Filomat 36:10 (2022), 3545–3549
https://doi.org/10.2298/FIL2210545K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Super Metric Spaces

Erdal Karapınar[a,b,c], **Farshid Khojasteh**[d]

[a]*Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot City 820000, Binh Duong Province, Vietnam*
[b]*Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan*
[d]*Department of Mathematics, Arak Branch, Islamic Azad University, Arak, Iran*

Abstract. The aim of this paper is to propose a new generalization of metric space which may open a new framework. As an application, we consider the analog of Banach contraction mapping principle that works properly.

1. Introduction

In an axiomatic framework, it would not be wrong to attribute the emergence of the concept of the distance to Euclid, maybe even earlier. The systematization and standardization of the distance in abstract mathematics were proposed by Frechét [2] under the name of ”L-function” which is known as ”metric” after Hausdorff:
For a nonempty set $\mathfrak{X}$, a distance function $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ is Euclidean metric or ”standard metric” or only ”metric” if it fulfills the following conditions:

($\mathfrak{c}$1) self-distance: $\mathfrak{d}(x, y) = 0$, if and only if, $x = y$ for all $x, y \in \mathfrak{X}$,
($\mathfrak{c}$2) symmetry: $\mathfrak{d}(x, y) = \mathfrak{d}(y, x)$, for all $x, y \in \mathfrak{X}$.
($\mathfrak{c}$3) triangular inequality: $\mathfrak{d}(x, y) \leq \mathfrak{d}(x, z) + \mathfrak{d}(z, y)$ for all $x, y, z \in \mathfrak{X}$,

Here, the pair $(\mathfrak{X}, \mathfrak{d})$ is called ”Euclidean metric space” or ”standard metric space” or simply ”metric space”.

The notion of metric space has been used not only in mathematics but qualitative sciences. For example, one of the interesting generalizations of metric, so called, partial metric [10] was given to solve the certain problems of Domain Theory of Computer Science. Later, the notion of partial metric was extended with the new notion, dislocated metric [13]. Besides these abstract constructions, the metric notion has been generalized and extended in various distinct ways. Among all, we shall recall some of these generalizations which are very familiar and mostly interesting. One of the early generalization is quasi-metric that is obtained by omitting the axiom ($\mathfrak{c}$2). Another early proposed notion is semi-metric [11] which satisfies only ($\mathfrak{c}$1) and ($\mathfrak{c}$2). On the other hand, Branciari [12] consider a distance which is obtained by changing the triangular inequality in metric with quadrilateral inequality. Another generalization of metric was obtained

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. super metric space, fixed point, contraction
Received: 05 April 2022; Revised: 14 May 2022; Accepted: 24 August 2022
Communicated by Dragan S. Djordjević
The second author wouldlike to thanks for financial support of the research Islamic Azad University of Arak
Email addresses: erdalkarapinar@tdmu.edu.vn,erdalkarapinar@yahoo.com (Erdal Karapınar), Fa.Khojasteh@iau.ac.ir,fr_khojasteh@yahoo.com (Farshid Khojasteh)

by replacing the triangle inequality with the modified version, $\mathfrak{d}(x,y) \leq s[\mathfrak{d}(x,z) + \mathfrak{d}(z,y)]$ for all $x, y, z \in \mathfrak{X}$ and for fixed $s \geq 1$. This new notion is known quasi-metric [16] in some sources, and b-metric [14],[15] in some other sources. In what follows, we recall cone metric [17] that was obtained by changing $[0,+\infty)$ in the definition of mapping $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \to [0,+\infty)$ by a cone of a Banach space, with certain properties. This idea was extended as complex-valued metric [18], and later, as quaternion-valued metric [22]. Another trend of the extension of metric is based on the geometry of three points instead of two points, such as, $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to [0,+\infty)$. The notion of 2-metric [6], D-metric [7], G-metric [8], S-metric [9] are the famous examples of this trend. There are more notions, such as multiplicative metric [5], ultra-metric, partial metric [23],m-metric[24–26], modular-metric [4], fuzzy metric [3], and so on (see [19–21] for more details). Unfortunately, not all of the above notions have a worth. As it shown in [1], some of them are equivalent to each others.

In this manuscript, we shall introduce a new generalization of metric notion which is Hausdorff. Further, in the topology of this new metric space, open (respectively, closed) ball is open (respectively, closed) set. Further, we consider the analog of Banach contraction mapping principle that works properly.

2. Main Result

In this section we introduce a new extension of a metric space and we examine slightly its topology.

