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Induction Functors for Hom-Doi-Hopf Module Categories

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Abstract. In this paper, we mainly investigate a revised induction funtor to study Hom-Doi-Hopf modules and the coinvariant modules for Hom-Hopf comodule algebras.

1. Introduction and Preliminaries

The authors in [1] introduced the notions of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted by a linear map providing a different way for constructing a subclass of quasi-Lie algebras. The coalgebra counterpart and the related notions of Hom-bialgebra and Hom-Hopf algebra were explored in [2-7, 9-10]. Afterwards, many important structures of Hopf algebra theory such as Doi-Hopf module, Yetter-Drinfeld modules, the Drinfeld doubles and other contexts have been replanted in Hom-setting.

Let H be a Hopf algebra with bijective antipode, and A/B an H-extension. Then it is well-known that the functor $(-)^{coH} : \mathcal{M}_A^H \to \mathcal{M}_B$ sending $M \in \mathcal{M}_A^H$ to the module of coinvariants M^{coH} has a left adjoint, which is called the induced functor $A\otimes_B - : \mathcal{M}_B \to \mathcal{M}_A^H$. (See [8, 11]).

Motivated by the study of Hom-Hopf structures and the work in [8, 11], the ultimate purpose of this paper is to investigate a modified induction functor $-\overline{\otimes}_B A$ which together with the coinvariant functor $(-)^{coH}$ determines an equivalence between $\mathcal{M}_B$ and a full subcategory of $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ for a Hom-Hopf algebra H with bijective antipode and a Haar integral.

Conventions Throughout the paper we work over a field k. We use Sweedler-type notation for coalgebras and comodules: for a coalgebra C, we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for $c \in C$; for a right C-comodule M, we denote its coaction by $\rho(m) = m_{[0]} \otimes m_{[1]}$.

Firstly we recall some basic definitions ([10]) needed in what follows. For further results see [1-7, 9].

Let C be a category. We introduce a new category $\widetilde{\mathcal{H}}(C)$ as follows: objects are couples (M, μ), with $M \in C$ and $\mu \in Aut_C(M)$. A morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in C satisfying $\nu f = f\mu$.

Let $\mathfrak{U}_k$ denote the category of k-modules. $\mathcal{H}(\mathfrak{U}_k)$ will be called the Hom-category associated to $\mathfrak{U}_k$. If $(M, \mu) \in \mathfrak{U}_k$, then $\mu : M \to M$ is obviously a morphism in $\mathcal{H}(\mathfrak{U}_k)$. It is easy to show that $\widetilde{\mathcal{H}}(\mathfrak{U}_k) = (\mathcal{H}(\mathfrak{U}_k), \otimes, (I, I), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is a monoidal category by Proposition 1.1 in [1]: the tensor product of (M, μ) and (N, ν) in $\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

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Assume that $(M,\mu),(N,\nu),(P,\pi)\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$. The associativity and unit constraints are given by the formulas: $\widetilde{a}_{M,N,P}((m\otimes n)\otimes p)=\mu(m)\otimes(n\otimes\pi^{-1}(p))$ and $\widetilde{l}_M(x\otimes m)=\widetilde{r}_M(m\otimes x)=x\mu(m)$. An algebra in $\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ will be called a monoidal Hom-algebra.

Definition 1.1. *A monoidal Hom-algebra* is an object $(A,\alpha)\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ together with a k-linear map $m_A:A\to A,\ m_A(a\otimes b)=ab$ and an element $1_A\in A$ such that for any $a,b,c\in A$:

$$\alpha(ab)=\alpha(a)\alpha(b); \tag{1}$$

$$1_Aa=a1_A=\alpha(a); \tag{2}$$

$$\alpha(a)(bc)=(ab)\alpha(c); \tag{3}$$

$$\alpha(1_A)=1_A. \tag{4}$$

Definition 1.2. *A monoidal Hom-coalgebra* is an object $(C,\beta)\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ together with k-linear maps $\Delta_C:C\to C\otimes C,\ \Delta_C(c)=c_1\otimes c_2$ (summation implicitly understood) and $\varepsilon_C:C\to k$ such that for any $x\in C$:

