Filomat 36:10 (2022), 3261–3278
https://doi.org/10.2298/FIL2210261R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Skew (A, m)-Symmetric Operators in a Hilbert Space

Rchid Rabaoui[a]

[a]*Department of Mathematics. University of Gabes. Faculty of Sciences. Gabes 6072. Tunisia*
University of Tunis El Manar. Faculty of Sciences of Tunis
Mathematical Analysis and Applications Laboratory LR11ES11, 2092, El Manar I, Tunis, Tunisia.

Abstract. In this paper, we study skew (A, m)-symmetric operators in a complex Hilbert space $\mathcal{H}$. Firstly, by introducing the generalized notion of left invertibility we show that if $T \in \mathcal{B}(\mathcal{H})$ is skew (A, m)-symmetric, then e^{isT} is left (A, m)-invertible for every $s \in \mathbb{R}$. Moreover, we examine some conditions for skew (A, m)-symmetric operators to be skew $(A, m-1)$-symmetric. The connection between c_0-semigroups of (A, m)-isometries and skew (A, m)-symmetries is also described. Next, we investigate the stability of a skew (A, m)-symmetric operator under some perturbation by nilpotent operators commuting with T. In addition, we show that if T is a skew (A, m)-symmetric operator, then T^n is also skew (A, m)-symmetric for odd n. Finally, we consider a generalization of skew (A, m)-symmetric operators to the multivariable setting. We introduce the class of skew (A, m)-symmetric tuples of operators and characterize the joint approximate point spectrum of such a family.

1. Introduction and preliminaries

Throughout this paper $\mathcal{H}$ stands for an infinite separable complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$. By $\mathcal{B}(\mathcal{H})$ we denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we write $\mathcal{R}(T)$, $\mathcal{N}(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ for the range space, the null space, the spectrum, the point spectrum and the approximate point spectrum of T, respectively. The cone of positive (semi-definite) operators is given by

$$\mathcal{B}(\mathcal{H})^+ := \Big\{ A \in \mathcal{B}(\mathcal{H}) : \langle Au \mid u \rangle \geq 0, \ \forall\, u \in \mathcal{H} \Big\}.$$

For $A \in \mathcal{B}(\mathcal{H})^+$, let

$$\mathcal{B}_A(\mathcal{H}) := \Big\{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subset \mathcal{R}(A) \Big\}.$$

Any $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form: $\langle \cdot \mid \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$, $\langle u \mid v \rangle_A := \langle Au \mid v \rangle$. By $\| \cdot \|_A$ we denote the semi-norm induced by $\langle \cdot \mid \cdot \rangle_A$, i.e. $\|u\|_A^2 = \langle u \mid u \rangle_A$. Observe that $\| \cdot \|_A$ is a norm if and only if A is injective.

2020 *Mathematics Subject Classification*. Primary 46C05, 47A10, 47A53, 47B20
Keywords. Skew (A, m)-symmetric operator, left (A, m)-invertible operator, (A, m)-Isometric operator, joint point spectrum, joint approximate point spectrum
Received: 03 February 2020; Accepted: 04 June 2022
Communicated by Dragan S. Djordjević
Email address: rchid.rabaoui@fsg.rnu.tn (Rchid Rabaoui)

For R, S, $X \in \mathcal{B}(\mathcal{H})$, let consider the map $\Theta_{R,S} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$, $\Theta_{R,S}(X) := RXS - X$. An induction argument shows that

$$\Theta^{(m)}_{R,S}(X) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} R^k X S^k, \quad m \geq 0. \tag{1}$$

The m-isometric operators were introduced by Agler back in the early nineties and were statued in detail by Agler and Stankus in a series of three papers ([1–3]). An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an m-isometry if $\Theta^{(m)}_{T^*,T}(I) := \Theta^{(m)}_{T}(I) = 0$. Equivalently, T is left m-invertible with T^* as a left inverse. Recently, Sid Ahmed et al. ([27]) generalized the concept of those operators on a Hilbert space. They introduced the (A,m)-isometric operators: For $m \geq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an (A,m)-isometry if $\Theta^{(m)}_{T}(A) = 0$, that is T is left (A,m)-invertible with T^* as a left inverse (see Definition 2.4). They showed many important results of such an operator (for more details see again [27–29]).

For R, $S \in \mathcal{B}(\mathcal{H})$, let $\mathcal{S}_{R,S} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be the generalized derivation operator defined by $\mathcal{S}_{R,S}(X) := RX - XS$. For every integer n, we have

$$\mathcal{S}^{(n)}_{R,S}(X) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} R^k X S^{n-k}. \tag{2}$$

The concept of the Helton class of an operator has been the object of some intensive study (see [24, 26]). For R, $S \in \mathcal{B}(\mathcal{H})$, we say that $S \in Helton_m(R)$ if $\mathcal{S}^{(m)}_{R,S}(I) = 0$. In [16], [17], J. W. Helton initiated the study of m-symmetries. A bounded linear operator T is said to be m-symmetric if $T \in Helton_m(T^*)$, that is $\mathcal{S}^{(m)}_{T^*,T}(I) := \mathcal{S}^{(m)}_{T}(I) = 0$. In light of complex symmetric operators, using the identity (2), M. Chō defined (m,C)-complex symmetric operators as follows: an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (m,C)-complex symmetric if there exists some conjugation C such that $\mathcal{S}^{(m)}_{T}(C)C = 0$. For more details, we refer the readers to [6–9, 13–15, 20–23, 25]. Recently, in [19], we have introduced the class of (A,m)-symmetric operators, i.e operators satisfying $\mathcal{S}^{(m)}_{T}(A) = 0$ where $A \in \mathcal{B}(\mathcal{H})^+$. We have shown that some properties and results related to m-symmetries and (m,C)-symmetries hold true also for the new class. Sid Ahmed et al. ([10]) extended the study of such a family to the multivariable setting. Particularly, they showed that if T is (A,m)-symmetric, then e^{itT} is (A,m)-isometric for every $t \in \mathbb{R}$ (Theorem 2.6).

We aim in this paper to study the class of skew (A,m)-symmetric operators introduced in [10]. We will show that some results for skew m-symmetric and skew m-complex symmetric operators remain true if we consider an additional semi-inner product defined by a positive operator A. Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be skew (A,m)-symmetric if $\zeta^{(m)}_{T}(A) = 0$, where

$$\zeta^{(m)}_{T}(A) := \sum_{k=0}^{m} \binom{m}{k} T^{*k} A T^{m-k}. \tag{3}$$

Remark 1.1. 1. *T is skew $(A,1)$-symmetric (or skew A-symmetric) if $AT = -T^*A$.*

2. *T is said to be skew $(A,2)$-symmetric if*

$$AT^2 + 2T^*AT + T^{*2}A = 0.$$

3. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be strict skew (A,m)-symmetric if T is (A,m)-symmetric, but it is not a skew $(A,m-1)$-symmetric operator.*

4. *For an invertible operator A, let*

$$R := \frac{1}{2}\Big(T + A^{-1}T^*A\Big), \qquad S := \frac{1}{2}\Big(T - A^{-1}T^*A\Big).$$

It is easy to see that R is A-symmetric, S is skew A-symmetric and $T = R + S$ ([19]).

5. *If T is (A, m)-symmetric, then (αT) is skew (A, m)-symmetric for α pure imaginary* ([19]).

The following example ensures that, in general, skew (A, m)-symmetric operators are not skew m-symmetric and vice versa ([19]).

Example 1.2. *Let $\mathcal{H} = \mathbb{C}^2$, $\|(x, y)\|^2 = |x|^2 + |y|^2$ for all $x, y \in \mathbb{C}$. Take $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, T is skew A-symmetric and not skew 3-symmetric. Moreover, S is skew symmetric and not skew $(A, 3)$-symmetric.*

A family $(T(t))_{t\geq 0}$ of bounded linear operators on $\mathcal{H}$ is called a strongly continuous (one-parameter) semigroup (or a C_0-semigroup) if it satisfies the functional equation

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t,\, s \geq 0, \\ T(0) = I, \end{cases}$$

and is strongly continuous in the following sense: the orbit maps $t \longmapsto T(t)x$ are continuous from $\mathbb{R}_+$ into $\mathcal{H}$ for every $x \in \mathcal{H}$ ([11]). The generator $B : \mathcal{D}(B) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$ is the operator

$$\begin{aligned} \mathcal{D}(B) &:= \Big\{x \in \mathcal{H} : \lim_{h\longrightarrow 0} \frac{T(h)x - x}{h} \text{ exists}\Big\}, \\ Bx &:= \lim_{h\longrightarrow 0} \frac{T(h)x - x}{h}, \quad x \in \mathcal{D}(B). \end{aligned}$$

Our study of skew (A, m)-symmetric operators is motivated by the connection between C_o-semigroups of (A, m)-isometries and their generators, as mentioned by the following theorem.

Theorem 1.3. *([19]) Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on $\mathcal{H}$ with generator $(X, \mathcal{D}(X))$. Assume that $T(t) \in \mathfrak{B}_A(\mathcal{H})$ for every $t \geq 0$. Then, the following properties are equivalent:*

1. *$(T(t))_{t\geq 0}$ is an (A, m)-isometry for every $t \geq 0$.*
2. *The generator X is skew (A, m)-symmetric on $\mathcal{D}(X^m)$.*

Inspired by [4] and [10], we extend (3) to commuting d-tuples. A d-tuple of commuting operators $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is said to be skew (A, m)-symmetric if

$$\sum_{k=0}^{m} \binom{m}{k} (T_1^* + \cdots + T_d^*)^k A (T_1 + \cdots + T_d)^{m-k} = 0,$$

equivalently, if $(T_1 + \cdots + T_d)$ is a skew (A, m)-symmetric bounded linear operator. In this framework, we aim to give some basic properties concerning such a family.

