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# Additive Properties of g-Drazin Invertible Linear Operators

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**Abstract.** In this paper, we investigate additive properties of generalized Drazin inverse for bounded linear operators. As an application we present new conditions under which a $2 \times 2$ operator matrix has g-Drazin inverse. These extend the main results of Dana and Yousefi (Int. J. Appl. Comput. Math., **4**(2018), page 9), Yang and Liu (J. Comput. Appl. Math., **235**(2011), 1412–1417) and Sun et al. (Filomat, **30**(2016), 3377–3388).

## 1. Introduction

Let $X$ be an arbitrary complex Banach space and $\mathcal{A}$ denote the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on $X$. An element $a$ in $\mathcal{A}$ has g-Drazin inverse, i.e., generalized Drazin inverse, provided that there exists $b \in R$ such that

$$b = bab, ab = ba, a - a^2 b \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}$ is invertible for every $\lambda \in \mathbb{C}\}$. As is well known, $a \in \mathcal{A}^{qnil} \Leftrightarrow \lim\limits_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0$. Such $b$, if exists, is unique, and is called the g-Drazin inverse of $a$, and denote it by $a^d$. We always use $\mathcal{A}^d$ to stand for the set of all g-Drazin invertible $a \in \mathcal{A}$. The g-Drazin inverse of operator matrix has various applications in singular differential equations, Markov chains and iterative methods (see [1–4, 6–8, 10, 12, 13]). The motivation of this paper is to explore wider conditions under which the sum of two generalized Drazin invertible operators on Banach spaces has generalized Drazin inverse. As an application we establish new conditions for the g-Drazin inverse of a $2 \times 2$ partitioned operator matrix.

In Section 2, we present wider conditions on generalized Drazin invertible operators $a$ and $b$ under which the sum $a + b$ has generalized Drazin inverse. These extend the main results of Dana and Yousefi [5, Theorem 4], Yang and Liu [17, Theorem 2.1] and Sun et al. [14, Theorem 3.1]. They are also the main tool in our following development.

In Section 3, we investigate the generalized Drazin inverse of a $2 \times 2$ operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

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where $A \in \mathcal{L}(X), B \in \mathcal{L}(X,Y), C \in \mathcal{L}(Y,X)$ and $D \in \mathcal{L}(Y)$. Here, $M$ is a bounded linear operator on $X \oplus Y$. This problem is quite complicated and was expensively studied by many authors. Our results contain many known results, e.g., [6] and [12].

If $a \in \mathcal{A}$ has g-Drazin inverse $a^d$, the element $a^\pi := 1 - aa^d \in \mathcal{A}$ is called the spectral idempotent of $a$. Finally, in Section 4, we present new conditions with the perturbation under which $M$ has generalized Drazin inverse. These also extend [5, Theorem 8] to the g-Drazin inverse of an operator matrix.

## 2. Additive results

The purpose of this section is to establish new conditions under which the sum of two g-Drazin invertible operators has g-Drazin inverse. We begin with

**Lemma 2.1.** *Let $a, b \in \mathcal{A}$ and $ab = 0$. If $a, b \in \mathcal{A}^d$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = (1 - bb^d)\Big(\sum_{n=0}^{\infty} b^n (a^d)^n\Big)a^d + b^d\Big(\sum_{n=0}^{\infty} (b^d)^n a^n\Big)(1 - aa^d).$$

*Proof.* See [8, Theorem 2.3]. □

**Lemma 2.2.** *Let $a, b \in \mathcal{A}$ have g-Drazin inverses. If $aba = 0$ and $ab^2 = 0$ and $b^3 = 0$, then $a + b \in \mathcal{A}^d$. In this case,*

$$(a+b)^d = a^d + b(a^d)^2 + b^2(a^d)^3 + (a^d)^2 b + b(a^d)^3 b + b^2(a^d)^4 b.$$