$$\beta(x)_1\otimes\beta(x)_2=\beta(x_1)\otimes\beta(x_2); \tag{5}$$

$$\varepsilon(\beta(x))=\varepsilon(x); \tag{6}$$

$$\varepsilon(x_1)x_2=\varepsilon(x_2)x_1=\beta^{-1}(x); \tag{7}$$

$$\beta^{-1}(x_1)\otimes x_{21}\otimes x_{22}=x_{11}\otimes x_{12}\otimes\beta^{-1}(x_2). \tag{8}$$

Definition 1.3. *A monoidal Hom-bialgebra* $H=(H,\gamma,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathcal{H}}(\mathfrak{U}_k)$. This means that (H,γ,m,η) is a monoidal Hom-algebra, $(H,\gamma,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra and that Δ and ε are morphisms of Hom-algebras, that is, for any $h,g\in H$:

$$(hg)_1\otimes(hg)_2=h_1g_1\otimes h_2g_2; \tag{9}$$

$$\Delta(1)=1\otimes1; \tag{10}$$

$$\varepsilon(hg)=\varepsilon(h)\varepsilon(g); \tag{11}$$

$$\varepsilon(1)=1. \tag{12}$$

Definition 1.4. *A monoidal Hom-Hopf algebra* is a monoidal Hom-bialgebra (H,γ) together with a linear map $S:H\to H\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ such that $S*I=I*S=\eta\varepsilon$ and $S\gamma=\gamma S$.

Definition 1.5. *Let (A,α) be a monoidal Hom-algebra. A right (A,α)-Hom-module* is an object $(M,\mu)\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ consists of a k-module M and a linear map $\mu:M\to M$ together with a morphism $\cdot:M\otimes A\to M\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ satisfying for any $a,b\in A$ and $m\in M$:

$$(m\cdot b)\cdot\alpha(a)=\mu(m)\cdot(ab); \tag{13}$$

$$m\cdot1=\mu(m); \tag{14}$$

$$\mu(m\cdot a)=\mu(m)\cdot\alpha(a). \tag{15}$$

The category of right (A,α)-Hom-modules is denoted by $\mathcal{M}_{\mathcal{A}}$.

Definition 1.6. *Let (C,β) be a monoidal Hom-coalgebra. A right (C,β)-Hom-comodule* is an object $(M,\mu)\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ together with a k-linear map $\rho_M: M\otimes M\to C,\ \rho_M(m)=m_{[0]}\otimes m_{[1]}\in\widetilde{\mathcal{H}}(\mathfrak{U}_k)$ satisfying for any $m\in M$:

$$\mu(m)_{[0]}\otimes\mu(m)_{[1]}=\mu(m_{[0]})\otimes\beta(m_{[1]});\tag{16}$$

$$m_{[0][0]}\otimes m_{[0][1]}\otimes\beta^{-1}(m_{[1]})=\mu^{-1}(m_{[0]})\otimes m_{[1]1}\otimes m_{[1]2};\tag{17}$$

$$\varepsilon(m_{[1]})m_{[0]}=\mu^{-1}(m).\tag{18}$$

The category of right (C,β)-Hom-comodules is denoted by $\mathcal{M}^C$. Define the coinvariant of (C,β) on (M,μ) as the set

$$M^{coH}=\{m\in M\mid m_{[0]}\otimes m_{[1]}=\mu^{-1}(m)\otimes 1\}.$$

Definition 1.7. *Let (H,γ) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A,α) is called a right (H,γ)-Hom-comodule algebra* if (A,α) is a right (H,γ)-Hom-comodule such that for any $a,b\in A$:

$$(ab)_{[0]}\otimes(ab)_{[1]}=a_{[0]}b_{[0]}\otimes a_{[1]}b_{[1]};\tag{19}$$

$$1_{[0]}\otimes 1_{[1]}=1\otimes 1.\tag{20}$$

Let $B=A^{coH}$, then A/B is called *a Hom-(H,γ)-extension.*

Definition 1.8. *Let (H,γ) be a monoidal Hom-Hopf algebra, and (A,α) a right (H,γ)-Hom-comodule algebra. A right (H,γ)-Hom-comodule (M,μ) is called a right $((H,\gamma),(A,\alpha))$-Hom-Doi-Hopf module* if it is also a right (A,α)-Hom-module such that for any $a\in A$ and $m\in M$,

$$(m\cdot a)_{[0]}\otimes(m\cdot a)_{[1]}=m_{[0]}\cdot a_{[0]}\otimes m_{[1]}a_{[1]}.\tag{21}$$

In the following, the category of right $((H,\gamma),(A,\alpha))$-Hom-Doi-Hopf modules will be denoted by $\mathcal{M}^{\mathcal{H}}_{\mathcal{A}}$.

Example 1.9. *Let (A,α) be a right (H,γ)-Hom-comodule algebra and (M,μ) a right (A,α)-Hom-module.*

Then $(M\otimes_B A,\mu\otimes\alpha)$ is a right $((H,\gamma),(A,\alpha))$-Hom-Doi-Hopf module with the right action $(m\otimes a)\cdot b=\mu(m)\otimes a\alpha^{-1}(b)$ and the right coaction $(m\otimes a)_{[0]}\otimes(m\otimes a)_{[1]}=\mu^{-1}(m)\otimes a_{[0]}\otimes\alpha(a_{[1]})$ for and $m\in M,\ a,b\in A$.

Obviously, $(M\otimes_B A,\mu\otimes\alpha)$ is both a right (A,α)-Hom-module and a right (H,γ)-Hom-comodule. The only thing left to prove is that the compatibility condition (21) is satisfied. Now for any $m\in M,\ a,b\in A$, we compute:

$$\begin{aligned}((m\otimes a)\cdot b)_{[0]}\otimes((m\otimes a)\cdot b)_{[1]}&=m\otimes a_{[0]}\alpha^{-1}(b_{[0]})\otimes\gamma(a_{[1]})b_{[1]}\\&=(m\otimes a)_{[0]}\cdot b_{[0]}\otimes(m\otimes a)_{[1]}b_{[1]}.\end{aligned}$$

Directly from Example 1.9, there exists an induction functor $F:\mathcal{M}_{\mathcal{B}}\to\mathcal{M}^{\mathcal{H}}_{\mathcal{A}},\ F(M)=M\otimes_B A$.

Proposition 1.10. *Let (A,α) be a right (H,γ)-Hom-comodule algebra. Then (F,G) is a pair of adjoint functors, where $G:\mathcal{M}^{\mathcal{H}}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{B}},\ G(N)=N^{coH}$.*

Proof. Firstly, we define the unit and counit as follows:

$$\eta_{(M,\mu)}: M\to(M\otimes_B A)^{coH},\ m\mapsto\mu^{-1}(m)\otimes_B 1;$$

$$\delta_{(N,\nu)}: N^{coH}\otimes_B A\to N,\ n\otimes_B a\mapsto n\cdot a.$$

Clearly, $\eta_{(M,\mu)}$ is well-defined. The fact that $\delta_{(N,\nu)}$ is reasonable follows from the computation $\delta_{(N,\nu)}(n\otimes ba)=n\cdot(ba)\overset{(13)}{=}\delta_{(N,\nu)}(n\cdot\nu^{-1}(b)\otimes\alpha(a)),\ n\in N,\ b\in B,\ a\in A$.