The present paper is organized as follows: In Section 2, the notion of left (A, m)-invertible operators is introduced. Theorem 2.5 shows that, under a suitable assumption, a left (A, m)-invertible operator is also left $(A, m-1)$-invertible. In Theorem 2.6, 2.7 and 2.8 we study the power, the product and the bounded perturbation of left (A, m)-invertible operators. Theorem 2.11 shows that if T is skew (A, m)-symmetric, then e^{isT} is left $(A, m-1)$-invertible. As a consequence of such a characterization, we prove in Theorem 2.12 that if T is skew (A, m)-symmetric, then it is skew $(A, m-1)$-symmetric. Moreover, it is shown in Theorem 2.15 that if R is skew (A, m)-symmetric and S is skew (A, n)-symmetric, then $e^{is(R+S)}$ is left $(A, m+n-1)$-invertible. The connection between c_0-semigroups of (A, m)-isometries and skew (A, m)-symmetric operators is described in Theorem 2.16. The aim of Section 3 is to study the stability of skew (A, m)-symmetries under bounded nilpotent perturbation. We show in Theorem 3.3 that if T and Q are commuting operators, T is skew (A, m)-symmetric and Q is l-nilpotent, then $(T + Q)$ is skew $(A, m + 2l - 2)$-symmetric. Moreover, we show that for n odd the power of a skew (A, m)-symmetry is also skew (A, m)-symmetric. Finally, in the closing section, we introduce the skew (A, m)-symmetric tuple of commuting operators. Especially, we characterize the joint spectrum and the joint approximate spectrum related to such a family.

2. Basic properties of skew (A, m)-symmetric operators

In this section we aim to give some properties and some characterizations of skew (A, m)-symmetric bounded linear operators.

In [10], the authors introduced the following polynomials

$$\begin{aligned}
\Big\{(y+x)^m\Big\}_{\mathbf{a}} &= \Big\{\sum_{k=0}^{m}\binom{m}{k} y^k x^{m-k}\Big\}_{\mathbf{a}} := \sum_{k=0}^{m}\binom{m}{k} y^k \mathbf{a} x^{m-k},\\
\Big\{(y-x)^n\Big\}_{\mathbf{a}} &= \Big\{\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} y^k x^{n-k}\Big\}_{\mathbf{a}} := \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} y^k \mathbf{a} x^{n-k},\\
\Big\{(yx-1)^m\Big\}_{\mathbf{a}} &= \Big\{\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} y^k x^{k}\Big\}_{\mathbf{a}} := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} y^k \mathbf{a} x^{k}.
\end{aligned}$$

For $T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$, it follows that

$$\zeta_T^{(m)}(A) = \Big\{(y+x)^m\Big\}_{\mathbf{a}}(T^*, T, A), \quad \mathcal{S}_T^{(m)}(A) = \Big\{(y-x)^m\Big\}_{\mathbf{a}}(T^*, T, A)$$
$$\textit{and} \quad \Theta_T^{(m)}(A) = \Big\{(yx-1)^m\Big\}_{\mathbf{a}}(T^*, T, A).$$

For nonzero $R, S \in \mathcal{B}(\mathcal{H})$, we denote by $R \otimes S$ the tensor product of R and S on the Hilbert space $\mathcal{H}\bar{\otimes}\mathcal{H}$ $\Big(\mathcal{H}\bar{\otimes}\mathcal{H}$ denotes the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}\Big)$. It is defined as follows:

$$\langle R \otimes S(x_1 \otimes y_1) \mid x_2 \otimes y_2\rangle := \langle Rx_1 \mid x_2\rangle \, \langle Sy_1 \mid y_2\rangle, \quad x_1, x_2, y_1, y_2 \in \mathcal{H}. \tag{4}$$

Proposition 2.1. *Let $T_i \in \mathcal{B}(\mathcal{H})$, $A_i \in \mathcal{B}(\mathcal{H})^+$ $(i = 1, 2)$. Then, the following statements hold.*

1. *If T_i is skew (A_i, m)-symmetric $(i = 1, 2)$, then $(T_1 \oplus T_2)$ is skew $(A_1 \oplus A_2, m)$-symmetric.*
2. *If T_1 is skew (A_1, m)-symmetric and T_2 is skew (A_2, n)-symmetric, then $(T_1 \otimes I)$ is skew $(A_1 \otimes A_2, m)$-symmetric and $(I \otimes T_2)$ is skew $(A_1 \otimes A_2, n)$-symmetric.*

Proof. 1. Assume that T_i is skew (A_i, m)-symmetric for $i = 1, 2$. Then, we have

$$\begin{aligned}
&\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big((T_1 \oplus T_2)^*, T_1 \oplus T_2, A_1 \oplus A_2\Big)\\
=\ &\underbrace{\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T_1^*, T_1, A_1\Big)}_{(=0)} \oplus \underbrace{\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T_2^*, T_2, A_2\Big)}_{(=0)} = 0.
\end{aligned}$$

Hence, $(T_1 \oplus T_2)$ is skew $(A_1 \oplus A_2, m)$-symmetric.

2. Let $u = u_1 \otimes u_2 \in \mathcal{H} \otimes \mathcal{H}$. Since $(R \otimes S)(A \otimes B) = RA \otimes SB$ and $(R \otimes S)^*(R \otimes S) = R^*R \otimes S^*S$, it holds

$$\begin{aligned}
&\Big\langle \Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big((T_1 \otimes I)^*, T_1 \otimes I, A_1 \otimes A_2\Big)u \mid u\Big\rangle\\
=\ &\underbrace{\Big\langle \Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T_1^*, T_1, A_1\Big)u_1 \mid u_1\Big\rangle}_{(=0)} \Big\langle A_2 u_2 \mid u_2\Big\rangle = 0.
\end{aligned}$$

Arguing in the same way as previously, we obtain

$$\begin{aligned}
&\Big\langle \Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big((I \otimes T_2)^*, I \otimes T_2, A_1 \otimes A_2\Big)u \mid u\Big\rangle\\
=\ &\underbrace{\Big\langle \Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T_2^*, T_2, A_2\Big)u_2 \mid u_2\Big\rangle}_{(=0)} \Big\langle A_1 u_1 \mid u_1\Big\rangle = 0.
\end{aligned}$$

□

Remark 2.2. *It follows from* (1)*-Proposition 2.1 that if T_i is skew (A_i, m)-symmetric $(i = 1, \cdots, n)$, then $\left(\bigoplus_{i=1}^{n} T_i\right)$ is skew $\left(\bigoplus_{i=1}^{n} A_i, m\right)$-symmetric.*

If there exists an integer $k \geq 1$ such that $S \in \mathcal{B}(\mathcal{H})$ satisfies $\mathcal{S}_{R,S}^{(k)}(A) = 0$, we say that S belongs to the $Helton_A$ class of an operator R with order k. We denote this by $S \in Helton_{A,k}(R)$. Recall that an operator $Q \in \mathcal{B}(\mathcal{H})$ is said to be nilpotent of order n if $Q^n = 0$ and $Q^{n-1} \neq 0$.
In the following theorem, we consider a class of (2×2) upper triangular operator matrices.

Theorem 2.3. *Let $T_i, X \in \mathcal{B}(\mathcal{H})$, $A_i \in \mathcal{B}(\mathcal{H})^+$ $(i = 1, 2)$, $R = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$, $S = \begin{pmatrix} T_1^* & X \\ 0 & T_2^* \end{pmatrix}$ and $\mathbb{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be operators on $\mathcal{H} \oplus \mathcal{H}$. Then, the following statements hold true.*

1. *If T_i is skew $(A_i, 2)$-symmetric $(i = 1, 2)$. Then, we have*

$$\Big(R \in Helton_{\mathbb{A},2}(S)\Big) \Longleftrightarrow \Big((A_1T_1 + 2T_1^*A_1)X + A_1XT_2 = -X(2A_2T_2 + T_2^*A_2) - T_1^*XA_2\Big).$$

2. *If T_1 is (A_1, m)-symmetric and T_2 is n-nilpotent, then R satisfies $R^{*n}\Big\{(y + x)^m\Big\}_{\mathbf{a}}\Big(R^*, R, A\Big)R^n = 0$.*

Proof. 1. The desired statement follows from the following equality

$$\begin{aligned}&\Big\{(y + x)^2\Big\}_{\mathbf{a}}\left(\begin{pmatrix} T_1^* & X \\ 0 & T_2^* \end{pmatrix}, \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right)\\ =& \begin{pmatrix} \Big\{(y + x)^2\Big\}_{\mathbf{a}}\Big(T_1^*, T_1, A_1\Big) & \mathbf{Y} \\ 0 & \Big\{(y + x)^2\Big\}_{\mathbf{a}}\Big(T_2^*, T_2, A_2\Big) \end{pmatrix},\end{aligned}$$

where

$$\mathbf{Y} = A_1(T_1X + XT_2) + 2(T_1^*A_1X + XA_2T_2) + (T_1^*X + XT_2^*)A_2.$$

2. Since $R^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j} \\ 0 & T_2^k \end{pmatrix}$ for $k \geq 1$, we get

$$R^{*n}\Big\{(y + x)^m\Big\}_{\mathbf{a}}\Big(R^*, R, A\Big)R^n = \begin{pmatrix} T_1^{*n}\Big\{(y + x)^m\Big\}_{\mathbf{a}}\Big(T_1^*, T_1, A_1\Big)T_1^n & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

□

We introduce below the notion of left (A, m)-invertibility, which will play a central role in this paper, and we give some of their useful properties.

