equivalent to

$$R_T \ge \frac{\frac{1}{\sqrt{3}}\Big(M_T + N_T \frac{\pi^{5/2}}{14} + \frac{2T^{1/6}}{\sqrt{\pi}}\Big) + g^I N_T}{1 - \Big(M_T + N_T \frac{\pi^{5/2}}{14} + \frac{2T^{1/6}}{\sqrt{\pi}} + N_T \frac{1}{2\sqrt{3}}\Big)},$$

and the right hand side value of the above inequality is finite, so for each $T \in \mathbb{N}$, $R_T > 0$ exists, which is the solution of the inequality in (H_2). Hence from the Theorem 4.1, the fractional order infinite point boundary value problem (30) has at least one solution in the Fr*é*chet space $\mathbb{C}^{2,\frac{1}{3}}(\mathbb{R}_+)$.

Acknowledgement

The authors express their sincere thanks to the unanimous reviewer for the careful reading of the article and the valuable constructive comments those improved the presentation of the article.

References

[1] R.P. Agarwal, M. Meehan, D. O'regan, Fixed point theory and applications, Cambridge University Press, Cambridge, 2001.
[2] A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, J. Comput. Appl. Math., Vol. 260, 68–77, 2014.
[3] A. Aghajani, Y. Jalilian, J.J. Trujillo, On the existence of solutions of fractional integro-difierential equations, Fract. Calc. Appl. Anal., 15(1), 44–69, 2012.
[4] A. Aghajani, N. Sabzali, Existence of coupled fixed points via measure of noncompactness and applications, J. Nonl. Conv. Anal., 15(5), 941–952, 2014.
[5] Z. Bai, H. Lu, Positive solutions for a boundary value problem of nonlinear fractional differential equations, J. Math. Anal. Appl., 311, 495–505, 2005.
[6] J. Banas, On measures of noncompactness in Banach spaces, Comment. Math. Univ. Carolinae, Vol. 21, 131–143, 1980.
[7] J. Banas, K. Goebel, Measure of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, 1980.
[8] M. Benchohra, J.R. Graef, F.Z. Mostafai, Weak solutions for nonlinear fractional difierential equations on reflexive Banach spaces, Electron. J. Qual. Theory., Vol. 54, 1–10, 2010.
[9] M. Borcut, Tripled fixed point theorems for monotone mappings in partially ordered metric spaces, Carpathian J. of Math., 28(2), 215–222, 2012.
[10] P. Borisut, P. Kumam, I. Ahmed K. Sitthithakerngkiet, Nonlinear Caputo Fractional Derivative with Nonlocal Riemann-Liouville Fractional Integral Condition Via Fixed Point Theorems, Symmetry, 11(6), 829, 2019.
[11] L. Cadariu, L. GavruţaL and P. Gavruça, Weighted space method for the stability of some nonlinear equations, Appl. Anal. Discrete Math., 6(1), 126–139, 2012.
[12] K. Cao, Y.Q. Chen, D. Stuart, A Fractional Micro-Macro Model for Crowds of Pedestrians Based on Fractional Mean Field Games, IEEE/CAA J. Automatica Sinica, 3(3), 261–270, 2016.
[13] S. S. Chang, Y. J. Cho, N. J. Huang, Coupled fixed point theorems with applications, J. Korean Math. Soc., Vol. 33, 575–585, 1996.
[14] G. Darbo, Punti uniti in trasformazioni a codominio non compatto. Rend. Sem. Mat. Un. Padova., Vol. 24, 84–92, 1955.
[15] L. Debanath, Recent Applications of Fractional Calculus to Science and Engineering, 3413–3442, 2003.
[16] C. Derbazi, H. Hammouche, M. Benchohra, Weak solutions for some nonlinear fractional differential equations with fractional integral boundary conditions in Banach space, J. Nonlinear funct. Anal., Vol. 2019, Article ID 7, 2019.
[17] M. Q. Feng, X. M. Zhang, W. G. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, Bound. Value Probl., Vol. 2011, Art. ID 720702, 20 pp.

[18] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc., Vol. 40, 604–608, 1955.

[19] M. Izadi, H. M. Srivastava, A novel matrix technique for multi-order pantograph differential equations of fractional order, Proc. Roy. Soc. London Ser. A: Math. Phys. Engrg. Sci., 477, Article ID 2021031, 1–21, 2021.

[20] H. A. Kayvanloo, M. Khanehgir, and R. Allahyari, A family of measures of noncompactness in the Hölder space $C^{n,\gamma}(\mathbb{R}_+)$ and its application to some fractional differential equations and numerical methods, J. Comp. Appl. Math., Vol. 363, 256–272, 2020.

[21] B. Hazarika, H. M. Srivastava, R. Arab, and M. Rabbani, Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it, J. Comput. Appl. Math. 343, 341–352, 2018.