We end the proof by checking the triangular identity:

$$\begin{aligned}(\delta_{F(M,\mu)}\circ F_{\eta_{(M,\mu)}})(m\otimes_B a)&=(\mu^{-1}(m)\otimes_B 1)\cdot a=m\otimes_B a;\\(G_{\delta_{(N,\nu)}}\circ\eta_{G(N,\nu)})(n)&=\nu^{-1}(n)\cdot 1=n,\end{aligned}$$

for any $m\in M,\ a\in A,\ n\in N$. $\square$

Definition 1.11. *Let (H,γ) be a monoidal Hom-Hopf algebra. An element $\lambda \in (H^*,\gamma^*)$ is called a left integral on* (H,γ) if for any $h \in H$,

$$h_1\lambda(h_2) = \lambda(h)1_H; \tag{22}$$

$$\lambda(\gamma(h)) = \lambda(h). \tag{23}$$

If in addition $\lambda(1) = 1$, then the left integral λ is said to be *normalized.*

Similarly, we can define a normalized right integral on (H,γ). *A Haar integral on* (H,γ) is a normalized two-sided integral.

Lemma 1.12. *Let (H,γ) be a monoidal Hom-Hopf algebra and λ a Haar integral on (H,γ). Then for any $h,g \in H$:*

$$h_1\lambda(g\gamma(h_2)) = \lambda(g_2\gamma^{-1}(h))S(g_1); \tag{24}$$

$$\gamma^{-1}(h_2)\lambda(h_1g) = \lambda(\gamma^{-2}(h)g_1)S(g_2). \tag{25}$$

Proof. (24) is a consequence of the following computation:

$$\begin{aligned} h_1\lambda(g\gamma(h_2)) &\overset{(7),(23)}{=} \varepsilon(g_1)h_1\lambda(gh_2) = (S(g_{11})g_{12})h_1\lambda(gh_2) \\ &\overset{(3),(8)}{=} S(g_1)(g_{21}\gamma^{-1}(h_1))\lambda(\gamma(g_{22})h_2) \\ &\overset{(22),(23)}{=} S(g_1)\lambda(g_2\gamma^{-1}(h_2)). \end{aligned}$$

The proof of (25) is similar and left to the reader. □

2. Induction Functor

In this section, we will investigate a revised induction functor for a Hom-Doi-Hopf module.

In what follows, we will first recall the notion of a torsion theory in [12].

Definition 2.1. *([12]) Let C be an abelian theory. A torsion theory* is given by a pair $(\mathcal{U},\mathcal{V})$ of full and replete (i.e. isomorphism closed) subcategories of C such that:

(I) For any object $X \in C$, there exists a short exact sequence:

$$0 \to T \to X \to F \to 0,$$

where 0 is the zero object in C, $T \in \mathcal{U}$, $F \in \mathcal{V}$.

(II) The only morphism $f : T \to F$ from $T \in \mathcal{U}$ to $F \in \mathcal{V}$ is the zero morphism.

When $(\mathcal{U},\mathcal{V})$ is a torsion theory, $\mathcal{U}$ is called the *torsion class of* C, and $\mathcal{V}$ its *torsion-free class.*

A torsion theory $(\mathcal{U},\mathcal{V})$ is *hereditary* if the torsion class $\mathcal{U}$ is closed under subobjects.

Lemma 2.2. *Let $M \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$. Then:*

(I) The map $\pi_M : M \to M^{coH}$, $m \mapsto \lambda(m_{[1]})\mu(m_{[0]})$ is an idempotent surjection of $\mathcal{M}_{\mathcal{B}}$.

(II) If we denote $\kappa(M) = \{m \in M \mid \pi_M(m\cdot a) = 0,\ \forall a \in A\}$, then $\kappa(M)$ is an object of $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$.

(III) $\kappa(M/\kappa(M)) = \overline{0}$.

(IV) $(M/\kappa(M))^{coH} \cong M^{coH}$ as right B-modules.