Definition 2.4. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T,\ S \in \mathcal{B}(\mathcal{H})$. We say that the operator T is left* (*resp. right*) *(A, m)-invertible by S, for some integer $m \geq 1$, if $\Big\{(yx - 1)^m\Big\}_{\mathbf{a}}\Big(S, T, A\Big) = 0$ $\Big($resp. $\Big\{(yx - 1)^m\Big\}_{\mathbf{a}}\Big(T, S, A\Big) = 0\Big)$.*

Sid Ahmed et al. have proved in [10, 27] that if T is (A, m)-isometric where m is even and T is invertible, then T is an $(A, m - 1)$-isometry. Under suitable assumptions, we extend a similar property to left invertible operators.

Theorem 2.5. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$ be invertible. Assume that $\Big\{(yx - 1)^{m-1}\Big\}_{\mathbf{a}}\Big(R, T, A\Big)$ and $R^{m-1}\Big\{(yx - 1)^{m-1}\Big\}_{\mathbf{a}}\Big(R^{-1}, T^{-1}, A\Big)T^{m-1}$ are nonnegative. If T is left (A, m)-invertible with inverse R where m is even, then T is left $(A, m - 1)$-invertible.*

Proof. Since m is even, we have $(-1)^k = -(-1)^{m-1-k}$. Then, it holds

$$\begin{aligned} R^{m-1}\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(R^{-1},T^{-1},A\Big)T^{m-1} &= \sum_{k=0}^{m}(-1)^{m-1-k}\binom{m}{k}R^{m-1-k}AT^{m-1-k} \\ &= -\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(R,T,A\Big) \geq 0. \end{aligned}$$

Therefore $\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(R,T,A\Big) = 0$. Hence, T is left $(A,m-1)$-invertible. $\square$

Theorem 2.6. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $R,S \in \mathcal{B}(\mathcal{H})$. If R is a left (A,m)-inverse of S, then R^n is a left (A,m)-inverse of S^n for each n.*

Proof. Since, for some constants λ_k $(k = 0,\cdots,m(n-1))$, it holds

$$(y^n x^n - 1)^m = \sum_{k=0}^{m(n-1)} \lambda_k y^{m(n-1)-k}(yx-1)^m y^{m(n-1)-k}.$$

Hence, we have

$$\Big\{(y^n x^n - 1)^m\Big\}_{\mathbf{a}} = \sum_{k=0}^{m(n-1)} \lambda_k y^{m(n-1)-k}\Big\{(yx-1)^m\Big\}_{\mathbf{a}} y^{m(n-1)-k}$$

and

$$\Big(\Big\{(y^n x^n - 1)^m\Big\}_{\mathbf{a}}\Big)\Big(R,S,A\Big) = \sum_{k=0}^{m(n-1)} \lambda_k R^{m(n-1)-k}\Big(\Big\{(yx-1)^m\Big\}_{\mathbf{a}}\Big)\Big(R,S,A\Big)S^{m(n-1)-k}.$$

If R is a left (A,m)-inverse of S, then $\Big(\Big\{(yx-1)^m\Big\}_{\mathbf{a}}\Big)\Big(R,S,A\Big) = 0$. From which we deduce that $\Big(\Big\{(y^n x^n - 1)^m\Big\}_{\mathbf{a}}\Big)\Big(R,S,A\Big) = 0$, and so R^n is a left (A,m)-inverse of S^n for each n. $\square$

Theorem 2.7. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and R_1, R_2, S_1, $S_2 \in \mathcal{B}(\mathcal{H})$ such that $R_1R_2 = R_2R_1$ and $S_1S_2 = S_2S_1$. Assume that R_1 is a left (A,m)-inverse of S_1 and R_2 is a left (A,m)-inverse of S_2, then R_1R_2 is a left $(A,m+n-1)$-inverse of S_1S_2.*

Proof. Fix $x,y \in \mathcal{H}$ and let $a_{i,j} = \langle R_1^i R_2^j A S_1^i S_2^j x \mid y\rangle$. Then, for all non-negative integers k and l, we have

$$a_{k+i,l} = \langle R_1^i A S_1^i\Big(S_1^k S_2^l x\Big) \mid \Big(R_1^{*k} R_2^{*l} y\Big)\rangle, \quad a_{k,l+j} = \langle R_2^j A S_2^j\Big(S_1^k S_2^l x\Big) \mid \Big(R_1^{*k} R_2^{*l} y\Big)\rangle.$$

The left (A,m)-invertibility of S_1 by R_1 and the left (A,n)-invertibility of S_2 by R_2 gives

$$\sum_{i=0}^{m}(-1)^{m-i}a_{k+i,l} = 0, \qquad \sum_{j=0}^{n}(-1)^{n-j}a_{k,l+j} = 0.$$

Applying [12, Corollary 2.5] we deduce that

$$0 = \sum_{s=0}^{m+n-1}(-1)^{m+n-1-s}a_{s,s} = \sum_{s=0}^{m+n-1}(-1)^{m+n-1-s}\binom{m+n-1}{s}\Big\langle (R_1R_2)^s A (S_1S_2)^s)x \Big| y\Big\rangle.$$

Since x and y are arbitrary in $\mathcal{H}$, it follows from that

$$\sum_{s=0}^{m+n-1}(-1)^{m+n-1-s}\binom{m+n-1}{s}(R_1R_2)^s A (S_1S_2)^s = 0.$$

Hence, R_1R_2 is a left $(A,m+n-1)$-inverse of S_1S_2. $\square$

Theorem 2.8. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $R, S, P, Q \in \mathcal{B}(\mathcal{H})$ such that Q is nilpotent with order l, P is nilpotent with order l', $RP = PR$ and $SQ = QS$. If R is a left (A, m)-inverse of S, then $(R + P)$ is a left $(A, m + l + l' - 2)$-inverse of $(S + Q)$.*

Proof. From the multinomial formula we get

$$\begin{aligned}\Big\{((y+s)(x+t)-1)^n\Big\}_{\mathbf{a}} &= \Big\{((yx-1)+(y+s)t+sx)^n\Big\}_{\mathbf{a}}\\ &= \Big\{\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(y+s)^k s^j (yx-1)^{n-k-j}x^j t^k\Big\}_a\\ &= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(y+s)^k s^j \Big\{(yx-1)^{n-k-j}\Big\}_a x^j t^k.\end{aligned}$$

By using this identity with x replaced by S, y replaced by R, s replaced by P, t replaced by Q, we obtain

$$\begin{aligned}&\Big\{((y+s)(x+t)-1)^n\Big\}_{\mathbf{a}}(R+P, S+Q, A)\\ =\ &\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(R+P)^k P^j \Big\{(yx-1)^{n-k-j}\Big\}_a (R, S, A) S^j Q^k.\end{aligned}$$

Let $n = m + l + l' - 2$. If $k \geq l$ or $j \geq l'$, then $Q^k = 0$ or $P^j = 0$. Hence $(R+P)^k P^j \Big\{(yx-1)^{n-k-j}\Big\}_a (R, S, A) S^j Q^k = 0$. On the other hand, if $k < l$ and $j < l'$, we have

$$n - k - j = m + l + l' - 2 - k - j \geq m + l + l' - 2 - (l-1) - (l'-1) = m.$$

Since R is a left (A, m)-inverse of S, we get $\Big\{(yx-1)^{n-k-j}\Big\}_a (R, S, A) = \Big\{(yx-1)^{m}\Big\}_a (R, S, A) = 0$. Hence, $(R+P)$ is a left $(A, m + l + l' - 2)$-inverse of $(S+Q)$. □

Exponential operators $\left(e^{isT},\ T \in \mathcal{B}(\mathcal{H}) \text{ and } s \in \mathbb{R}\right)$ are with some interest since they act on a wave function to move it in time and space (see [5], [30]). Note that $\left(e^{iT}\right)$ is a function of an operator $f(T)$ which is defined by its expansion in a Taylor series. Due to this, we devote most of our interest to distinguish some connection between skew (A, m)-symmetric operators and their associated exponential operator.

Using the formula $e^{\alpha M} := \sum_{k\geq 0} \dfrac{\alpha^k M^k}{k!}\ \left(M \in \mathcal{B}(\mathcal{H}),\ \alpha \in \mathbb{C}\right)$, we can write

$$e^{isT} = I + (is)T + \frac{(is)^2}{2!}T^2 + \frac{(is)^3}{3!}T^3 + \cdots, \qquad e^{isT^*} = I + (is)T^* + \frac{(is)^2}{2!}T^{*2} + \frac{(is)^3}{3!}T^{*3} + \cdots. \tag{5}$$

and it follows from that

$$\begin{aligned}&R_A(s) := e^{isT^*} A e^{isT}\\ =\ &A + (is)\Big(AT + T^*A\Big) + \frac{(is)^2}{2!}\Big(AT^2 + 2T^*AT + T^{*2}A\Big) + \frac{(is)^3}{3!}\Big(AT^3 + 3T^*AT^2 + 3T^{*2}AT + T^{*3}A\Big) + \cdots\\ =\ &A + (is)\zeta_T^{(1)}(A) + \frac{(is)^2}{2!}\zeta_T^{(2)}(A) + \frac{(is)^3}{3!}\zeta_T^{(3)}(A) + \cdots.\end{aligned} \tag{6}$$

Let consider the following algebraic condition on T

$$\widetilde{\mathbf{POL_m}}(A): \qquad R_A(s) = A + \sum_{k=1}^{m} s^k E_k, \qquad E_k := E_k(A, T, T^*) \in \mathcal{B}(\mathcal{H}). \tag{7}$$

Note that

$$\frac{d^n}{ds^n}R_A(s) = i^n e^{isT^*}\,\zeta_T^{(n)}(A)\,e^{isT}, \quad n \in \mathbb{N}. \tag{8}$$

The $\widetilde{\mathbf{POL_m}}(A)$ condition is equivalent to $\frac{d^{m+1}}{ds^{m+1}}\Big(e^{isT^*}A\,e^{isT}\Big) = 0$ and consequently to $\zeta_T^{(m+1)}(A) = 0$. As a consequence of (8), $T \in \mathcal{B}(\mathcal{H})$ is skew (A, m)-symmetric if it satisfies $\widetilde{\mathbf{POL_{m-1}}}(A)$.

