Filomat 36:10 (2022), 3231–3239
https://doi.org/10.2298/FIL2210231D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Non-Linear Bi-Skew Jordan Derivations on $*$-Algebra

Vahid Darvish[a], **Mojtaba Nouri**[b], **Mehran Razeghi**[c]

[a] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, China*
Reading Academy, Nanjing University of Information Science and Technology, Nanjing, China
[b] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[c] *Department of Mathematics, Farhangian University, Tehran, Iran*

Abstract. Let $\mathcal{A}$ be a prime $*$-algebra. In this paper, we suppose that $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$$

where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$. Then, Φ is an additive $*$-derivation.

1. Introduction

Let $\mathcal{R}$ be a $*$-algebra. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [3, 8, 10, 14]).

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if $\Phi(A + B) = \Phi(A) + \Phi(B)$ and $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{R}$. A map Φ is an additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, and have been studied intensively ([2, 11–13, 17]).

Let us define λ-Jordan $*$-product by $A \bullet_\lambda B = AB + \lambda BA^*$. We say that the map Φ with the property of $\Phi(A \bullet_\lambda B) = \Phi(A) \bullet_\lambda B + A \bullet_\lambda \Phi(B)$ is a λ-Jordan $*$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [1, 15].

A von Neumann algebra $\mathcal{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies the double commutant property: $\mathcal{A}'' = \mathcal{A}$ where $\mathcal{A}' = \{T \in B(H), TA = AT, \forall A \in \mathcal{A}\}$ and $\mathcal{A}'' = \{\mathcal{A}'\}'$. Denote by $\mathcal{Z}(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A}$ the center of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a factor if its center is trivial, that is, $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$. For $A \in \mathcal{A}$, recall that the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P such that $PA = A$. It is not difficult to see that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint,

2020 *Mathematics Subject Classification*. 46J10, 47B48, 46L10
Keywords. New product derivation, Prime $*$-algebra, Additive map
Received: 15 September 2019; Accepted: 24 August 2022
Communicated by Dragan S. Djordjević
Corresponding author: Vahid Darvish
The second author is supported by the Talented Young Scientist Program of Ministry of Science and Technology of China (Iran-19-001).
Email addresses: vahid.darvish@mail.com, vdarvish@nuist.edu.cn (Vahid Darvish), mojtaba.nori2010@gmail.com (Mojtaba Nouri), razeghi.mehran19@yahoo.com (Mehran Razeghi)

then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$ (see [9]). It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, [5, 6].

Recently, Yu and Zhang in [18] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [7] have investigated a non-linear λ-Jordan $*$-derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scalar, then $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

On the other hand, many mathematician devoted themselves to study the $*$-Jordan product $A \bullet B = AB + BA^*$. In [19], F. Zhang proved that every non-linear $*$-Jordan derivation map $\Phi : \mathcal{A} \to \mathcal{A}$ on a factor von neumann algebra with $I_{\mathcal{A}}$ the identity of it is an additive $*$-derivation.

In [16], we showed that $*$-Jordan derivation map on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is an additive $*$-derivation.

Very recently the authors of [4] discussed some bijective maps preserving the new product $A^*B + B^*A$ between von Neumann algebras with no central abelian projections. In other words, Φ holds in the following condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Motivated by the above results, in this paper, we prove that if $\mathcal{A}$ is a prime $*$-algebra then $\Phi : \mathcal{A} \to \mathcal{A}$ which holds in the following condition

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$$

where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$, is an additive $*$-derivation.

We say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

2. Main Results

In this section, we show that Φ which satisfies in the following assumption is an $*$-additive derivation.

Assumption 1. *Let $\mathcal{A}$ be a prime $*$-algebra and*

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B) \tag{1}$$

for $\Phi : \mathcal{A} \to \mathcal{A}$ where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$.