Proof. (I) Firstly, π_M is surjective because λ is normalized. Now we show that π_M is well-defined. In fact, for any $m \in M$,

$$\begin{aligned} \pi_M(m)_{[0]} \otimes \pi_M(m)_{[1]} &= \mu(m_{[0][0]}) \otimes \alpha(m_{[0][1]})\lambda(m_{[1]}) \\ &\overset{(17)}{=} m_{[0]} \otimes \alpha(m_{[1]1})\lambda\alpha(m_{[1]2}) = \mu^{-1}(\pi_M(m)) \otimes 1. \end{aligned}$$

Then we immediately obtain that $\pi_M \circ \pi_M = \pi_M$. Meanwhile, for any $b \in B$,

$$\begin{aligned}\pi_M(m \cdot b) &= \lambda(m_{[1]}b_{[1]})\mu(m_{[0]}) \cdot \beta((b_{[0]}) \\ &= \lambda(m_{[1]})\mu(m_{[0]}) \cdot b = \pi_M(m) \cdot b,\end{aligned}$$

finishing the proof of (I).

(II) Obviously, $\kappa(M)$ is an (A, α)-Hom-submodule of M. On the other hand, for any $a \in A$ and $m \in \kappa(M)$,

$$\begin{aligned}&\pi_M(m_{[0]} \cdot a) \otimes m_{[1]} = \lambda(m_{[0][1]}a_{[1]})\mu(m_{[0][0]}) \cdot \alpha(a_{[0]}) \otimes m_{[1]} \\ &\overset{(17)}{=} \lambda(m_{[1]1}a_{[1]})m_{[0]} \cdot \alpha(a_{[0]}) \otimes \gamma(m_{[1]2}) \\ &\overset{(25),(17)}{=} \lambda(m_{[1]}\gamma^2(a_{[0][1]}))m_{[0]} \cdot \alpha^2(a_{[0][0]}) \otimes S(\gamma(a_{[1]})) \\ &= \mu^{-1}(m \cdot \alpha^2(a_{[0]})) \otimes S(\gamma(a_{[1]})) = 0,\end{aligned}$$

therefore $m_{[0]} \otimes m_{[1]} \in \kappa(M) \otimes H$, implying that $\kappa(M)$ is a subobject of M in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$.

(III) First we need to show that $\pi_M(\kappa(M)) \subseteq \kappa(M)$. Indeed, for any $a \in A$ and $m \in \kappa(M)$,

$$\begin{aligned}&\pi_M(\pi_M(m) \cdot a) = \lambda(\gamma(m_{[0][1]})a_{[1]})\lambda(m_{[1]})\mu^2(m_{[0][0]}) \cdot \alpha(a_{[0]}) \\ &\overset{(17)}{=} \lambda(\gamma(m_{[1]1})a_{[1]})\lambda(m_{[1]2})\mu(m_{[0]}) \cdot \alpha(a_{[0]}) \\ &\overset{(25),(17)}{=} \lambda(\gamma^{-1}(m_{[1]})a_{[0][1]})\lambda(S(a_{[1]}))\mu(m_{[0]}) \cdot \alpha^2(a_{[0][0]}) \\ &= \pi_M(m \cdot \alpha(a_{[0]}))\lambda(S(a_{[1]})) = 0.\end{aligned}$$

Let $m + \kappa(M) \in M/\kappa(M)$ such that $\pi_{M/\kappa(M)}(m \cdot a + \kappa(M)) = \overline{0}$. Since $\pi_{M/\kappa(M)}(m \cdot a + \kappa(M)) = \pi_M(m \cdot a) + \kappa(M)$, thus $\pi_M(m \cdot a) \in \kappa(M)$. By (I) we have $\pi_M(m \cdot a) = \pi_M{}^2(m \cdot a) = 0$, implying that $m \in \kappa(M)$. Hence $\kappa(M/\kappa(M)) = \overline{0}$.