We begin our study of exponential operators with the following result in which we characterize $\cos\Big(sT\Big) := \frac{e^{isT}+e^{-isT}}{2}$ and $\sin\Big(sT\Big) := \frac{e^{isT}-e^{-isT}}{2i}$, for $s \in \mathbb{R}$.

Proposition 2.9. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. For $s \in \mathbb{R}$, if e^{isT} and e^{-isT} are skew A-symmetric operators, then $\cos\Big(sT\Big)$ is skew A-symmetric and $\sin\Big(sT\Big)$ is A-symmetric.*

Proof. Since e^{isT} and e^{-isT} are skew A-symmetric, we get

$$\begin{aligned}
A\cos(sT) + \Big(\cos(sT)\Big)^* A &= \frac{1}{2}\Big\{Ae^{isT} + Ae^{-isT} + e^{-isT^*}A + e^{isT^*}A\Big\}\\
&= \frac{1}{2}\Big\{-e^{-isT^*}A + Ae^{-isT} + e^{-isT^*}A + e^{isT^*}A\Big\} = 0,\\
A\sin(sT) - \Big(\sin(sT)\Big)^* A &= \frac{1}{2i}\Big\{Ae^{isT} - Ae^{-isT} + e^{-isT^*}A - e^{isT^*}A\Big\}\\
&= \frac{1}{2i}\Big\{-e^{-isT^*}A + e^{isT^*}A + e^{-isT^*}A - e^{isT^*}A\Big\} = 0
\end{aligned}$$

and this allows to conclude. □

Proposition 2.10. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. If T is skew (A, m)-symmetric, then for each positive integer k, the following identity holds*

$$\Big(e^{isT^*}\Big)^k A\Big(e^{isT}\Big)^k = \sum_{j=0}^{m-1}\frac{(isk)^j}{j!}\zeta_T^j(A), \quad \textit{with } \zeta_T^{(0)}(A) = A. \tag{9}$$

Proof. It follows from (6) that

$$\Big(e^{isT^*}\Big)^k A\Big(e^{isT}\Big)^k = A + (is)k\zeta_T^{(1)}(A) + \frac{(is)^2k^2}{2!}\zeta_T^{(2)}(A) + \cdots + \frac{(is)^{m-1}k^{m-1}}{(m-1)!}\zeta_T^{(m-1)}(A) + \frac{(is)^m k^m}{m!}\zeta_T^{(m)}(A) + \cdots. \tag{10}$$

Since T is skew (A, m)-symmetric, $\zeta_T^{(n)}(A) = 0$ for $n \geq m$, and this finishes the proof. □

In [10, Theorem 2.6], the authors proved that if T is an (A, m)-symmetric operator, then e^{isT} is (A, m)-isometric for every $t \in \mathbb{R}$. By using the generalized notion of left invertible operators introduced above we give a characterization related to skew (A, m)-symmetries.

Theorem 2.11. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. If T is skew (A, m)-symmetric, then for every $s \in \mathbb{R}$, the operator e^{isT} is left (A, m)-invertible with left inverse e^{isT^*}.*

Proof. By (9), it holds that

$$\Big\{(yx-1)^m\Big\}_{\mathbf{a}}\Big(e^{isT^*},e^{isT},A\Big) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(e^{isT^*}\Big)^k A\Big(e^{isT}\Big)^k$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big\{A+(is)k\zeta_T^{(1)}(A)+\frac{(is)^2k^2}{2!}\zeta_T^{(2)}(A)+\frac{(is)^3k^3}{3!}\zeta_T^{(3)}(A)+\cdots+\frac{(is)^{m-1}k^{m-1}}{(m-1)!}\zeta_T^{(m-1)}(A)\Big\}$$

$$= A\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big)+(is)\zeta_T^{(1)}(A)\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k\Big)+\frac{(is)^2}{2!}\zeta_T^{(2)}(A)\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^2\Big)+\cdots+$$

$$+\cdots+\frac{(is)^{m-1}}{(m-1)!}\zeta_T^{(m-1)}(A)\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^{m-1}\Big).$$

Since

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^j = \begin{cases} 0, & \text{if } 0\le j\le m-1;\\ m!, & \text{if } j=m,\end{cases} \tag{11}$$

we get $\Big\{(yx-1)^m\Big\}_{\mathbf{a}}\Big(e^{isT^*},e^{isT},A\Big)=0$ and this allows to conclude. □

In [19] we have shown that if T is (A,m)-symmetric and m is even, then T is always $(A,m-1)$-symmetric. In the case of skew (A,m)-symmetries, we establish the following characterization.

Theorem 2.12. *Let $A\in\mathcal{B}(\mathcal{H})^+$ and $T\in\mathcal{B}(\mathcal{H})$. Assume that $\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(e^{isT^*},e^{isT},A)$ and $(e^{isT^*})^{m-1}\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(e^{isT^*},e^{-isT},A)(e^{isT})^{m-1}$ are nonnegative. If T is skew (A,m)-symmetric where m is even, then T is skew $(A,m-1)$-symmetric.*

Proof. Since T is skew (A,m)-symmetric, $\zeta_T^{(n)}(A)=0$ for all $n\ge m$ and, by Theorem 2.11, e^{isT} is left (A,m)-invertible. On the other hand, since m is even and e^{isT} is invertible, it follows from Theorem 2.5 and the hypothesis that e^{isT} is left $(A,m-1)$-invertible for all $s\in\mathbb{R}$. On the other hand, Equation (10) gives

$$0=\Big\{(yx-1)^{m-1}\Big\}_{\mathbf{a}}\Big(e^{isT^*},e^{isT},A\Big)=\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\Big(e^{isT^*}\Big)^kA\Big(e^{isT}\Big)^k$$

$$= \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\Big\{A+(is)(m-1-k)\zeta_T^{(1)}(A)+\frac{(is)^2(m-1-k)^2}{2!}\zeta_T^{(2)}(A)$$

$$+\frac{(is)^3(m-1-k)^3}{3!}\zeta_T^{(3)}(A)+\cdots+\frac{(is)^{m-1}(m-1-k)^{m-1}}{(m-1)!}\zeta_T^{(m-1)}(A)\Big\}$$

$$= A\Big(\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\Big)+(is)\zeta_T^{(1)}(A)\Big(\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}(m-1-k)\Big)$$

$$+\frac{(is)^2}{2!}\zeta_T^{(2)}(A)\Big(\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}(m-1-k)^2\Big)+\cdots+$$

$$+\cdots+\frac{(is)^{m-1}}{(m-1)!}\zeta_T^{(m-1)}(A)\Big(\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}(m-1-k)^{m-1}\Big).$$

By using the identity (11), we obtain $\zeta_T^{(m-1)}(A)=0$. Hence T is skew $(A,m-1)$-symmetric. □

Proposition 2.13. *Let $A\in\mathcal{B}(\mathcal{H})^+$ and $T\in\mathcal{B}(\mathcal{H})$. Then the following statements hold.*

1. *If* $(T_n)_n$ *is a sequence of skew* (A,m)*-symmetric operators such that* $\lim\limits_{n\longrightarrow\infty}\left\|e^{isT_n}-e^{isT}\right\|=0$ *then* e^{isT} *is left* (A,m)*-invertible.*
2. *If* T *is skew* (A,m)*-symmetric, then* e^{isnT} *is left* (A,m)*-invertible with left inverse* e^{isnT^*} *for all* $n\in\mathbb{N}$*.*
3. *If* T *is* (A,m)*-symmetric, then* $e^{is\alpha T}$ *is left* (A,m)*-invertible for* α *pure imaginary.*
4. *If* T *is skew* (A,m)*-symmetric and* A*-symmetric, then the operator* e^{isT} *belongs to the Helton*$_A$ *class of* e^{isT^*}*.*

Proof. 1. Since T_n is skew (A,m)-symmetric, e^{isT_n} is left (A,m)-invertible. On the other hand, the fact that the class of left (A,m)-invertible operators is closed in norm and $\lim\limits_{n\longrightarrow\infty}\left\|e^{isT_n}-e^{isT}\right\|=0$ allow to deduce that e^{isT} is left (A,m)-invertible.