It is easy to prove $\Phi(0) = 0$. We need the following lemmas:

Lemma 2.1. *Let Φ satisfy in Assumption 1, then we show that $\Phi(\frac{I}{2}) = 0$, $\Phi(-\frac{I}{2}) = 0$ and $\Phi(i\frac{I}{2}) = 0$.*

Proof. For showing that $\Phi\left(\frac{I}{2}\right) = 0$, we write

$$\Phi\left(\frac{I}{2} \diamond \frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right) \diamond \frac{I}{2} + \frac{I}{2} \diamond \Phi\left(\frac{I}{2}\right).$$

So,

$$\Phi\left(\frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^*. \tag{2}$$

So $\Phi(\frac{I}{2})^* = 0$ then $\Phi(\frac{I}{2}) = 0$

For proving $\Phi\left(-\frac{I}{2}\right) = 0$, we can write

$$\Phi\left(\frac{I}{2} \diamond -\frac{I}{2}\right) = \frac{I}{2} \diamond \Phi\left(-\frac{I}{2}\right).$$

It follows that

$$\Phi\left(-\frac{I}{2}\right) = \frac{\Phi\left(-\frac{I}{2}\right) + \Phi\left(-\frac{I}{2}\right)^*}{2}. \tag{3}$$

It follows from the above equation that $\Phi\left(-\frac{I}{2}\right)$ is self-adjoint.
On the other hand, we have

$$\Phi\left(-\frac{I}{2} \diamond -\frac{I}{2}\right) = -\frac{I}{2} \diamond \Phi\left(-\frac{I}{2}\right) + \Phi\left(-\frac{I}{2}\right) \diamond -\frac{I}{2}.$$

Then

$$\Phi\left(\frac{I}{2}\right) = -\left(\Phi\left(-\frac{I}{2}\right) + \Phi\left(-\frac{I}{2}\right)^*\right). \tag{4}$$

Since $\Phi\left(-\frac{I}{2}\right)$ is self-adjoint, from (4) we have the result.

For showing $\Phi\left(i\frac{I}{2}\right) = 0$, we have the following

$$\Phi\left(i\frac{I}{2} \diamond i\frac{I}{2}\right) = \Phi\left(i\frac{I}{2}\right) \diamond i\frac{I}{2} + i\frac{I}{2}\Phi\left(i\frac{I}{2}\right).$$

Hence,

$$\Phi\left(\frac{I}{2}\right) = -i\Phi\left(i\frac{I}{2}\right) + i\Phi\left(i\frac{I}{2}\right)^*.$$

So,

$$\Phi\left(i\frac{I}{2}\right)^* - \Phi\left(i\frac{I}{2}\right) = 0 \tag{5}$$

Also, we have

$$\Phi\left(\frac{I}{2} \diamond i\frac{I}{2}\right) = \frac{I}{2} \diamond \Phi\left(i\frac{I}{2}\right).$$

So,

$$\Phi(0) = \frac{\Phi\left(i\frac{I}{2}\right)^* + \Phi\left(i\frac{I}{2}\right)}{2}.$$

Then, we have

$$\Phi\left(i\frac{I}{2}\right)^* + \Phi\left(i\frac{I}{2}\right) = 0. \tag{6}$$

From (5) and (6), we have $\Phi\left(i\frac{I}{2}\right) = 0$. □

Lemma 2.2. *Let Φ satisfy in Assumption 1 then we show that*

1. $\Phi(-iA) = -i\Phi(A)$.
2. $\Phi(iA) = i\Phi(A)$.

Proof. By Lemma 2.1, we can check that

$$\Phi\left(-iA \diamond \frac{I}{2}\right) = \Phi\left(A \diamond i\frac{I}{2}\right).$$

So,

$$\Phi(-iA) \diamond \frac{I}{2} = \Phi(A) \diamond i\frac{I}{2}.$$

It follows that

$$\Phi(-iA)^* + \Phi(-iA) = i\Phi(A)^* - i\Phi(A). \tag{7}$$

On the other hand, one can check that

$$\Phi\left(-iA \diamond i\frac{I}{2}\right) = \Phi\left(-\frac{I}{2} \diamond A\right).$$

So,

$$\Phi(-iA) \diamond i\frac{I}{2} = -\frac{I}{2} \diamond \Phi(A).$$

It follows that

$$i\Phi(-iA)^* - i\Phi(-iA) = -\Phi(A) - \Phi(A)^*. \tag{8}$$

Equivalently, we obtain

$$-\Phi(-iA)^* + \Phi(-iA) = -i\Phi(A) - i\Phi(A)^*. \tag{9}$$

By adding equations (7) and (9) we have

$$\Phi(-iA) = -i\Phi(A).$$

Similarly, we can show that $\Phi(iA) = i\Phi(A)$. □

Our main theorem is as follows:

Theorem 2.3. *Let $\mathcal{A}$ be a prime $*$-algebra. Let $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies in*

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B) \tag{10}$$

where $A \diamond B = A^*B + B^*A$ for all $A, B \in \mathcal{A}$, then Φ is an $*$-additive derivation.

Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$.

Proof of Theorem 2.3. We give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i, j = 1, 2$.

Claim 1. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$ we have*

$$\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).$$

Let $T = \Phi(A_{11} + A_{12}) - \Phi(A_{11}) - \Phi(A_{12})$, we should prove that $T = 0$.
For $X_{21} \in \mathcal{A}_{21}$ we can write that

$$\begin{aligned}
&\Phi(A_{11} + A_{12}) \diamond X_{21} + (A_{11} + A_{12}) \diamond \Phi(X_{21}) = \Phi((A_{11} + A_{12}) \diamond X_{21})\\
&= \Phi(A_{11} \diamond X_{21}) + \Phi(A_{12} \diamond X_{21}) = \Phi(A_{11}) \diamond X_{21} + A_{11} \diamond \Phi(X_{21})\\
&+\Phi(A_{12}) \diamond X_{21} + A_{12} \diamond \Phi(X_{21})\\
&= (\Phi(A_{11}) + \Phi(A_{12})) \diamond X_{21} + (A_{11} + A_{12}) \diamond \Phi(X_{21}).
\end{aligned}$$

So, we obtain

$$T \diamond X_{21} = 0.$$

Since $T = T_{11} + T_{12} + T_{21} + T_{22}$ we have

$$T^*_{21}X_{21} + T^*_{22}X_{21} + X^*_{21}T_{21} + X^*_{21}T_{22} = 0.$$

From the above equation and primeness of $\mathcal{A}$ we have $T_{22} = 0$ and

$$T^*_{21}X_{21} + X^*_{21}T_{21} = 0. \tag{11}$$

On the other hand, similarly by applying iX_{21} instead of X_{21} in above, we obtain

$$iT^*_{21}X_{21} + iT^*_{22}X_{21} - iX^*_{21}T_{21} - iX^*_{21}T_{22} = 0.$$

Since $T_{22} = 0$ we obtain from the above equation that

$$-T^*_{21}X_{21} + X^*_{21}T_{21} = 0. \tag{12}$$

From (11) and (12) we have

$$X^*_{21}T_{21} = 0.$$

Since $\mathcal{A}$ is prime, then we get $T_{21} = 0$.

It suffices to show that $T_{12} = T_{11} = 0$. For this purpose for $X_{12} \in \mathcal{A}_{12}$ we write

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1 + ((A_{11} + A_{12}) \diamond X_{12}) \diamond \Phi(P_1)\\
&= (\Phi(A_{11} + A_{12}) \diamond X_{12} + (A_{11} + A_{12}) \diamond \Phi(X_{12})) \diamond P_1 + (A_{11} + A_{12}) \diamond X_{12} \diamond \Phi(P_1)\\
&= \Phi(A_{11} + A_{12}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1 + A_{12} \diamond \Phi(X_{12}) \diamond P_1\\
&+A_{11} \diamond X_{12} \diamond \Phi(P_1) + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned}$$

So, we showed that

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi(A_{11} + A_{12}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1\\
&+A_{12} \diamond \Phi(X_{12}) \diamond P_1 + A_{11} \diamond X_{12} \diamond \Phi(P_1) + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned} \tag{13}$$

