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Corrigendum to "Weyl type theorems for selfadjoint operators on Krein spaces" [Filomat 32:17 (2018), 6001–6016]

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The purpose of this note is to point out a mistake in the definition of the dimension of the $\mathcal{J}$-kerenl in section 3 of this paper. Because of this mistake, there are some errors in examples and remarks.

Throughout this note, $(\mathcal{K}, J)$ denotes a Krein space equipped with an indefinite inner product $\langle\cdot,\cdot\rangle_J$, unless specified otherwise.

In section 3 of the paper, we defined the *$\mathcal{J}$-kernel* of T, $\mathcal{J}\text{-}\ker(T)$, by

$$\mathcal{J}\text{-}\ker(T) := \{x \in \mathcal{K} : \langle Tx, Tx\rangle_J = 0\}.$$

In general, unlike the kernel, the $\mathcal{J}$-kernel is not a subspace of $\mathcal{K}$.

For example, consider the finite dimensional space $\mathcal{K} = \mathbb{C}^3$ equipped with the standard inner product $\langle\cdot,\cdot\rangle$. For $J = \mathrm{diag}(1, -1, -1)$, $(\mathbb{C}^3, J)$ becomes a 3-dimensional Krein space. Let $T = I_3$ be the identity operator on $\mathbb{C}^3$. For $u_1 := (1, 1, 0),\ u_2 := (1, 0, 1) \in \mathbb{C}^3$, we have that

$$\langle Tu_i, Tu_i\rangle_J = \langle JTu_i, Tu_i\rangle = \langle Ju_i, u_i\rangle = 0 \quad (i = 1, 2),$$

so that $u_1, u_2 \in \mathcal{J}\text{-}\ker(T)$. On the other hand, $\langle T(u_1 + u_2), T(u_1 + u_2)\rangle_J = 2 \neq 0$, which shows that the $\mathcal{J}$-kernel is not a subspace.

Let $\mathcal{V}_\mu$ be a chain of neutral subspaces in $\mathcal{J}\text{-}\ker(T)$ containing $\ker(T)$ and let V_μ be the maximal element of $\mathcal{V}_\mu$. We define *the dimension of* $\mathcal{J}\text{-}\ker(T)$ by

$$\dim \mathcal{J}\text{-}\ker(T) := \sup_\mu \dim V_\mu$$

where the supremum is taken over all maximal elements in such all chains.

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Using this definition, we can define the $\mathcal{J}$*-ascent* as before, that is,

$$\varphi(T) := \sup_k \dim(\mathcal{J}\text{-}\ker(T^k)).$$

Remark 3.5 should be revised as follows;

Remark 3.5. Unlike the Fredholm index, in general, the index product formula does not hold for the $\mathcal{J}$-index. More precisely, even if $T, S \in \mathcal{L}(\mathcal{K})$ are $\mathcal{J}$-Fredholm, we may see that $\mathcal{J}\text{-ind}(ST) \neq \mathcal{J}\text{-ind}(S) + \mathcal{J}\text{-ind}(T)$. We have an example which does not satisfy the index product formula for the $\mathcal{J}$-index as follows;

The Krein space $\mathcal{K}$ given in Example 3.2 is finite dimensional, so that T is $\mathcal{J}$-Fredholm. We see that T^2 is also $\mathcal{J}$-Fredholm. If $x = (x_1, x_2, x_3)$ belongs to $\mathcal{J}\text{-}\ker(T)$, then we have

$$0 = \langle Tx, Tx\rangle_J = \langle JTx, Tx\rangle = -x_2^2 - x_3^2 + x_1^2,$$

so that $x_1^2 = x_2^2 + x_3^2$. Let $\mathcal{V}_\mu$ be a chain of neutral subspaces in $\mathcal{J}\text{-}\ker(T)$ contaning $\ker(T) = \{0\}$. Then a maximal element of $\mathcal{V}_\mu$ is spanned by a set $\{x\}$. This means that $\dim \mathcal{J}\text{-}\ker(T) = 1$. However, if $y = (y_1, y_2, y_3)$ belongs to $\mathcal{J}\text{-}\ker(T^2)$, then we also have

$$0 = \langle T^2y, T^2y\rangle_J = \langle JT^2y, T^2y\rangle = -y_3^2 - y_1^2 + y_2^2.$$

Hence this implies that $\dim \mathcal{J}\text{-}\ker(T^2) = 1$, so that $\mathcal{J}\text{-ind}(T^2) = 1$. Thus the index product formula does not hold for the $\mathcal{J}$-index since $\mathcal{J}\text{-ind}(T) + \mathcal{J}\text{-ind}(T) = 2 \neq 1 = \mathcal{J}\text{-ind}(T^2)$. □

For the accuracy of the exposure, we use this opportunity to make the following corrections:

- On the page 6003, line −2, "the $\mathcal{J}$-kernel $\mathcal{J}\text{-}\ker(T)$ is not an invariant subspace of T" should be replaced by "the $\mathcal{J}$-kernel $\mathcal{J}\text{-}\ker(T)$ is not invariant under T".
- On the page 6004, line 4 and 6, "$\mathcal{J}\text{-}\ker(T)$ is an invariant subspace of T" should be replaced by "$\mathcal{J}\text{-}\ker(T)$ is invariant under T".
- The first two lines of the proof of Theorem 3.10 (i) must be revised as follows;

  If T is $\mathcal{J}$-Weyl and selfadjoint, then we have that $\dim \mathcal{J}\text{-}\ker(T) = \dim \ker(T)$. Assume that $x \notin \ker(T)$ and $x \in \mathcal{J}\text{-}\ker(T)$. Then $\alpha x \in \mathcal{J}\text{-}\ker(T)$ for arbitrary $\alpha \in \mathbb{C}$. This means that $x \oplus \ker(T) \subset \mathcal{J}\text{-}\ker(T)$. However this contradicts because $\dim\{x \oplus \ker(T)\} \geq \dim \mathcal{J}\text{-}\ker(T)$. Thus we get the equality $\mathcal{J}\text{-}\ker(T) = \ker(T)$.
- In Example 4.7.2, "$\dim \ker(V^* - I) = 2$" should be "$\dim \ker(V^* - I) = 1$". In line 6 of Example 4.7.2, "Thus we see that $\dim \mathcal{J}\text{-}\ker(V - I) = 2$" should be replaced by "The maximal element of a chain of neutral subspace in $\mathcal{J}\text{-}\ker(V - I)$ containing $\ker(V - I)$ is spanned by a set $\{e_1\}$. Thus, we see that $\dim \mathcal{J}\text{-}\ker(V - I) = 1$".