2. Since T is skew (A,m)-symmetric, it holds from Theorem 2.11 that e^{isT} is left (A,m)-invertible with left inverse e^{isT^*}. Applying Theorem 2.6 we can conclude.
3. The claim follows immediately from Remark 1.1.
4. It follows from (5) that

$$\begin{aligned} Ae^{isT} &= A+(is)AT+\frac{(is)^2}{2!}AT^2+\frac{(is)^3}{3!}AT^3+\cdots, \\ e^{isT^*}A &= A+(is)T^*A+\frac{(is)^2}{2!}T^{*2}A+\frac{(is)^3}{3!}T^{*3}A+\cdots. \end{aligned}$$

If T is A-symmetric (that is $AT=T^*A$), then $AT^k=T^{*k}A$ for all $k\geq 0$. Hence, from the two above equalities we get $Ae^{isT}=e^{isT^*}A$.
□

Proposition 2.14. *Let* $s\in\mathbb{R}$, $A\in\mathcal{B}(\mathcal{H})^+$ *and* $T\in\mathcal{B}(\mathcal{H})$ *be a skew* A*-symmetric operator. Then, the following statements hold.*

1. $\cos(sT)$ *is* A*-isometric if and only if* $\mathcal{R}\Big(I-\cos(2sT)\Big)\subset\mathcal{N}(A)$*.*
2. $\sin(sT)$ *is* A*-isometric if and only if* $\mathcal{R}\Big(3I-\cos(2sT)\Big)\subset\mathcal{N}(A)$*.*

Proof. Since T is skew A-symmetric, e^{isT} is left A-invertible with left inverse e^{isT^*}, that is $A=e^{isT^*}Ae^{isT}$. Therefore, we obtain

$$\begin{aligned} A-\Big(\cos(sT)\Big)^*A\cos(sT) &= A-\frac{1}{4}\Big\{Ae^{2isT}+2A+Ae^{-2isT}\Big\}=\frac{1}{2}A\Big\{I-\cos(2sT)\Big\}, \\ A-\Big(\sin(sT)\Big)^*A\sin(sT) &= A-\frac{1}{4}\Big\{Ae^{2isT}-2A+Ae^{-2isT}\Big\}=\frac{1}{2}A\Big\{3I-\cos(2sT)\Big\} \end{aligned}$$

and the above two identities allows to conclude. □

Theorem 2.15. *Let* $A\in\mathcal{B}(\mathcal{H})^+$ *and* $R,S\in\mathcal{B}(\mathcal{H})$ *such that* $RS=SR$*. If* R *is skew* (A,m)*-symmetric and* S *is skew* (A,n)*-symmetric, then* $e^{is(R+S)}$ *is left* $(A,m+n-1)$*-invertible with left inverse* $e^{is(R+S)^*}$*.*

Proof. Since R is skew (A,m)-symmetric and S is skew (A,n)-symmetric, it holds from Theorem 2.11 that $S_1=e^{isR}$ is left (A,m)-invertible with left inverse $R_1=e^{isR^*}$ and $S_2=e^{isS}$ is left (A,n)-invertible with left inverse $R_2=e^{isS^*}$, for every $s\in\mathbb{R}$. It is clear that $R_1R_2=R_2R_1$ and $S_1S_2=S_2S_1$. Applying Theorem 2.7, we deduce that $S_1S_2=e^{is(S+R)}$ is left $(A,m+n-1)$-invertible with left inverse $R_1R_2=e^{is(R+S)^*}$. □

As an immediate consequence of Theorem 1.3 and Theorem 2.11, we have the following result in which we examine the relation between a semigroup of (A,m)-isometries and the exponential operator of its infinitesimal generator.

Theorem 2.16. *Let* $A\in\mathcal{B}(\mathcal{H})^+$ *and* $(T(t))_{t\geq0}$ *be a* C_0*-semigroup on* $\mathcal{H}$ *with generator* $(X,\mathcal{D}(X))$*. Assume that* $T(t)\in\mathfrak{B}_A(\mathcal{H})$ *for every* $t\geq0$*. If* $(T(t))_{t\geq0}$ *is* (A,m)*-isometric for every* $t\geq0$*, then* e^{isX} *is left* (A,m)*-invertible on* $\mathcal{D}(X^m)$*.*

3. Sum with a nilpotent operator and power of a skew (A, m)-symmetry

In this section, we investigate the stability of a skew (A, m)-symmetric operator T under some perturbation by a nilpotent operator. Moreover, in the closing part of this section, we examine the integer power of a skew (A, m)-symmetry.

Lemma 3.1. *Let $T,\ Q \in \mathcal{B}(\mathcal{H})$ with $TQ = QT$. Then the following identities hold.*

1.
$$\zeta^{(l)}_{T+Q}(A) = \sum_{k=0}^{l}\sum_{j=0}^{l-k}\binom{l}{k}\binom{l-k}{j} Q^{*j}\zeta_T^{(l-k-j)}(A)Q^k. \tag{12}$$

2.
$$\mathcal{S}^{(l)}_{T+Q}(A) = \sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k\binom{l}{k}\binom{l-k}{j} Q^{*j}\zeta_T^{(l-k-j)}(A)Q^k. \tag{13}$$

Proof. 1. The desired identity follows from the following

$$\begin{aligned}
&\Big\{((y+s)+(x+t))^m\Big\}_{\mathbf{a}}(T^*,T,A) = \Big\{((y+x)+s+t)^m\Big\}_{\mathbf{a}}(T^*,T,A)\\
=\ &\Big\{\sum_{k=0}^{l}\sum_{j=0}^{l-k}\binom{l}{k}\binom{l-k}{j} s^{*j}(y+x)^{(l-k-j)}t^k\Big\}_{\mathbf{a}}(T^*,T,A)\\
=\ &\Big(\sum_{k=0}^{l}\sum_{j=0}^{l-k}\binom{l}{k}\binom{l-k}{j} s^{*j}\Big\{(y+x)^{(l-k-j)}\Big\}_{\mathbf{a}}t^k\Big)(T^*,T,A)\\
=\ &\sum_{k=0}^{l}\sum_{j=0}^{l-k}\binom{l}{k}\binom{l-k}{j} Q^{*j}\zeta_T^{(l-k-j)}(A)Q^k.
\end{aligned}$$

2. We have

$$\begin{aligned}
&\Big\{((y+s)-(x+t))^m\Big\}_{\mathbf{a}}(T^*,T,A) = \Big\{((y-x)-(t-s))^m\Big\}_{\mathbf{a}}(T^*,T,A)\\
=\ &\Big\{\sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k\binom{l}{k}\binom{l-k}{j} s^{*j}(y-x)^{(l-k-j)}t^k\Big\}_{\mathbf{a}}(T^*,T,A)\\
=\ &\Big(\sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k\binom{l}{k}\binom{l-k}{j} s^{*j}\Big\{(y-x)^{(l-k-j)}\Big\}_{\mathbf{a}}t^k\Big)(T^*,T,A)\\
=\ &\sum_{k=0}^{l}\sum_{j=0}^{l-k}(-1)^k\binom{l}{k}\binom{l-k}{j} Q^{*j}\zeta_T^{(l-k-j)}(A)Q^k.
\end{aligned}$$

□

In the following theorem, we examine conditions for the operator $(T+Q)$ to be skew $(A, m+2l-2)$-symmetric $\Big($resp. $(A, m+2l-2)$-symmetric ([10, 19])$\Big)$, where Q is l-nilpotent.

Theorem 3.2. *Let $T,\ Q \in \mathcal{B}(\mathcal{H})$ satisfying $TQ = QT$ and Q is l-nilpotent. Then the following statements hold.*

1. *If T is skew (A, m)-symmetric, then the following claims hold true:*
   (a) *The operator $(T+Q)$ is skew $(A, m+2l-2)$-symmetric.*

(b) *The operator* $(T+Q)$ *is strict skew* $(A, m+2l-2)$*-symmetric if and only if* $Q^{*l-1}\zeta_T^{(m-1)}(A)Q^{l-1} \neq 0$.

2. *If* T *is* (A, m)*-symmetric, then the following assertions hold true:*
   (a) *The operator* $(T+Q)$ *is* $(A, m+2l-2)$*-symmetric.*
   (b) *The operator* $(T+Q)$ *is strict* $(A, m+2l-2)$*-symmetric if and only if* $Q^{*l-1}\mathcal{S}_T^{(m-1)}(A)Q^{l-1} \neq 0$.

Proof. Let $n = m+2l-2$.

1. Assume that T is skew (A, m)-symmetric, that is $\zeta_T^{(m)}(A) = 0$.
   (a) According to Lemma 3.1, identity (12), it follows that

$$\zeta_{T+Q}^{(l)}(A) = \sum_{k=0}^{l}\sum_{j=0}^{l-k}\binom{l}{k}\binom{l-k}{j}Q^{*j}\zeta_T^{(l-k-j)}(A)Q^k.$$

   If $j \geq l$ or $k \geq l$, then $Q^{*j} = Q^k = 0$. On the other hand, if $j < l$ and $k < l$, then $n-k-j \geq m$ and so $\zeta_T^{(n-k-j)}(A) = 0$. Hence, $\zeta_{T+Q}^{(n)}(A) = 0$, that is T is skew (A, n)-symmetric.
   (b) Since Q is l-nilpotent, $Q^l = 0$ and $Q^{l-1} \neq 0$. Hence, we have

$$\zeta_{T+Q}^{(n-1)}(A) = \binom{n-1}{l-1}\binom{n-l}{l-1}Q^{*l-1}\zeta_T^{(m-1)}(A)Q^{l-1},$$

   and the claim follows from that.
2. Using the identity (13) and arguing as in the previous statement, we can show easily the assertion. For more details, we also refer the readers to [19, Theorem 4.2].