Definition 2.1. *Let $\mathfrak{X}$ be a nonempty set. We say that $\mathfrak{m} : \mathfrak{X} \times \mathfrak{X} \to [0,+\infty)$ is super-metric or super metric if*

1. *if $\mathfrak{m}(x,y) = 0$, then $x = y$ for all $x, y \in \mathfrak{X}$*
2. *$\mathfrak{m}(x,y) = \mathfrak{m}(y,x)$, for all $x, y \in \mathfrak{X}$,*
3. *there exists $s \geq 1$ such that for all $y \in \mathfrak{X}$ there exist distinct sequences $(x_n), (y_n) \subset \mathfrak{X}$, with $\mathfrak{m}(x_n, y_n) \to 0$ when n tends to infinity, such that*

$$\limsup_{n\to\infty} \mathfrak{m}(y_n, y) \leq s \limsup_{n\to\infty} \mathfrak{m}(x_n, y).$$

Then, we call $(\mathfrak{X}, \mathfrak{m})$ a super metric space.

Example 2.2. *Let $X = [0,+\infty]$ and define*

$$\mathfrak{m}(x,y) = \begin{cases} \dfrac{x+y}{1+x+y} & x \neq y, x \neq 0, y \neq 0 \\ 0 & x = y \\ \max\left\{\frac{x}{2}, \frac{y}{2}\right\} & \text{otherwise} \end{cases}$$

Suppose that $y \in \mathfrak{X}$ and $(x_n), (y_n)$ are two distinct sequences in $\mathfrak{X}$ such that $\mathfrak{m}(x,y) \to 0$ as $n \to \infty$. Since the sequences are distinct we have $\mathfrak{m}(x_n, y_n) = \frac{x_n+y_n}{x_n+y_n+1} \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Therefore, there exists $N > 0$ such that for all $n \geq N$, we have

$$\limsup_{n\to\infty} \mathfrak{m}(y_n, y) = \limsup_{n\to\infty} \frac{y_n + y}{y_n + y + 1} = \frac{y}{y+1} \leq s\frac{y}{y+1} = s \limsup_{n\to\infty} \frac{x_n + y}{x_n + y + 1}.$$

In the case $y = 0$ the proof is straightforward. Thus, $(\mathfrak{X}, \mathfrak{m})$ is a super metric space. It is worth mentioning that the space is not JS metric because considering a sequence (x_n) such that $x_n \to x$, in this metric is possible while $x_n \to 0$ and $x = 0$. Hence, for all $y \in \mathfrak{X}$, we have $\mathfrak{m}(x,y) = \frac{y}{2}$. Also, $\mathfrak{m}(x_n, y) = \frac{x_n+y}{x_n+y+1}$. If there exists $s \geq 1$ such that

$$\frac{y}{2} = \mathfrak{m}(x,y) \leq s \limsup_{n\to\infty} \frac{x_n + y}{x_n + y + 1} = s\left(\frac{y}{y+1}\right).$$

Then, for all $y \in \mathfrak{X}$ we have $s \geq \frac{y}{y+1}$ and this is a contradiction because $\mathfrak{X}$ is unbounded. Therefore, $(\mathfrak{X}, \mathfrak{m})$ is not JS-metric space.

Definition 2.3. *Let* $(\mathfrak{X}, \mathfrak{m})$ *be a super metric space and let* $\{x_n\}$ *be a sequence in* $\mathfrak{X}$. *We say that* $\{x_n\}$ *converges to* x *in* $\mathfrak{X}$, *if and only if,* $\mathfrak{m}(x_n, x) \to 0$, *as* $n \to \infty$.

Definition 2.4. *Let* $(\mathfrak{X}, \mathfrak{m})$ *be a super metric space and let* $\{x_n\}$ *be a sequence in* $\mathfrak{X}$. *We say that* $\{x_n\}$ *is a Cauchy sequence in* $\mathfrak{X}$, *if and only if,* $\lim_{n\to\infty} \sup\{\mathfrak{m}(x_n, x_m) : m > n\} = 0$.

Definition 2.5. *Let* $(\mathfrak{X}, \mathfrak{m})$ *be a super metric space. We say that* $(\mathfrak{X}, \mathfrak{m})$ *is a complete super metric space, if and only if, every Cauchy sequence is convergent in* $\mathfrak{X}$.

2.1. *Banach fixed point theorem in super metric spaces*

In this section, we prove the Banach fixed point theorem in super metric spaces.