*Proof.* Let $p = a^2 + ab$ and $q = ba + b^2$. Since $(ab)^2 = 0$, we see that $ab \in \mathcal{A}^d$. By Cline's formula, $ba \in \mathcal{A}^d$. Clearly, $(ab)a^2 = (ab)b^2 = 0$, it follows by Lemma 2.1 that $p, q \in \mathcal{A}^d$. Furthermore, we check that

$$pq = (a^2 + ab)(ba + b^2) = a^2ba + a^2b^2 + ab^2(a + b) = 0,$$

and then $(a + b)^2 = p + q \in \mathcal{A}^d$ by Lemma 2.1. According to [11, Corollary 2.2], $a + b \in \mathcal{A}^d$. The detailed formula of the g-Drazin inverse $(a + b)^d$ can be derived by the straightforward computation according to the preceding discussion. □

In [5], Dana and Yousefi considered the Drazin inverse of $P + Q$ under the conditions that $PQP = 0, QPQ = 0, P^2Q^2 = 0$ and $PQ^3 = 0$ for complex matrices $P$ and $Q$. We note that every complex matrix has Drazin inverse which coincides with its g-Drazin inverse. We now extend this result to g-Drazin inverse of bounded linear operators as follows.

**Theorem 2.3.** *Let $a, b \in \mathcal{A}^d$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix}, M^d = F^d + G(F^d)^2 + G^2(F^d)^3 + (F^d)^2G + G(F^d)^3G + G^2(F^d)^4G,$$

*where*

$$F^d = (I - KK^d)\Big[\sum_{n=0}^{\infty} K^n (H^d)^n\Big]H^d + K^d\Big[\sum_{n=0}^{\infty} (K^d)^n H^n\Big](I - HH^d);$$
$$H^d = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}, G^3 = 0.$$

*Proof.* Set

$$M = \begin{pmatrix} a^3 + a^2b + ab^2 & a^3b \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^3 = 0, FG^2 =$ and $FGF = 0$. Moreover, we have

$$\begin{aligned} F &= \begin{pmatrix} a^3 & 0 \\ a^2+ab+ba+b^2 & b^3 \end{pmatrix} \\ &= \begin{pmatrix} a^3 & 0 \\ a^2+ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2+ab & b^3 \end{pmatrix} \\ &:= H+K. \end{aligned}$$

One easily checks that

$$H = \begin{pmatrix} a^3 & 0 \\ a^2+ba & 0 \end{pmatrix} = \begin{pmatrix} a^2 \\ a+b \end{pmatrix}(a,0).$$

Since $(a,0)\begin{pmatrix} a^2 \\ a+b \end{pmatrix} = a^3 \in \mathcal{A}^d$, it follows by Cline's formula (see [9, Theorem 2.1]), we see that

$$\begin{aligned} H^d &= \begin{pmatrix} a^2 \\ a+b \end{pmatrix}((a^3)^d)^2(a,0) = \begin{pmatrix} a^2 \\ a+b \end{pmatrix}(a^d)^6(a,0) \\ &= \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 + b(a^d)^5 & 0 \end{pmatrix}. \end{aligned}$$

Likewise, We have

$$K^d = \begin{pmatrix} 0 \\ b \end{pmatrix}(b^d)^4(1,b) = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Clearly, $HK = 0$. In light of Lemma 2.1,

$$F^d = (I - KK^d)\Big[\sum_{n=0}^{\infty} K^n (H^d)^n\Big]H^d + K^d\Big[\sum_{n=0}^{\infty}(K^d)^n H^n\Big](I - HH^d).$$

As $G^d = 0$, by Lemma 2.2, we have

$$M^d = F^d + G(F^d)^2 + G^2(F^d)^3 + (F^d)^2G + G(F^d)^3G + G^2(F^d)^4G.$$

Clearly, $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)\right)^3$. By using Cline's formula,

$$(a+b)^d = \left((1,b)\begin{pmatrix} a \\ 1 \end{pmatrix}\right)^d = (a+b, ab+b^2)M^d\begin{pmatrix} a \\ 1 \end{pmatrix}.$$

as asserted. □

**Corollary 2.4.** *Let $a, b, ab \in \mathcal{A}^d$ have g-Drazin inverses. If $a^2b = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^d$.*