Since $A_{12} \diamond X_{12} \diamond P_1 = 0$ we have

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi((A_{11} \diamond X_{12}) \diamond P_1) + \Phi((A_{12} \diamond X_{12}) \diamond P_1)\\
&= \Phi(A_{11} \diamond X_{12}) \diamond P_1 + (A_{11} \diamond X_{12}) \diamond \Phi(P_1) + \Phi(A_{12} \diamond X_{12}) \diamond P_1 + (A_{12} \diamond X_{12}) \diamond \Phi(P_1)\\
&= (\Phi(A_{11}) \diamond X_{12} + A_{11} \diamond \Phi(X_{12})) \diamond P_1 + (A_{11} \diamond X_{12}) \diamond \Phi(P_1)\\
&+(\Phi(A_{12}) \diamond X_{12} + A_{12} \diamond \Phi(X_{12})) \diamond P_1 + (A_{12} \diamond X_{12}) \diamond \Phi(P_1)\\
&= \Phi(A_{11}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1 + A_{11} \diamond X_{12} \diamond \Phi(P_1)\\
&+\Phi(A_{12}) \diamond X_{12} \diamond P_1 + A_{12} \diamond \Phi(X_{12}) \diamond P_1 + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned}$$

So,

$$\begin{aligned}&\Phi(((A_{11}+A_{12})\diamond X_{12})\diamond P_1)=\Phi(A_{11})\diamond X_{12}\diamond P_1+A_{11}\diamond\Phi(X_{12})\diamond P_1\\&+A_{11}\diamond X_{12}\diamond\Phi(P_1)+\Phi(A_{12})\diamond X_{12}\diamond P_1\\&+A_{12}\diamond\Phi(X_{12})\diamond P_1+A_{12}\diamond X_{12}\diamond\Phi(P_1).\end{aligned}\tag{14}$$

From (13) and (14) we have

$$\Phi(A_{11}+A_{12})\diamond X_{12}\diamond P_1=\Phi(A_{11})\diamond X_{12}\diamond P_1+\Phi(A_{12})\diamond X_{12}\diamond P_1.$$

It follows that $T\diamond X_{12}\diamond P_1=0$, so $T_{11}^*X_{12}+X_{12}^*T_{11}=0$. We have $T_{11}^*X_{12}=0$ or $T_{11}XP_2=0$ for all $X\in\mathcal{A}$, then we have $T_{11}=0$. Similarly, we can show that $T_{12}=0$ by applying P_2 instead of P_1 in above.

Claim 2. *For each $A_{11}\in\mathcal{A}_{11}, A_{12}\in\mathcal{A}_{12}, A_{21}\in\mathcal{A}_{21}$ and $A_{22}\in\mathcal{A}_{22}$ we have*

1. $\Phi(A_{11}+A_{12}+A_{21})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})$.
2. $\Phi(A_{12}+A_{21}+A_{22})=\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})$.

We show that

$$T=\Phi(A_{11}+A_{12}+A_{21})-\Phi(A_{11})-\Phi(A_{12})-\Phi(A_{21})=0.$$

So, we have

$$\begin{aligned}&\Phi(A_{11}+A_{12}+A_{21})\diamond X_{21}+(A_{11}+A_{12}+A_{21})\diamond\Phi(X_{21})\\&=\Phi((A_{11}+A_{12}+A_{21})\diamond X_{21})=\Phi(A_{11}\diamond X_{21})+\Phi(A_{12}\diamond X_{21})+\Phi(A_{21}\diamond X_{21})\\&(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}))\diamond X_{21}+(A_{11}+A_{12}+A_{21})\diamond\Phi(X_{21}).\end{aligned}$$

It follows that $T\diamond X_{21}=0$. Since $T=T_{11}+T_{12}+T_{21}+T_{22}$ we have

$$T_{22}^*X_{21}+T_{21}^*X_{21}+X_{21}^*T_{22}+C_{21}^*T_{21}=0.$$

Therefore, $T_{22}=T_{21}=0$.
From Claim 1, we obtain

$$\begin{aligned}&\Phi(A_{11}+A_{12}+A_{21})\diamond X_{12}+(A_{11}+A_{12}+A_{21})\diamond\Phi(X_{12})\\&=\Phi((A_{11}+A_{12}+A_{21})\diamond X_{12})=\Phi((A_{11}+A_{12})\diamond X_{12})+\Phi(A_{21}\diamond X_{12})\\&=\Phi(A_{11}\diamond X_{12})+\Phi(A_{12}\diamond X_{12})+\Phi(A_{21}\diamond X_{12})\\&=(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21}))\diamond X_{12}+(A_{11}+A_{12}+A_{21})\diamond\Phi(X_{12}).\end{aligned}$$