[22] B. Hazarika, H. M. Srivastava, R. Arab, and M. Rabbani, Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution, Appl. Math. Comput., 360, 131–146, 2019.

[23] M. Khuddush, K. R. Prasad, Infinitely many positive solutions for an iterative system of conformable fractional order dynamic boundary value problems on time scales, Turk. J. Math., (2021). DOI: 10.3906/mat-2103-117

[24] M. Khuddush, K. R. Prasad, Existence, uniqueness and stability analysis of a tempered fractional order thermistor boundary value problems, J. Anal (2022). https://doi.org/10.1007/s41478-022-00438-6

[25] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204, Elsevier Science B. V., Amsterdam 2006.

[26] K. Kuratowski, Sur les espaces complets, Fund. Math., Vol. 15, 301–309, 1930.

[27] H. Lee, S. Kim, Multivariate coupled fixed point theorems on ordered partial metric spaces, J. Korean Math. Soc., 51, 1189–1207, 2014.

[28] K. Li, J. Peng, J. Gao, Nonlocal fractional semilinear Differential equations in separable Banach spaces, E. J. Differential Equations, (2013)(7), 1–7, 2013.

[29] J. Liang, Z. Liu, X. Wang, Solvability for a couple system of nonlinear fractional differential equations in a Banach space, Fract. Calc. Appl. Anal., 6(1), 51–63, 2013.

[30] S. A. Mohiuddine, H. M. Srivastava, and A. Alotaibi, Application of measures of noncompactness to the infinite system of second-order differential equations in ℓ_p spaces, Adv. Difference Equations 2016, Article ID 317, 1–13, 2016.

[31] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[32] K. R. Prasad, M. Khuddush, D. Leela, Existence of solutions for n-dimensional fractional order hybrid BVPs with integral boundary conditions by an application of n-fixed point theorem, J. Anal., (2021). https://doi.org/10.1007/s41478-020-00291-5

[33] K. R. Prasad, D. Leela, M. Khuddush, Existence and uniqueness of positive solutions for system of (p, q, r)-Laplacian fractional order boundary value problems, Adv. Theory Nonlinear Anal. Appl., 5(1), 138–157, 2021. https://doi.org/10.31197/atnaa.703304

[34] K. R. Prasad, M. Khuddush, D. Leela, Existence, uniqueness and Hyers–Ulam stability of a fractional order iterative two-point boundary value Problems, Afr. Mat., (2021). https://doi.org/10.1007/s13370-021-00895-5

[35] K. R. Prasad, M. Khuddush, D. Leela, Existence of solutions for infinite systems of regular fractional Sturm-Liouville problems in the spaces of tempered sequences, Tbilisi Math. J., 13 (4), 193–209, 2020. DOI: 10.32513/tbilisi/1608606058

[36] A. Salem, H. M. Alshehri, L. Almaghamsi, Measure of noncompactness for an infinite system of fractional Langevin equation in a sequence space, Adv. Differ. Equ. 2021(132), 2021. https://doi.org/10.1186/s13662-021-03302-2

[37] H. M. Srivastava, A. Das, B. Hazarika, and S. A. Mohiuddine, Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces c0 and l1 via the measure of noncompactness, Math. Methods Appl. Sci., 41, 3558–3569, 2018.

[38] H. M. Srivastava, Diabetes and its resulting complications: Mathematical modeling via fractional calculus, Public Health Open Access, 4 (3), 1–5, 2020.

[39] H. M. Srivastava, K. M. Saad, M. M. Khader, An efficient spectral collocation method for the dynamic simulation of the fractional epidemiological model of the Ebola virus, Chaos, Solitons and Fractals, 140, 1–7, 2020.

[40] H. M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, Kyungpook Math. J., 60, 73–116, 2020.

[41] V. E. Tarasov, V. V. Tarasova, Time dependent fractional dynamics with memory in quantum and economic physics, Annals of Physics, 383, 579–599, 2017.

[42] C. C. Tisdell, A. Zaidi, Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling, Nonlinear Anal., Vol. 68, 3504–3524, 2008.

[43] F. J. Torres, Existence of a positive solution for a boundary value problem of a nonlinear fractional differential equation, Bull. Iran. Math. Soc., 39(2), 307–323, 2013.

[44] D. W. J. Victor, M. Khuddush, Existence of solutions for n-dimensional fractional order bvp with ∞-point boundary conditions via the concept of measure of noncompactness, Advanced Studies: Euro-Tbilisi Mathematical Journal, 15(1), 19–37, 2022.

[45] J.R. Wang, Y. Zhou, M. Feckan, Abstract Cauchy problem for fractional differential equations, Nonlinear Dyn., 71:685 (2013) DOI 10.1007/s11071-012-0452-9.

[46] S. Q. Zhang, Existence results of positive solutions to fractional differential equation with integral boundary conditions, Math. Bohem., 135(2), 299–317, 2010.