(IV) We begin by proving that the functor $(-)^{coH}$ is exact. By Proposition 1.10, we know that it is left exact. We only need to show that it is also right exact. In fact, for any surjective morphism $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ and $n \in N^{coH}$, there exists $m \in M$ such that $f(m) = n$. Since f is right (H, γ)-colinear, we obtain that $f(m_{[0]}) \otimes m_{[1]} = \nu^{-1}(n) \otimes 1$. Applying λ on the second tensorand of both sides, we get $f(\pi_M(m)) = n$ as required.

Next, for any $M \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$, $\kappa(M)^{coH} = 0$. Indeed, for any $m \in \kappa(M)^{coH}$, then $m \in \kappa(M) \cap M^{coH}$, thus $m = \pi_M(m) = 0$, implying that $f|_{M^{coH}} : M^{coH} \to N^{coH}$ is surjective in $\mathcal{M}_{\mathcal{B}}$. So the functor $(-)^{coH}$ is exact.

Now, since $(-)^{coH}$ is exact,

$$(M/\kappa(M))^{coH} \cong M^{coH}/\kappa(M)^{coH} = M^{coH},$$

thus we complete the proof. □

As a consequence of Lemma 2.2, we immediately obtain the following result:

Lemma 2.3. *There exists a hereditary torsion theory $(\mathcal{U}, \mathcal{V})$ in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$, where $\mathcal{U} = \{T \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}} \mid \kappa(T) = T\}$ and $\mathcal{V} = \{F \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}} \mid \kappa(F) = 0\}$.*

By Lemma 2.2 *(II), we can induce a factor Hom-Doi-Hopf module*

$$N \overline{\otimes}_B A = (N \otimes_B A)/\kappa(N \otimes_B A),\ N \in \mathcal{M}_B.$$

We denote by $n \overline{\otimes}_B a$ the image of the element $n \otimes_B a \in N \otimes_B A$ in $N \overline{\otimes}_B A$. Moreover, if $M, P \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ and $f : M \to P$ is a morphism in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$, then f maps $\kappa(M)$ to $\kappa(P)$. So it induces a morphism $\overline{f} : M/\kappa(M) \to P/\kappa(P)$ in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$. For simplicity, we write $M/\kappa(M)$ as $\overline{M}$.

Proposition 2.4. *As mentioned above, $(- \overline{\otimes}_B A, (-)^{coH})$ is an adjoint pair of functors.*

Proof. Let $(M,\mu)\in\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ and $(N,\nu)\in\mathcal{M}_B$, we define

$$\chi: Hom_{(A,H)}(N\overline{\otimes}_B A,\overline{M})\to Hom_B(N,M^{coH}),\ \chi(f)(n)=f(n\overline{\otimes}_B 1),$$

for any $n\in N$ and $f\in Hom_{(A,H)}(N\overline{\otimes}_B A,\overline{M})$. It is easy to see that $\chi(f)$ is right B-Hom-linear and $\chi(f)(n)\in\overline{M}^{coH}$, hence it also belongs to M^{coH} by Lemma 2.2 (IV). Therefore χ is well-defined.

Consider another map $\omega: Hom_B(N,M^{coH})\to Hom_{(A,H)}(N\overline{\otimes}_B A,\overline{M})$ by $\omega(g)(n\overline{\otimes}_B a)=\overline{\mu^{-1}(g(n))\cdot\alpha^{-1}(a)}$ for any $g\in Hom_B(N,M^{coH})$ and $n\overline{\otimes}_B a\in N\overline{\otimes}_B A$.