Hence,

$$T_{11}^*X_{12}+T_{12}^*X_{12}+X_{12}^*T_{11}+X_{12}^*T_{12}=0.$$

Then $T_{11}=T_{12}=0$. Similarly

$$\Phi(A_{12}+A_{21}+A_{22})=\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}).$$

Claim 3. *For each $A_{11}\in\mathcal{A}_{11}, A_{12}\in\mathcal{A}_{12}, A_{21}\in\mathcal{A}_{21}$ and $A_{22}\in\mathcal{A}_{22}$ we have*

$$\Phi(A_{11}+A_{12}+A_{21}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}).$$

We show that

$$T=\Phi(A_{11}+A_{12}+A_{21}+A_{22})-\Phi(A_{11})-\Phi(A_{12})-\Phi(A_{21})-\Phi(A_{22})=0.$$

From Claim 2, we have

$$\begin{aligned}
&\Phi(A_{11}+A_{12}+A_{21}+A_{22})\diamond X_{12}+(A_{11}+A_{12}+A_{21}+A_{22})\diamond\Phi(X_{12})\\
&=\Phi((A_{11}+A_{12}+A_{21}+A_{22})\diamond X_{12})\\
&=\Phi((A_{11}+A_{12}+A_{21})\diamond X_{12})+\Phi(A_{22}\diamond X_{12})\\
&=\Phi(A_{11}\diamond X_{12})+\Phi(A_{12}\diamond X_{12})+\Phi(A_{21}\diamond X_{12})+\Phi(A_{22}\diamond X_{12})\\
&=(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))\diamond X_{12}\\
&+(A_{11}+A_{12}+A_{21}+A_{22})\diamond\Phi(X_{12}).
\end{aligned}$$

So, $T\diamond X_{12}=0$. It follows that

$$T_{11}^*X_{12}+T_{12}^*X_{12}+X_{12}^*T_{11}+X_{12}^*T_{12}=0.$$

Then $T_{11}=T_{12}=0$.

Similarly, by applying X_{21} instead of X_{12} in above, we obtain $T_{21}=T_{22}=0$.

Claim 4. *For each $A_{ij},B_{ij}\in\mathcal{A}_{ij}$ such that $i\neq j$, we have*

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

It is easy to show that

$$(P_i+A_{ij})(P_j+B_{ij})+(P_j+B_{ij}^*)(P_i+A_{ij}^*)=A_{ij}+B_{ij}+A_{ij}^*+B_{ij}^*.$$

So, we can write

$$\begin{aligned}
&\Phi(A_{ij}+B_{ij})+\Phi(A_{ij}^*+B_{ij}^*)=\Phi((P_i+A_{ij}^*)\diamond(P_j+B_{ij}))\\
&=\Phi(P_i+A_{ij}^*)\diamond(P_j+B_{ij})+(P_i+A_{ij}^*)\diamond\Phi(P_j+B_{ij})\\
&=(\Phi(P_i)+\Phi(A_{ij}^*))\diamond(P_j+B_{ij})+(P_i+A_{ij}^*)\diamond(\Phi(P_j)+\Phi(B_{ij}))\\
&=\Phi(P_i)\diamond B_{ij}+P_i\diamond\Phi(B_{ij})+\Phi(A_{ij}^*)\diamond P_j+A_{ij}^*\diamond\Phi(P_j)\\
&=\Phi(P_i\diamond B_{ij})+\Phi(A_{ij}^*\diamond P_j)\\
&=\Phi(B_{ij})+\Phi(B_{ij}^*)+\Phi(A_{ij})+\Phi(A_{ij}^*).
\end{aligned}$$

Therefore, we show that

$$\Phi(A_{ij}+B_{ij})+\Phi(A_{ij}^*+B_{ij}^*)=\Phi(A_{ij})+\Phi(B_{ij})+\Phi(A_{ij}^*)+\Phi(B_{ij}^*). \tag{15}$$

By an easy computation, we can write

$$(P_i+A_{ij})(iP_j+iB_{ij})+(-iP_j-iB_{ij}^*)(P_i+A_{ij}^*)=iA_{ij}+iB_{ij}-iA_{ij}^*-iB_{ij}^*.$$