□

Theorem 3.3. *Let* $A \in \mathcal{B}(\mathcal{H})^+$*,* $T \in \mathcal{B}(\mathcal{H})$ *and* $Q \in \mathcal{B}(\mathcal{H})$ *be l-nilpotent. Then, the following statements hold true.*

1. *If* T *is skew* (A, m)*-symmetric, then* $T \otimes I_{\mathcal{H}} + I_{\mathcal{H}} \otimes Q$ *is a skew* $(A \otimes A, m+2l-2)$*-symmetric operator.*
2. *If* T *is* (A, m)*-symmetric, then* $T \otimes I_{\mathcal{H}} + I_{\mathcal{H}} \otimes Q$ *is* $(A \otimes A, m+2l-2)$*-symmetric.*

Proof.
1. Since T is skew (A, m)-symmetric, it follows from Lemma 2.1 that $T \otimes I_{\mathcal{H}}$ is skew (A, m)-symmetric. Moreover, $I_{\mathcal{H}} \otimes Q \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is a l-nilpotent operator. Hence, applying Theorem 3.2 we can conclude.
2. Arguing in the same way as in the previous statement, we prove the desired claim.

□

As a consequence of Theorem 2.11 and [10, Theorem 2.6], we have the following result.

Corollary 3.4. *Let* $A \in \mathcal{B}(\mathcal{H})^+$*,* $T \in \mathcal{B}(\mathcal{H})$ *and* $Q \in \mathcal{B}(H)$ *be an l-nilpotent operator. Then, we have:*

1. *If* T *is skew* (A, m)*-symmetric, then* $e^{is(T\otimes I_{\mathcal{H}}+I_{\mathcal{H}}\otimes Q)}$ *is left* $(A \otimes A, m+2l-2)$*-invertible.*
2. *If* T *is* (A, m)*-symmetric, then* $e^{is(T\otimes I_{\mathcal{H}}+I_{\mathcal{H}}\otimes Q)}$ *is* $(A \otimes A, m+2l-2)$*-isometric.*

It is known from [19, Theorem 4.4] that if $T \in \mathcal{B}(\mathcal{H})$ is an (A, m)-symmetric operator, then T^n $(n \geq 2)$ is also (A, m)-symmetric. We will give a simple proof of such a result ([10]). For skew (A, m)-symmetric operators, we have the following result.

Theorem 3.5. *Let* $A \in \mathcal{B}(H)^+$ *and* $T \in \mathcal{B}(H)$*. Then, the following statements hold true.*

1. *If* T *is skew* (A, m)*-symmetric, then* T^n *is skew* (A, m)*-symmetric for n odd.*
2. *If* T *is* (A, m)*-symmetric, then* T^n *is* (A, m)*-symmetric for all* $n \in \mathbb{N}$*.*

Proof. 1. Assume that T is skew (A,m)-symmetric. For an odd integer n, the following formula holds true

$$\begin{aligned}\Big\{(y^n+x^n)^m\Big\}_{\mathbf{a}} &= \Big\{\Big((y+x)(y^{n-1}-y^{n-2}x+\cdots-yx^{n-2}+x^{n-1}\Big)^m\Big\}_{\mathbf{a}} \\ &= \Big\{\sum_{k=0}^{m(n-1)} \xi_k y^{m(n-1)-k}(y+x)^m x^k\Big\}_{\mathbf{a}} \\ &= \sum_{k=0}^{m(n-1)} \xi_k y^{m(n-1)-k}\Big\{(y+x)^m\Big\}_{\mathbf{a}} x^k, \end{aligned} \tag{14}$$

where ξ_k are constants for $k=0,\cdots,m(n-1)$. It follows from (14) that

$$\Big\{(y^n+x^n)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big) = \sum_{k=0}^{m(n-1)} \xi_k T^{*m(n-1)-k}\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big)T^k. \tag{15}$$

By (15), if $\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big)=0$, then $\Big\{(y^n+x^n)^m\Big\}_{\mathbf{a}}\Big(T^*,T\text{'}A\Big)=0$. Hence, T^n is skew (A,m)-symmetric.

2. Assume that T is (A,m)-symmetric. The following formula hold true

$$\begin{aligned}\Big\{(y^n-x^n)^m\Big\}_{\mathbf{a}} &= \Big\{\Big((y-x)(y^{n-1}+y^{n-2}x+\cdots+yx^{n-2}+x^{n-1}\Big)^m\Big\}_{\mathbf{a}} \\ &= \Big\{\sum_{k=0}^{m(n-1)} \eta_k y^{m(n-1)-k}(y-x)^m x^k\Big\}_{\mathbf{a}} \\ &= \sum_{k=0}^{m(n-1)} \eta_k y^{m(n-1)-k}\Big\{(y-x)^m\Big\}_{\mathbf{a}} x^k, \end{aligned} \tag{16}$$

where η_k are constants for $k=0,\cdots,m(n-1)$. It follows from (16) that

$$\Big\{(y^n-x^n)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big) = \sum_{k=0}^{m(n-1)} \eta_k T^{*m(n-1)-k}\Big\{(y-x)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big)T^k. \tag{17}$$

By (17), if $\Big\{(y-x)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big)=0$, then $\Big\{(y^n-x^n)^m\Big\}_{\mathbf{a}}\Big(T^*,T,A\Big)=0$. Hence, T^n is (A,m)-symmetric.
□

Remark 3.6. *When n is even,* (1)*-Theorem 3.5 is not true. Let $\mathcal{H}=\mathbb{C}^2$, $\|(x,y)\|^2=|x|^2+|y|^2$ for all $x,y\in\mathbb{C}$, $A=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ and $T=\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}$. Then, T is skew A-symmetric. Moreover, it is easy to verify that $\Big\{(y+x)\Big\}_{\mathbf{a}}\Big(T^{*2},T^2,A\Big)\neq 0$, so T^2 is not skew A-symmetric.*

4. Skew (A,m)-symmetric tuples and spectral properties

In this section, we introduce the class of skew (A,m)-symmetric commuting tuple of bounded linear operators. Some of their spectral properties are studied.

For $m\in\mathbb{N}$, $A\in\mathcal{B}(\mathcal{H})^+$ and $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$, set

$$\Phi_A^m(\mathbf{T}) := \sum_{k=0}^m (-1)^{m-k}\binom{m}{k}(T_1^*+\cdots+T_d^*)^k A(T_1+\cdots+T_d)^{m-k}, \tag{18}$$

$$\Psi_A^m(\mathbf{T}) := \sum_{k=0}^m \binom{m}{k}(T_1^*+\cdots+T_d^*)^k A(T_1+\cdots+T_d)^{m-k}. \tag{19}$$

Definition 4.1. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$. A d-tuple of commuting operators $\mathbf{T}$ is said to be skew (A, m)-symmetric tuple (resp. an (A, m)-symmetric tuple) if $\Psi_A^m(\mathbf{T}) = 0$ $\big($resp. $\Phi_A^m(\mathbf{T}) = 0\big)$, equivalently if $(T_1 + \cdots + T_d)$ is a skew (A, m)-symmetric bounded linear operator (resp. an (A, m)-symmetric operator), that is $\zeta^{(m)}_{T_1+\cdots+T_d}(A) = 0$ $\big($resp. $\mathcal{S}^{(m)}_{T_1+\cdots+T_d}(A) = 0\big)$.*

Remark 4.2.
1. *If $A = I$, then $\boldsymbol{T}$ is a skew m-symmetric tuple if and only if $\boldsymbol{T}$ is a skew (A, m)-symmetric tuple.*
2. *If $A = 0$, then any commuting d-tuple of operators is skew (A, m)-symmetric.*
3. *Let $T = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators. Then:*
   (a) *T is a skew A-symmetric tuple if*
   $$A(T_1 + \cdots + T_d) = -(T_1^* + \cdots + T_d^*)A.$$
   (b) *T is a skew $(A, 2)$-symmetric tuple if*
   $$\begin{aligned} 0 &= A(T_1 + \cdots + T_d)^2 + 2(T_1^* + \cdots + T_d^*)A(T_1 + \cdots + T_d) + (T_1^* + \cdots + T_d^*)^2 A \\ &= \sum_{i=1}^{d}(AT_i^2 + T_i^{*2}A + 2T_i^*AT_i) + 2\Big(\sum_{1\le i<j\le d} AT_iT_j + T_i^*T_j^*A\Big) + 2\sum_{1\le i\ne j\le d} T_i^*AT_j^*. \end{aligned}$$

Example 4.3.
1. *Take $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $T_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $T_2 = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$. Then, $\boldsymbol{T} = (T_1, T_2)$ is a skew A-symmetric tuple of bounded linear operators.*
2. *If T is skew (A, m)-symmetric, then $\boldsymbol{T} = (T, \cdots, T)$ is a skew (A, m)-symmetric d-tuple.*

Set $R_n := T_{1n}^* + \cdots + T_{dn}^*$, $R := T_1^* + \cdots + T_d^*$, $S_n := T_{1n} + \cdots + T_{dn}$ and $S := T_1 + \cdots + T_d$.