*Proof.* Since $ab \in \mathcal{A}^d$, we see that $ba \in \mathcal{A}^d$ by Cline's formula. As $a^2(ab) = 0$, it follows by Lemma 2.1 that $p := a^2 + ab \in \mathcal{A}^d$. Likewise, $q := ba + b^2 \in \mathcal{A}^d$. One easily checks that

$$pqp = 0, qpq = 0, p^2q^2 = 0 \text{ and } pq^3 = 0.$$

In light of Theorem 2.2, $(a + b)^2 = p + q \in \mathcal{A}^d$. According to [11, Corollary 2.2], $a + b \in \mathcal{A}^d$, as asserted. □

Let $a, b \in \mathcal{A}^d$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $a^3b = 0$, then $a + b \in \mathcal{A}^d$. This is a symmetrical result of Theorem 2.1, and can be proved by a similar route.

In [17], Sun et al. considered the Drazin inverse of $P + Q$ in the case of $PQ^2 = 0, P^2QP = 0, (QP)^2 = 0$ for two square matrices over a skew field. As is well known, every square matrix over skew fields has Drazin inverse. We are now ready to extend [17, Theorem 3.1] to g-Drazin inverses of bounded linear operators and prove:

**Theorem 2.5.** *Let $a, b \in \mathcal{A}^d$. If $ab^2 = 0, a^2ba = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix}, M^d = F^d + G(F^d)^2 + G^2(F^d)^3 + (F^d)^2G + G(F^d)^3G + G^2(F^d)^4G,$$

*where*

$$F^d = (I - KK^d)\Big[\sum_{n=0}^{\infty} K^n (H^d)^n\Big]H^d + K^d\Big[\sum_{n=0}^{\infty} (K^d)^n H^n\Big](I - HH^d);$$
$$H^d = \begin{pmatrix} (a^d)^2 & 0 \\ (a^d)^3 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}, G^4 = 0.$$

*Proof.* Set

$$M = \begin{pmatrix} a^3 + a^2b + aba & a^3b + abab \\ a^2 + ab + ba + b^2 & a^2b + bab + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + aba & a^3b + abab \\ 0 & a^2b + bab \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^3 = 0, FGF = 0$ and $FG^2 = 0$. Moreover, we have

$$\begin{aligned} F &= \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2 + ab & b^3 \end{pmatrix} \\ &:= H + K. \end{aligned}$$

As in the proof of Theorem 2.2, One easily checks that

$$H^d = \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 + b(a^d)^5 & 0 \end{pmatrix}, K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^3 & (b^d)^2 \end{pmatrix}.$$

Further,

$$F^d = (I - KK^d)\Big[\sum_{n=0}^{\infty} K^n (H^d)^n\Big]H^d + K^d\Big[\sum_{n=0}^{\infty} (K^d)^n H^n\Big](I - HH^d)$$

In light of Lemma 2.2,

$$M^d = F^d + G(F^d)^2 + G^2(F^d)^3 + (F^d)^2G + G(F^d)^3G + G^2(F^d)^4G.$$

Obviously, $M = \Big(\begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)\Big)^3$. By virtue of Cline's formula,

$$(a+b)^d = \Big((1, b)\begin{pmatrix} a \\ 1 \end{pmatrix}\Big)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix},$$

as desired. □

Let $a, b \in \mathcal{A}^d$. If $a^2b = 0, aba^2 = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^d$. This can be proved in a symmetric way as in Theorem 2.5.