Now we have to show that χ is the inverse of ω. In fact, for any $a\in A$, $n\in N$, $g\in Hom_B(N,M^{coH})$ and $f\in Hom_{(A,H)}(N\overline{\otimes}_B A,\overline{M})$,

$$\begin{aligned}&(\chi\circ\omega)(g)(n)=\chi(\omega(g))(n)=\overline{\mu^{-1}(g(n))\cdot 1}\overset{(14)}{=}\overline{g(n)}=g(n);\\&(\omega\circ\chi)(f)(n\overline{\otimes}_B a)=\overline{\mu^{-1}(\chi(f)(n))\cdot\alpha^{-1}(a)}\\&=\overline{\mu^{-1}(f(n\overline{\otimes}_B 1))\cdot\alpha^{-1}(a)}=\overline{f(\nu^{-1}(n)\overline{\otimes}_B 1)\cdot\alpha^{-1}(a)}=f(n\overline{\otimes}_B a),\end{aligned}$$

so $\chi\circ\omega=id_{Hom_B(N,M^{coH})}$ and $\omega\circ\chi=id_{Hom_{(A,H)}(N\overline{\otimes}_B A,\overline{M})}$ if we notice that $\overline{M^{coH}}=M^{coH}$. □

The remainder of this section will be denoted to the discussion of equivalence between the full subcategory of all 0-generated Hom-Doi-Hopf modules which are torsion free and $\mathcal{M}_{\mathcal{B}}$.

Definition 2.5. *Let $(M,\mu)\in\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$. It is called 0-generated* if $M=M^{coH}$.

If every object in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ is 0-generated, then the category $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ is called 0-*generated* .

Lemma 2.6. *Assume that $(M,\mu)\in\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ is a 0-generated Hom-Doi-Hopf module and $(N,\nu)\in\mathcal{M}_{\mathcal{B}}$. Then for any morphism $\theta\in Hom_B(M^{coH},N)$, there exists a unique morphism $\vartheta\in Hom_{(A,H)}(M,N\overline{\otimes}_B A)$ such that $\vartheta|_{M^{coH}}=\theta\overline{\otimes}_B 1_A$.*

Proof. By Proposition 1.10 the counit $\delta_{(M,\mu)}: M^{coH}\otimes_B A\to M$ is a morphism in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$, and its kernel $ker(\delta_{(M,\mu)})$ is a torsion Hom-Doi-Hopf submodule of $M^{coH}\otimes_B A$. Since (M,μ) is 0-generated, $\delta_{(M,\mu)}$ is an epimorphism. It implies that the statements hold if and only if there is a unique morphism $\phi: M^{coH}\otimes_B A\to N\overline{\otimes}_B A$ such that $\phi(ker(\delta_{(M,\mu)}))=0$ and $\phi\circ(id_{M^{coH}}\otimes_B 1_A)=\theta\overline{\otimes}_B 1_A$.

It is clear that

$$\phi: M^{coH}\otimes_B A\overset{\theta\otimes_B id_A}{\longrightarrow}N\otimes_B A\longrightarrow N\overline{\otimes}_B A$$

is a morphism in $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ such that the above statements are true. We still have to prove that $ker(\delta_{(M,\mu)})\subseteq ker(\phi)$. It is obviously satisfied because $ker(\delta_{(M,\mu)})\subseteq\kappa(M^{coH}\otimes_B A)\subseteq ker(\phi)$ since $N\overline{\otimes}_B A$ is torsion free. □

Lemma 2.7. *For any $(N,\nu)\in\mathcal{M}_{\mathcal{B}}$, the factor unit $\overline{\eta_N}: N\to\overline{(N\overline{\otimes}_B A)}^{coH}$ is bijective.*