Proposition 4.4. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$. Then, the following assertions hold true.*
1. *$\mathcal{N}\big(S\big) \subset \mathcal{N}\big(AS^{*m}\big)$.*
2. *If $\boldsymbol{T}$ is skew (A, m)-symmetric, then $\boldsymbol{T}$ is skew (A, n)-symmetric for all $n \ge m$.*
3. *If S is invertible, then*
   (a) *$\boldsymbol{T}$ is skew (A, m)-symmetric if and only if S^{-1} is skew (A, m)-symmetric.*
   (b) *If $S^{*k}AS^{m-k} = S^{m-k}AS^{*k}$ for $k = 0, 1, \cdots, m$, then $\boldsymbol{T}$ is skew (A, m)-symmetric if and only if S^{*-1} is skew (A, m)-symmetric.*
4. *If $\boldsymbol{T}$ is skew (A, m)-symmetric, then $(\boldsymbol{T} - \lambda)$ is skew (A, m)-symmetric, for $\lambda = (\lambda_1, \cdots, \lambda_d)$.*

Proof.
1. Since $\mathbf{S}$ is skew (A, m)-symmetric, we obtain for $x \in \mathcal{H}$
$$S^{*m}Ax + \sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k} S^{*k}AS^{m-k}x = 0. \tag{20}$$
Let $x \in \mathcal{N}\big(S\big)$. The identity (20) gives $S^{*m}Ax = 0$, and hence $x \in \mathcal{N}\big(S^{*m}A\big)$.
2. The statement follows immediately from the equality below
$$\begin{aligned} \Big\{(y + x)^{m+1}\Big\}_{\mathbf{a}}(S^*, S, A) &= \Big\{y(y + x)^m + (y + x)^m x\Big\}_{\mathbf{a}}(S^*, S, A) \\ &= \Big(y\Big\{(y + x)^m\Big\}_{\mathbf{a}} + \Big\{(y + x)^m\Big\}_{\mathbf{a}} x\Big)_{\mathbf{a}}(S^*, S, A) \\ &= S^*\zeta_S^{(m)}(A) + \zeta_S^{(m)}(A)S \\ &= S^*\Psi_A^m(\mathbf{T}) + \Psi_A^m(\mathbf{T})S. \end{aligned}$$
3. (a) Assume that $\mathbf{T}$ is skew (A, m)-symmetric. Then it follows that
$$0 = S^{*-m}\Psi_A^m(\mathbf{T})S^{-m} = \sum_{k=0}^{m}\binom{m}{k}(S^{-1})^{*(m-k)}A(S^{-1})^k,$$
which completes the proof.

(b) Assume that $S^{*k}AS^{m-k} = S^{m-k}AS^{*k}$ for $k = 0, 1, \cdots, m$. Then, we have

$$\begin{aligned} 0 = S^{*m}\zeta^{(m)}_{(S^{-1})^*}(A)S^m &= S^{*m}\Big(\sum_{k=0}^{m}\binom{m}{k}(S^{-1})^k A (S^{-1})^{*(m-k)}\Big)S^m \\ &= S^{*m}\Big(\sum_{k=0}^{m}\binom{m}{k}(S^{-1})^{*(m-k)} A (S^{-1})^k\Big)S^m \\ &= \Psi_A^m(\mathbf{T}). \end{aligned}$$

4. The proof follows from

$$\Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(S^* - \sum_{i=1}^{d}\lambda_i, S - \sum_{i=1}^{d}\lambda_i, A\Big) = \Big\{(y+x)^m\Big\}_{\mathbf{a}}\Big(S^*, S, A\Big).$$

□

In the following theorem we show that the class of skew (A, m)-symmetric commuting tuples of operators is closed in norm.

Theorem 4.5. *Let $\Big(\mathbf{T}_n = (T_{1n}, \cdots, T_{dn})\Big)_n$ be a sequence of skew (A, m)-symmetric commuting tuples such that $T_{jn} \longrightarrow T_j$ for each $j = 1, \cdots, d$ as $n \longrightarrow \infty$ in the strong topology of $\mathcal{L}(\mathcal{H})$. Then, the commuting tuple $\mathbf{T} = (T_1, \cdots, T_d)$ is skew (A, m)-symmetric.*

Proof. Since $\mathbf{T}_n = (T_{1n}, \cdots, T_{dn})$ is skew (A, m)-symmetric, we have

$$\begin{aligned} &\|\Psi_A^m(\mathbf{T})\| = \|\Psi_A^m(\mathbf{T}_n) - \Psi_A^m(\mathbf{T})\| \\ = \;& \Big\|\sum_{k=0}^{m}\binom{m}{k}R_n^k A S_n^{m-k} - \sum_{k=0}^{m}\binom{m}{k}R^k A S^{m-k}\Big\| \\ \leq \;& \Big\|\sum_{k=0}^{m}\binom{m}{k}R_n^k A S_n^{m-k} - \sum_{k=0}^{m}\binom{m}{k}R_n^k A S^{m-k}\Big\| \\ & + \Big\|\sum_{k=0}^{m}\binom{m}{k}R_n^k A S^{m-k} - \sum_{k=0}^{m}\binom{m}{k}R^k A S^{m-k}\Big\| \\ \leq \;& \Big\|\sum_{k=0}^{m}\binom{m}{k}R_n^k A\Big(S_n^{m-k} - S^{m-k}\Big)\Big\| + \Big\|\sum_{k=0}^{m}\binom{m}{k}\Big(R_n^k - R^k\Big)AS^{m-k}\Big\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned}$$

So $\Psi_A^m(\mathbf{T}) = 0$ and hence the commuting tuple $\mathbf{T}$ is skew (A, m)-symmetric. □

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the single valued extension property at λ (abbreviated SVEP at λ) if for every open set D containing λ the only analytic function $f : D \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$ on D. We say T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$ ([18], [24]).

Theorem 4.6. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple. Assume that $0 \notin \sigma_p(A)$. If S^* has the SVEP and **T** is skew (A, m)-symmetric, then S has the SVEP.*

Proof. Let $f : D \longrightarrow \mathcal{H}$ be an analytic function such that $(\lambda - S)f(\lambda) = 0$ for all $\lambda \in D$. By (4)-Proposition 4.4, we have

$$\begin{aligned}
& \sum_{k=0}^{m}\binom{m}{k} S^{*k}AS^{m-k}f(\lambda)-(S^*-\lambda)^mAf(\lambda)\\
= & \sum_{k=0}^{m}\binom{m}{k}(S^*-\lambda)^kA(S-\lambda)^{m-k}f(\lambda)-(S^*-\lambda)^mAf(\lambda)\\
= & \Big\{\sum_{k=0}^{m-1}\binom{m}{k}(S^*-\lambda)^kA(S-\lambda)^{m-k-1}\Big\}(S-\lambda)f(\lambda)=0.
\end{aligned}$$

Since $\mathbf{T}$ is skew (A,m)-symmetric, we obtain $(S^*-\lambda)^mAf(\lambda)=0$. Using an induction argument, we get $Af(\lambda)\equiv 0$. Since $0\notin\sigma_p(A)$, it holds $f(\lambda)\equiv 0$ for all $\lambda\in D$. Hence, S has the single valued extension property. □

In the rest of this section, we provide spectral properties of skew (A,m)-symmetric operators. Recall that two vectors x and y are A-orthogonal if $\langle Ax\mid y\rangle=0$.

For a commuting tuple $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$, we denote by $\sigma_{ja}(\mathbf{T})$ (resp. $\sigma_{jp}(\mathbf{T})$) the joint approximate point spectrum (resp. the joint point spectrum) of $\mathbf{T}$.

Definition 4.7. *Let $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$. We say that :*

1. *$\lambda=(\lambda_1,\cdots,\lambda_d)\in\sigma_{ja}(\mathbf{T})$ if there exists a sequence x_n of unit vectors such that*
$$(T_i-\lambda_i)x_n\longrightarrow 0\quad \text{as } n\longrightarrow\infty\ \text{ for all } i=1,\cdots,d.$$
2. *$\lambda=(\lambda_1,\cdots,\lambda_d)\in\sigma_{jp}(\mathbf{T})$ if there exists a nonzero vector $x\in\mathcal{H}$ such that*
$$(T_i-\lambda_i)x=0\quad \text{for all } i=1,\cdots,d.$$

Theorem 4.8. *Let $A\in\mathcal{B}(\mathcal{H})^+$ and $\mathbf{T}=(T_1,\cdots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple. Assume that $0\notin\sigma_{ap}(A)$. If $\mathbf{T}$ is skew (A,m)-symmetric, then the following statements are satisfied.*

1. *If $\lambda=(\lambda_1,\cdots,\lambda_d)\in\sigma_{ja}(\mathbf{T})$, then $-\Big(\lambda_1+\cdots+\lambda_d\Big)\in\sigma_{ap}(S^*)$.*
2. *If $\lambda=(\lambda_1,\cdots,\lambda_d)\in\sigma_{jp}(\mathbf{T})$, then $-\Big(\lambda_1+\cdots+\lambda_d\Big)\in\sigma_{p}(S^*)$.*
3. *Let $\lambda=(\lambda_1,\cdots,\lambda_d)$ and $\mu=(\mu_1,\cdots,\mu_d)$ be to two joint eigenvalues of $\mathbf{T}$ corresponding to the eigenvectors u and v. If $\Big(\sum_{j=1}^{d}(\lambda_j+\mu_j)\Big)\neq 0$, then u and v are A-orthogonal.*
4. *Let $\lambda=(\lambda_1,\cdots,\lambda_d)$ and $\mu=(\mu_1,\cdots,\mu_d)$ be to two joint eigenvalues of $\mathbf{T}$ such that $\Big(\sum_{j=1}^{d}(\lambda_j+\mu_j)\Big)\neq 0$. If $\{u_n\}_n$, $\{v_n\}_n$ are two sequences of unit vectors such that $(T_j-\lambda_j)u_n\longrightarrow 0$ and $(T_j-\mu_j)v_n\longrightarrow 0$ (as $n\longrightarrow+\infty$), $j=1,\cdots,d$, then $\langle Au_n\mid v_n\rangle\longrightarrow 0$ (as $n\longrightarrow+\infty$).*