Theorem 2.6. *Let* $(\mathfrak{X}, \mathfrak{m})$ *be a complete super-metric space and let* $T : \mathfrak{X} \to \mathfrak{X}$ *be a mapping. Suppose that* $0 < \alpha < 1$ *such that*

$$\mathfrak{m}(Tx, Ty) \leq \alpha \mathfrak{m}(x, y), \tag{1}$$

for all $x, y \in \mathfrak{X}$. *Then* T *has a unique fixed point in* $\mathfrak{X}$.

Proof. Let $x_0 \in \mathfrak{X}$ and let $x_1 = Tx_0$. If $x_0 = x_1$ then x_1 is the fixed point and the proof is completed. So suppose that $x_0 \neq x_1$. Thus, $\mathfrak{m}(x_0, x_1) > 0$. Thus, without loss of generality, we can define $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$. So $\mathfrak{m}(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. So we have

$$\begin{aligned} \mathfrak{m}(x_n, x_{n+1}) &\leq \alpha \mathfrak{m}(x_n, x_{n-1}) \\ &\leq \alpha^2 \mathfrak{m}(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n \mathfrak{m}(x_1, x_0), \end{aligned} \tag{2}$$

Taking limit from both side of 9 implies that

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0. \tag{3}$$

Now suppose that, $m, n \in \mathbb{N}$ and $m > n$. If $x_n = x_m$, we have $T^m(x_0) = T^n(x_0)$. Thus we have, $T^{m-n}(T^n(x_0)) = T^n(x_0)$. Thus, we have $T^n(x_0)$ is the fixed point of T^{m-n}. Also,

$$T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0)).$$

It means that, $T(T^n(x_0))$ is the fixed point of T^{m-n}. Thus, $T(T^n(x_0)) = T^n(x_0)$. So $T^n(x_0)$ is the fixed point of T. So without loss of generality we can suppose that $x_n \neq x_m$. Therefore,

$$\limsup_{n\to\infty} \mathfrak{m}(x_n, x_{n+2}) \leq s \limsup_{n\to\infty} \mathfrak{m}(x_{n+1}, x_{n+2}). \tag{4}$$

Thus, since $\limsup_{n\to\infty} \mathfrak{m}(x_n, x_{n+2}) = 0$, we

$$\limsup_{n\to\infty} \mathfrak{m}(x_n, x_{n+3}) \leq s \limsup_{n\to\infty} \mathfrak{m}(x_{n+2}, x_{n+3}) = 0. \tag{5}$$

Inductively, one can conclude that $\lim_{n\to\infty} \sup\{\mathfrak{m}(x_n, x_m) : m > n\} = 0$. It means that $\{x_n\}$ is a Cauchy sequence. Since $(\mathfrak{X}, \mathfrak{m}$ is a complete super metric space, the sequence $\{x_n\}$ converges to $z \in \mathfrak{X}$. We claim that z is the fixed point of T. On the contrary, assume $\mathfrak{m}(z, Tz) > 0$. Note that

$$\mathfrak{m}(x_{n+1}, Tz) = \mathfrak{m}(Tx_n, Tz) \leq \alpha \mathfrak{m}(x_n, z) \to 0 \ \ (as\ n \to \infty). \tag{6}$$

Thus, $\lim_{n\to\infty} \mathfrak{m}(x_{n+1}, Tz) = 0$. If there $N > 0$ such that for all $n > N$, $x_{N+1} = z$, (6) concludes that $\mathfrak{m}(z, Tz) = 0$ and so we have z is the fixed point for T. Otherwise, suppose that for all $n \in \mathbb{N}$, $x_n \neq z$. Thus we have,

$$\mathfrak{m}(z, Tz) \leq s \limsup_{n\to\infty} \mathfrak{m}(x_{n+1}, Tz), \tag{7}$$

and one can conclude that $\mathfrak{m}(z, Tz) = 0$, which is a contradiction. Thus, $z = Tz$ is the fixed point of T in $\mathfrak{X}$. Also, the uniqueness of the fixed point is straightforward from (8). □

Example 2.7. *Let* $\mathfrak{X} = [2,3]$ *and define*

$$\mathfrak{m}(x,y) = \begin{cases} xy & x \neq y, \\ 0 & x = y. \end{cases}$$

Let $(x_n), (y_n)$ *be two distinct sequences such that* $\mathfrak{m}(x_n, y_n) \to 0$ *as* $n \to \infty$*. Since the sequences are distinct we have* $\mathfrak{m}(x_n, y_n) = x_n y_n \to 0$*, and we can choose* $y_n \to 0$ *and* $x_n \to u$ *as* $n \to \infty$*, where* $u \in \mathfrak{X}$*. Moreover, for any* $y \in \mathfrak{X}$*,*

$$\limsup_{n\to\infty} \mathfrak{m}(y_n, y) = \limsup_{n\to\infty} y_n y = 0 \leq s \limsup_{n\to\infty} \mathfrak{m}(x_n, y) = \limsup_{n\to\infty} x_n y = u \cdot y,$$

and it follows that $(\mathfrak{X}, \mathfrak{m})$ *is a super-metric space. Now consider* $T : \mathfrak{X} \to \mathfrak{X}$ *as follows*

$$Tx = \begin{cases} 2 & , \quad x \neq 3, \\ \frac{3}{2} & , \quad x = 3. \end{cases}$$