Then, we have

$$\begin{aligned}
&\Phi(iA_{ij}+iB_{ij})+\Phi(-iA_{ij}^*-iB_{ij}^*)=\Phi((P_i+A_{ij}^*)\diamond(iP_j+iB_{ij}))\\
&=\Phi(P_i+A_{ij}^*)\diamond(iP_j+iB_{ij})+(P_i+A_{ij}^*)\diamond\Phi(iP_j+iB_{ij})\\
&=(\Phi(P_i)+\Phi(A_{ij}^*))\diamond(iP_j+iB_{ij})+(P_i+A_{ij}^*)(\Phi(iP_j)+\Phi(iB_{ij}))\\
&=\Phi(P_i)\diamond iB_{ij}+P_i\diamond\Phi(iB_{ij})+\Phi(A_{ij}^*)\diamond iP_j+A_{ij}^*\diamond\Phi(iP_j)\\
&=\Phi(P_i\diamond iB_{ij})+\Phi(A_{ij}^*\diamond iP_j)\\
&=\Phi(iB_{ij})+\Phi(-iB_{ij}^*)+\Phi(iA_{ij})+\Phi(-iA_{ij}^*).
\end{aligned}$$

We showed that

$$\Phi(iA_{ij} + iB_{ij}) + \Phi(-iA_{ij}^* - iB_{ij}^*) = \Phi(iB_{ij}) + \Phi(-iB_{ij}^*) + \Phi(iA_{ij}) + \Phi(-iA_{ij}^*).$$

From Lemma 2.2 and the above equation, we have

$$\Phi(A_{ij} + B_{ij}) - \Phi(A_{ij}^* + B_{ij}^*) = \Phi(B_{ij}) - \Phi(B_{ij}^*) + \Phi(A_{ij}) - \Phi(A_{ij}^*). \tag{16}$$

By adding equations (15) and (16), we obtain

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Claim 5. *For each* $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ *such that* $1 \leq i \leq 2$, *we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

We show that

$$T = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0.$$

We can write

$$\begin{aligned}
&\Phi(A_{ii} + B_{ii}) \diamond P_j + (A_{ii} + B_{ii}) \diamond \Phi(P_j) = \Phi((A_{ii} + B_{ii}) \diamond P_j) \\
&= \Phi(A_{ii} \diamond P_j) + \Phi(B_{ii} \diamond P_j) \\
&\Phi(A_{ii}) \diamond P_j + A_{ii} \diamond \Phi(P_j) + \Phi(B_{ii}) \diamond P_j + B_{ii} \diamond \Phi(P_j) \\
&= (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond P_j + (A_{ii} + B_{ii}) \diamond \Phi(P_j).
\end{aligned}$$

So, we have

$$T \diamond P_j = 0.$$

Therefore, we obtain $T_{ij} = T_{ji} = T_{jj} = 0$.
On the other hand, for every $X_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}
&\Phi(A_{ii} + B_{ii}) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}) = \Phi((A_{ii} + B_{ii}) \diamond X_{ij}) \\
&= \Phi(A_{ii} \diamond X_{ij}) + \Phi(B_{ii} \diamond X_{ij}) = \Phi(A_{ii}) \diamond X_{ij} + A_{ii} \diamond \Phi(X_{ij}) \\
&+\Phi(B_{ii}) \diamond X_{ij} + B_{ii} \diamond \Phi(X_{ij}) \\
&= (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}).
\end{aligned}$$

So,

$$(\Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii})) \diamond X_{ij} = 0.$$

It follows that $T \diamond X_{ij} = 0$ or $T_{ii}X_{ij} = 0$. By knowing that $\mathcal{A}$ is prime, we have $T_{ii} = 0$.
Hence, the additivity of Φ comes from the above claims.

In the rest of this paper, we show that Φ is $*$-derivation.

Claim 6. Φ *preserves star.*

Since $\Phi(I) = 0$ then we can write

$$\Phi\,(I \diamond A) = I \diamond \Phi(A).$$

Then

$$\Phi(A + A^*) = \Phi(A) + \Phi(A)^*.$$

So, we showed that Φ preserves star.