Proof. 1. Assume that $0\notin\sigma_{ap}(A)$. Let $\lambda=(\lambda_1,\cdots,\lambda_d)\in\sigma_{ja}(\mathbf{T})$. Then there exists a sequence $(x_n)_n$ with $\|x_n\|=1$ such that $\lim\limits_{n\longrightarrow\infty}(T_i-\lambda_i)x_n=0,\ i=1\cdots,d$. Since $\mathbf{T}$ is skew (A,m)-symmetric, we get

$$\begin{aligned}
0 &= \lim_{n\longrightarrow\infty}\Big\|\Big(\sum_{k=0}^{m}\binom{m}{k}S^{*k}AS^{m-k}\Big)x_n\Big\|\\
&= \lim_{n\longrightarrow\infty}\Big\|\Big(\sum_{k=0}^{m}\binom{m}{k}S^{*k}A\Big(\lambda_1+\cdots+\lambda_d\Big)^{m-k}\Big)x_n\Big\|\\
&= \lim_{n\longrightarrow\infty}\Big\|\Big(S^*+\Big(\lambda_1+\cdots+\lambda_d\Big)\Big)^mAx_n\Big\|\\
&= \lim_{n\longrightarrow\infty}\Big\|\Big(S^*+\Big(\lambda_1+\cdots+\lambda_d\Big)\Big)^m\frac{Ax_n}{\|Ax_n\|}\Big\|.
\end{aligned}$$

Since $\left(\frac{Ax_n}{\|Ax_n\|}\right)_n$ is a sequence of unit vectors, it holds that $-\left(\lambda_1 + \cdots + \lambda_d\right) \in \sigma_{ap}(S^*)$.

2. Let $\lambda = (\lambda_1, \cdots, \lambda_d) \in \sigma_{jp}(\mathbf{T})$, that is $\bigcap_{i=1}^{d} \mathcal{N}(T_i - \lambda_i) \neq \{0\}$. Then, there exists a nonzero $x \in \mathcal{H}$ such that $(T_i - \lambda_i)x = 0$, $i = 1 \cdots, d$. Since $\mathbf{T}$ is skew (A, m)-symmetric, it follows $\left(S^* + \left(\lambda_1 + \cdots + \lambda_d\right)\right)^m Ax = 0$. If $-\left(\lambda_1 + \cdots + \lambda_d\right) \notin \sigma_p(S^*)$, then $S^* + \left(\lambda_1 + \cdots + \lambda_d\right)$ is injective. Hence $Ax = 0$, which is a contradiction. Hence $-\left(\lambda_1 + \cdots + \lambda_d\right) \in \sigma_p(S^*)$.
3. Let $\lambda = (\lambda_1, \cdots, \lambda_d)$ and $\mu = (\mu_1, \cdots, \mu_d)$ be to two joint eigenvalues of $\mathbf{T}$ corresponding to the eigenvectors u and v $\left(\text{i.e } \left(T_j - \lambda_j\right)u = (T_j - \lambda_j)v = 0, \text{ for } j = 1, \cdots, d\right)$. By using the statement (2), we obtain $-(\lambda_1 + \cdots + \lambda_d) \in \sigma_p(S^*)$ and $-(\mu_1 + \cdots + \mu_d) \in \sigma_p(S^*)$. Since $\mathbf{T}$ is skew (A, m)-symmetric, it holds

$$\begin{aligned} 0 &= \left\langle\Big(\sum_{k=0}^{m} \binom{m}{k} S^{*k} A S^{m-k}\Big)u \mid v\right\rangle = \sum_{k=0}^{m} \binom{m}{k} \left\langle AS^{m-k}u \mid S^k v\right\rangle \\ &= \sum_{k=0}^{m} \binom{m}{k} \left\langle\left(\lambda_1 + \cdots + \lambda_d\right)^{m-k} Au \mid \left(\mu_1 + \cdots + \mu_d\right)^k v\right\rangle \\ &= \left(\sum_{j=1}^{d} (\lambda_j + \mu_j)\right)^m \langle Au \mid v\rangle. \end{aligned}$$

Since $\left(\sum_{j=1}^{d} (\lambda_j + \mu_j)\right) \neq 0$, we get $\langle Au \mid v\rangle = 0$.
4. By arguing in the same way as in the assertion (3), we can prove the desired claim.
□

Applying arguments similar to those established in proving statements (1) and (2) of Theorem 4.8, we get the following result in which we characterize the joint approximate point spectrum and the joint point spectrum of a commuting d-tuple of operators.

Theorem 4.9. *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple. Assume that $0 \notin \sigma_{ap}(A)$. If $\mathbf{T}$ is skew (A, m)-symmetric, then the following statements hold true.*

1. $\sigma_{ja}(\boldsymbol{T}) \subset \left\{(\lambda_1, \cdots, \lambda_d) \in \mathbb{C}^d : \Re e \left(\sum_{k=1}^{d} \lambda_k\right) = 0\right\}$.
2. $\sigma_{jp}(\boldsymbol{R}) \subset \left\{(\lambda_1, \cdots, \lambda_d) \in \mathbb{C}^d : \Re e \left(\sum_{k=1}^{d} \lambda_k\right) = 0\right\}$.

References

[1] J. Agler, M. Stankus, *m-Isometric Transformations of Hilbert Spaces I*, Integral Equations and Operator Theory 21:4 (1995) 383–429.
[2] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space II*, Integral Equations and Operator Theory 23:1 (1995) 1–48.
[3] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space III*, Integral Equations and Operator Theory 24:4 (1996) 379–421.
[4] M. M. Chō, H. Motoyoshi, B. N. Nastovska, *On the joint spectra of commuting tuples of operators and a conjugation*, Functional Analysis, Approximation and Computation 9:2 (2017) 21–26.
[5] M. Chō, E. Ko, J. E. Lee, *Skew m-Complex Symmetric Operators*, Filomat 33:10 (2019) 2975–2983.
[6] M. Chō, E. Ko, J. Lee, *Properties of m-complex symmetric operators*, Stud. Univ. Babeş-Bolyai Math. 62:2 (2017) 233–248.
[7] M. Chō, E. Ko, J. Lee, *On m-complex symmetric operators*, Mediterranean Journal of Mathematics 13:4 (2016) 2025–2038.
[8] M. Chō, E. Ko, J. Lee, *On m-complex symmetric operators II*, Mediterranean Journal of Mathematics 13:5 (2016) 3255–3264.
[9] M. Chō, J. E. Lee, K. Tanahashi, J. Tomiyama, *On $[m, C]$-symmetric Operators*, KYUNGPOOK Math. J. 58 (2018), 637–650.
[10] M. Chō, S. A. o Mahmoud, *(A, m)-Symmetric commuting tuples of operators on a Hilbert Space*, MIA 22:3 (2019) 931–947.
[11] K. J. Engel, R. Nagel, *A Short Course on Operator Semigroups*, Springer. 2006th edition (2006).

[12] B. P. Duggal, V. Müller, *Tensor product of left n-invertible operators*, STUD. MATH. 215:2 (2013).
[13] S. R. Garcia, M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. 358 (2006) 1285–1315.
[14] S. R. Garcia, M. Putinar, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. 359 (2007) 3913–3931.
[15] S. R. Garcia, W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. 362 (2010) 6065–6077.
[16] J. W. Helton, *Jordan operators in infinite dimensions and Surm Liouville conjugate point theory*, Bul. Amer. Math. Soc **78:1** (1972) 57–61.
[17] J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany, Hungary (1970) 279–287.
[18] Y. Han, J. Lee, *Local Spectral Properties for the Helton class of an operator*, Filomat 24:3 (2010) 83–93.
[19] N. Jeridi, R. Rabaoui, *On (A, m)-Symmetric Operators in a Hilbert Space*, Results Math 74:124 (2019) https://doi.org/10.1007/s00025-019-1049-0.
[20] S. Jung, E. Ko, M. Lee, J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. 379 (2011) 325–333.
[21] S. Jung, E. Ko, J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. 382 (2011) 252–260.
[22] S. Jung, E. Ko, *On complex symmetric operator matrices*, J. Math. Anal. Appl. 406 (2013) 373–385.
[23] E. Ko, E. J. Ko, J. Lee, *Skew complex symmetric operator and Weyl type theorems*, Bull. Korean Math. Soc. 52:4 (2015) 1269–1283.
[24] Y. Kim, E. Ko, J. Lee, *Operators with the single valued extension property*, Bull. Korean Math. Soc. 43:3 (2006) 509–517.
[25] C. G. Li, S. Zhu, *Skew symmetric normal operators*, Proc. Amer. Math. Soc. 141:8 (2013) 2755–2762.
[26] S. A. McCullough, L. Rodman, *Hereditary classes of operators and matrices*, Amer. Math. Monthly 104:5 (1997) 415–430.
[27] O. A. M. Sid Ahmed, A. Saddi, *A-m-Isometric operators in semi-Hilbertian spaces*, Linear Algebra and its Applications 436:10(2012) 3930–3942.
[28] R. Rabaoui, A. Saddi, *On the Orbit of an A-m-isometry*, Annales Mathematicae Silesianae 26 (2012) 75–91.
[29] R. Rabaoui, A. Saddi, *(A, m)-Isometric Unilateral Weighted Shifts in Semi-Hilbertian Spaces*, Bull. Malays. Math. Sci. Soc. DOI 10.1007/s40840-016-0307-5
[30] A. Tokmakoff, *Exponential operators, Time-Dependent Quantum Mechanics and Spectroscopy*, Reference material, Univ. of Chicago, https://tdqms.uchicago.edu/