Considering $s = \frac{9}{4}$*,* $\alpha = \frac{1}{2}$*,* $x \neq 3$ *and* $y = 3$*, we have*

$$\mathfrak{m}(Tx, Ty) = \mathfrak{m}(2, \frac{3}{2}) = 2 \times \frac{3}{2} = 3 \leq \frac{1}{2} \times 3x = \alpha \mathfrak{m}(x, y),$$

because $2 \leq x < 3$*. The other cases are straightforward and so the mapping* T *has a unique fixed point by Theorem 2.6 for* $\alpha = \frac{1}{2}$*. But note that if we consider the Euclidean metric* $D(x, y) = |x - y|$*, then for all* $\alpha \in [0, 1]$*, if* $x_n = 3 - \frac{1}{n}$ *and* $y = 3$*, we have*

$$|Tx - Ty| = |2 - \frac{3}{2}| = \frac{1}{2} > \alpha|(3 - \frac{1}{n}) - 3| = \frac{\alpha}{n},$$

since $0 \leq \alpha \leq 1 < \frac{n}{2}$*, for all* $n \geq 2$*. Thus,* T *is not a Banach contraction with respect to* $(\mathfrak{X}, D)$*.*

Theorem 2.8. *Let* $(\mathfrak{X}, \mathfrak{m})$ *be a complete super-metric space and let* $T : \mathfrak{X} \to \mathfrak{X}$ *be a continuous mapping. Suppose that* $\varphi : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ *is an upper-semi continuous mapping such that*

$$\mathfrak{m}(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{8}$$

for all $x, y \in \mathfrak{X}$*. Then* T *has a fixed point in* $\mathfrak{X}$*.*

Proof. Let $x_0 \in \mathfrak{X}$ and let $x_1 = Tx_0$. If $x_0 = x_1$ then x_1 is the fixed point and the proof is completed. So suppose that $x_0 \neq x_1$. Thus, $\mathfrak{m}(x_0, x_1) > 0$. Thus, without loss of generality, we can define $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$. So $\mathfrak{m}(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. So we have

$$\mathfrak{m}(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \tag{9}$$

so we have

$$\sum_{n=1}^{k} \mathfrak{m}(x_n, x_{n+1}) \leq \sum_{n=1}^{k} \varphi(x_n) - \varphi(x_{n+1}) = \varphi(x_1) - \varphi(x_{k+1}) < \varphi(x_1). \tag{10}$$

Thus, $\sum_{n=1}^{\infty} \mathfrak{m}(x_n, x_{n+1}) < \infty$. So we have

$$\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0. \tag{11}$$

Now suppose that, $m, n \in \mathbb{N}$ and $m > n$. If $x_n = x_m$, we have $T^m(x_0) = T^n(x_0)$. Thus we have, $T^{m-n}(T^n(x_0)) = T^n(x_0)$. Thus, we have $T^n(x_0)$ is the fixed point of T^{m-n}. Also,

$$T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0)).$$

It means that, $T(T^n(x_0))$ is the fixed point of T^{m-n}. Thus, $T(T^n(x_0)) = T^n(x_0)$. So $T^n(x_0)$ is the fixed point of T. So without loss of generality we can suppose that $x_n \neq x_m$. Therefore,

$$\limsup_{n\to\infty} \mathfrak{m}(x_n, x_{n+2}) \leq s\mathfrak{m}(x_{n+1}, x_{n+2}). \tag{12}$$

Thus, since $\mathfrak{m}(x_n, x_{n+2}) \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \mathfrak{m}(x_n, x_{n+3}) \leq s\mathfrak{m}(x_{n+2}, x_{n+3}). \tag{13}$$

Again, inductively, one can found that $\lim_{n\to\infty} \sup\{\mathfrak{m}(x_n, x_m) : m > n\} = 0$. It means that $\{x_n\}$ is a Cauchy sequence. Since $(\mathfrak{X}, \mathfrak{m})$ is a complete super metric space, the sequence $\{x_n\}$ converges to $z \in \mathfrak{X}$.