Proof. By Lemma 2.2 (Iv), it suffice to prove that the unit

$$\eta_N: N\to(N\otimes_B A)^{coH}$$

is an isomorphism for any $(N,\nu)\in\mathcal{M}_{\mathcal{B}}$. We define the map

$$\xi_N:(N\otimes_B A)^{coH}\to N,\ \xi_N(n\otimes_B a)=n\cdot\pi_A(a),$$

and claim that it is the inverse of η_N. Indeed, for any $n\in N$, $a\in A$, we compute:

$$\begin{aligned}&(\xi_N\circ\eta_N)(n)=\xi_N(\nu^{-1}(n)\otimes_B 1)=\nu^{-1}(n)\cdot 1=n;\\&(\eta_N\circ\xi_N)(n\otimes_B a)=\nu^{-1}(n\cdot\pi_A(a))\otimes_B 1_A=\nu^{-1}(n)\cdot a_{[0]}\lambda(a_{[1]})\otimes_B 1_A\\&=\nu^{-1}(n)\cdot\alpha^{-1}(a)\otimes_B 1_A=n\otimes_B a,\end{aligned}$$

finishing the proof. □

Lemma 2.8. *For any* 0*-generated* $(M, \mu) \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$*, the factor counit*

$$\overline{\delta_{(M,\mu)}} : M^{coH}\overline{\otimes}_B A \to \overline{M}$$

is bijective.

Proof. *We first suppose that* (M, μ) *is torsion free, that is* $\overline{M} = M$*. By Lemma 2.6, there is a unique Hom-Doi-Hopf module morphism* ϑ *such that* $\vartheta|_{M^{coH}} = id_{M^{coH}}\overline{\otimes}_B 1_A$*. Meanwhile since* (M, μ) *is torsion free, by the definition of* χ*, which is bijective in Proposition 2.4, there exists a unique Hom-Doi-Hopf module morphism* $\varphi : M^{coH}\overline{\otimes}_B A \to M$ *such that* $\varphi \circ (id_{M^{coH}\overline{\otimes}_B 1_A}) = id_{M^{coH}}$*. Note that* $\varphi = \overline{\delta_{(M,\mu)}}$ *by the uniqueness. Combining the foregoing two assertions, we can obtain a Hom-Doi-Hopf morphism* $\overline{\delta_{(M,\mu)}} \circ \vartheta$ *that restricts to the identity on* M^{coH}*.*

For any $m \in M$*, there exists* $x \in M^{coH}$ *and* $a \in A$ *such that* $m = x \cdot a$ *since* (M, μ) *is* 0*-generated. Thus* $(\overline{\delta_{(M,\mu)}} \circ \vartheta)(m) = (\overline{\delta_{(M,\mu)}} \circ \vartheta)(x \cdot a) = (\overline{\delta_{(M,\mu)}} \circ \vartheta)(x) \cdot a = x \cdot a = m$*, which implies that* ϑ *is injective. On the other hand, by the construction of* ϑ *in Lemma 2.6, we know that* ϕ *is the surjection from* $M^{coH}\otimes_B A$ *to* $M^{coH}\overline{\otimes}_B A$ *and* $\phi = \vartheta \circ \delta_{(M,\mu)}$*. Thus* ϑ *is surjective and hence an isomorphism. Then so is* $\overline{\delta_{(M,\mu)}}$*.*

Now for any 0*-generated* $(M, \mu) \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$*, the factor* $(\overline{M} = M/\kappa(M), \overline{\mu}) \in \mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ *is torsion free and* 0*-generated too. So* $\overline{M} \cong \overline{M}^{coH}\overline{\otimes}_B A \cong M^{coH}\overline{\otimes}_B A$*.* □

As a consequence of the above argument, we have the following main result.

Theorem 2.9. *The pair of functors* $(-\overline{\otimes}_B A, (-)^{coH})$ *forms an equivalence between the full subcategory of all* 0*-generated Hom-Doi-Hopf modules which are torsion free and* $\mathcal{M}_{\mathcal{B}}$*.*

Corollary 2.10. *If* $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ *is* 0*-generated and torsion free, then* $(-\overline{\otimes}_B A, (-)^{coH})$ *forms an equivalence between* $\mathcal{M}_{\mathcal{A}}^{\mathcal{H}}$ *and* $\mathcal{M}_{\mathcal{B}}$*.*

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