Claim 7. *we prove that Φ is derivation.*

For every $A, B \in \mathcal{A}$ we have

$$\begin{aligned}\Phi(AB + B^*A^*) &= \Phi(A^* \diamond B)\\ &= \Phi(A^*) \diamond B + A^* \diamond \Phi(B)\\ &= \Phi(A^*)^*B + \Phi(B)^*A^* + B^*\Phi(A^*) + A\Phi(B).\end{aligned}$$

On the other hand, since Φ preserves star, we have

$$\Phi(AB + B^*A^*) = \Phi(A)B + A\Phi(B) + B^*\Phi(A^*) + \Phi(B)^*A^*. \tag{17}$$

So, from (17), we have

$$\begin{aligned}&\Phi(i(AB - B^*A^*) = \Phi(A(iB) + (iB)^*A^*)\\ &= \Phi(A)(iB) + A\Phi(iB) + (iB)^*\Phi(A^*) + \Phi(iB)^*A^*.\end{aligned}$$

Therefore, from Lemma 2.2 we have

$$\Phi(AB - B^*A^*) = \Phi(A)B + A\Phi(B) - B^*\Phi(A^*) - \Phi(B^*)A^*. \tag{18}$$

By adding equations (17) and (18), we have

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

This completes the proof.

References

[1] Z. Bai, S. Du, The structure of non-linear Lie derivations on factor von Neumann algebras, Linear Algebra Appl. 436 (2012) 2701-2708.
[2] E. Christensen, Derivations of nest algebras, Ann. Math. 229 (1977) 155-161.
[3] J. Cui, C.K. Li, Maps preserving product $XY - YX^*$ on factor von Neumann algebras, Linear Algebra Appl. 431 (2009) 833-842.
[4] C. Li, F. Zhao, Q. Chen, Nonlinear maps preserving product $X^*Y + Y^*X$ on von Neumann algebras, Bull. Iran. Math. Soc. 44 (2018) 729–738.
[5] R. V. Kadison, J. R. Ringrose, Fundamentals of the theory of operator algebras I, New York, Academic Press (1983).
[6] R. V. Kadison, J. R. Ringrose, Fundamentals of the theory of operator algebras II, New York, Academic Press (1986).
[7] C. Li, F. Lu, X. Fang, Nonlinear ξ−Jordan $*$-derivations on von Neumann algebras, Linear and Multilinear Algebra. 62 (2014) 466-473.
[8] C. Li, F. Lu, X. Fang, Nonlinear mappings preserving product $XY + YX^*$ on factor von Neumann algebras, Linear Algebra Appl. 438 (2013), 2339-2345.
[9] C. R. Miers, Lie homomorphisms of operator algebras, Pacific J Math. 38 (1971) 717–735.
[10] L. Molnár, A condition for a subspace of B(H) to be an ideal, Linear Algebra Appl. 235 (1996), 229-234.
[11] S. Sakai, Derivations of W^*-algebras, Ann. Math. 83 (1966) 273-279.
[12] P. Šemrl, Additive derivations of some operator algebras, Illinois J. Math. 35 (1991) 234-240.
[13] P. Šemrl, Ring derivations on standard operator algebras, J. Funct. Anal. 112 (1993) 318-324.
[14] A. Taghavi, V. Darvish, H. Rohi, Additivity of maps preserving products $AP \pm PA^*$ on C^*-algebras, Math. Slovaca 67 (2017) 213–220.
[15] A. Taghavi, M. Nouri, M. Razeghi, V. Darvish, A note on non-linear $*$-Jordan derivations on $*$-algebras, Math. Slovaca 69 (3) (2019) 639-646.
[16] A. Taghavi, H. Rohi, V. Darvish, Non-linear $*$-Jordan derivations on von Neumann algebras, Linear Multilinear Algebra 64 (2016)426–439.
[17] A. Taghavi, M. Nouri, M. Razeghi, V. Darvish, Non-linear λ-Jordan triple $*$-derivation on prime $*$-algebras, Rocky Mountain J. Math. 48 (8) (2018) 2705–2716.
[18] W. Yu, J. Zhang, Nonlinear $*$-Lie derivations on factor von Neumann algebras, Linear Algebra Appl. 437 (2012) 1979-1991.
[19] F. Zhang, Nonlinear skew Jordan derivable maps on factor von Neumann algebras, Linear Multilinear Algebra 64 (2016) 2090–2103.