Taking into account the continuity assumption of the mapping T, it follows that $z = Tz$ is the fixed point of T in $\mathfrak{X}$. □

References

[1] E.Karapınar, Recent advances on metric fixed point theory: A review, Appl. Comput. Math., (in press)
[2] M.R. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo **22** (1906) 1-74. doi: 10.1007/BF03018603.
[3] L. A. Zadeh, Fuzzy sets, Information and Control, vol. 8, no. 3, pp. 338-353, 1965.
[4] V.V. Chistyakov, Modular metric spaces, I: basic concepts, Nonlinear Anal., 72 (2010), 1–14.
[5] A.E. Bashirov, E.M. Kurpınar and A. Ozyapici, Multiplicative calculus and its applications J. Math. Anal. Appl.,337 (2008), 36-48.
[6] S. Gahler, 2-metriche raume und ihre topologische strukture, Math. Nachr. **26**, (1963), 115-148.
[7] B.C. Dhage, *Generalized Metric Space and Mapping With Fixed Point*, Bull. Cal. Math. Soc. **84**, (1992), 329–336
[8] Z. Mustafa and B. Sims, A New Approach to Generalized Metric Spaces, *Journal of Nonlinear and Convex Analysis*, 7, No. 2 (2006). 289–297.
[9] S. Sedghi, N. V. Dung, Fixed point theorems on Super metric spaces, Mat. Vesnik 66,1(2014), 113-124
[10] S.G. Matthews, Partial metric topology, Proc. 8th Summer of Conference on General Topology and Applications, Ann. New York Aced. Sci., **728** (1994) 183–197.
[11] W. A. Wilson, On Semi-Metric Spaces, American Journal of Mathematics, Vol. 53, No. 2 (Apr., 1931), pp. 361-373 (13 pages) https://doi.org/10.2307/2370790
[12] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type *Int. J. Math. Math. Sci.*, 29,(2002),531-536
[13] P. Hitzler and A. Seda, Mathematical Aspects of Logic Programming Semantics, Studies in Informatics Series, Chapman and Hall, CRC Press, Taylor and Francis Group (2011).
[14] Bakhtin, I.A., (1989), The contraction mapping principle in quasimetric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. **30**, 26–37.
[15] Czerwik, S., (1993), Contraction mappings in b-metric spaces. Acta. Math. Inform. Univ. Ostraviensis. **1**, 5–11.
[16] V. Berinde, *Generalized contractions in quasi-metric spaces*, Seminar on Fixed Point Theory, Babeş-Bolyai University, Research Sem., (1993), 3–9.
[17] L.-G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468-1476.
[18] A. Azam, B. Fisher, M. Khan, Common Fixed point theorems in complex valued metric spaces.Numerical Functional Analysis and Optimization, **32** (2011), 243–253.
[19] F. Khojasteh, E. Karapınar, H. Khandani, Some applications of Caristi's fixed point theorem in metric spaces, Fixed Point Theory and Applications, 2016 (1), 1–10.
[20] R.P. Agarwal, H.H. Alsulami, E. Karapınar, F. Khojasteh, Remarks on some recent fixed point results on quaternion-valued metric spaces, Abstract and Applied Analysis 2014, 14 (2014)
[21] F. Khojasteh, A. Razani, S. Moradi, A fixed point of generalized TF-contraction mappings in cone metric spaces, Fixed Point Theory and Applications, **1** (2011), 1–15.
[22] A. E-S. Ahmed, A. Asad, S. Omran, Fixed point theorems in quaternion-valued metric spaces, Abstract and Applied Analysis,(2014) Article Id. 258985
[23] Matthews, S.G.: Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728**, 183–197 (1994)
[24] M. Asadi, E. Karapinar and P. Salimi, New extension of p-metric spaces with some fixed point results on m-metric spaces, Journal of Inequalities and Applications, 2014, 2014:18
[25] M. Asadi, M. Azhini, E.Karapinar, and H. Monfared, Simulation Functions Over M-Metric Spaces, East Asian Math. J. Vol. 33 (2017), No. 5, 559–570. http://dx.doi.org/10.7858/eamj.2017.039
[26] E. Karapınar, Shimaa I. Moustafa , Ayman Shehata, Ravi P. Agarwal Fractional Hybrid Differential Equations and Coupled Fixed-Point Results for α-Admissible $F(\psi_1, \psi_2)$ Contractions in M-Metric Spaces Volume 2020, Article ID 7126045, 13 pages; https://doi.org/10.1155/2020/7126045