## 3. Block operator matrices

To illustrate the preceding results, we are concerned with the generalized Drazin inverse for a block operator matrix. Throughout this section, the operator matrix $M$ is given by (1.1), i.e.,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{L}(X)^d, B \in \mathcal{L}(X,Y), C \in \mathcal{L}(Y,X)$ and $D \in \mathcal{L}(Y)^d$. Using different splitting approach, we shall obtain various conditions for the g-Drazin inverse of $M$. In fact, the explicit g-Drazin inverse of $M$ could be computed by the formula in Theorem 2.5.

**Theorem 3.1.** *If $ABC = 0, DCA = 0, DCB = 0, CBCA = 0$ and $CBCB = 0$, then $M$ has g-Drazin inverse.*

*Proof.* Write $M = p + q$, where

$$p = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

It is obvious by [8, Lemma 2.2] that $p$ and $q$ have g-Drazin inverses. Clearly, $q^2 = 0$, and so $pq^2 = 0$. As $ABC = 0, DCA = 0$ and $DCB = 0$, then $p^2qp = 0$. It follows from $CBCA = 0$ and $CBCB = 0$ that $(qp)^2 = 0$. Then by applying Theorem 2.5, $p + q = M$ has g-Drazin inverse. □

**Corollary 3.2.** *( [6, Theorem 3]) If $BC = 0$ and $DC = 0$, then $M$ has g-Drazin inverse.*

*Proof.* It is obvious by Theorem 3.1. □

**Theorem 3.3.** *If $ABC = 0, ABD = 0, DCB = 0, BCBC = 0$ and $BCBD = 0$, then $M$ has g-Drazin inverse.*

*Proof.* Write $M = p + q$, where

$$p = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

By using [8, Lemma 2.2] it is clear that $p, q$ have g-Drazin inverses. Obviously, $pq^2 = 0$. Also by the assumptions $ABC = 0, ABD = 0, DCB = 0$ we have $p^2qp = 0$. By using $BCBC = 0$ and $BCBD = 0$, we have $(qp)^2 = 0$. Then we get the result by Theorem 2.5. □

**Corollary 3.4.** *If $ABC = 0, ABD = 0, BCB = 0$ and $DCB = 0$, then $M$ has g-Drazin inverse.*

*Proof.* It is special case of Theorem 3.3. □

If $AB = 0$ and $CB = 0$, we claim that $M$ has g-Drazin inverse (see [6, Theorem 2]). This is a direct consequence of Corollary 3.4.

**Example 3.5.** *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \text{ and } D = 0$$

*are complex matrices. Then $ABC = 0, ABD = 0, BCB = 0$ and $DCB = 0$. In this case, $AB, CB \neq 0$.*

**Lemma 3.6.** *If $CBCB = 0$, then $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ has g-Drazin inverse.*

*Proof.* Write

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. In view of [8, Lemma 2.2], $p$ has g-Drazin inverse. By virtue of Lemma 3.6, $q$ has g-Drazin inverse. It is obvious that $pq^2 = 0$, $p^2qp = 0$ and $(qp)^2 = 0$. Then by Theorem 2.5, $M$ has g-Drazin inverse. □

**Lemma 3.7.** *If $ABC = 0$ and $CBCB = 0$, then $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ has g-Drazin inverse.*

*Proof.* Write

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. It is obvious that $pq^2 = 0$, $p^2qp = 0$ and $(qp)^2 = 0$. Then by Theorem 2.5, it has g-Drazin inverse. □

**Theorem 3.8.** *If $ABC = 0, DCA = 0, DCB = 0$ and $CBCB = 0$, then $M$ has g-Drazin inverse.*

*Proof.* Write

$$M = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Let $p = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ and $q = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. Then $p$ has g-Drazin inverse as $p^2 = 0$. In light of Lemma 3.7, $q$ has g-Drazin inverse. Also $pq^2 = 0$, $p^2qp = 0$ and $(qp)^2 = 0$. Then by Theorem 2.5, $M$ has g-Drazin inverse. □

**Corollary 3.9.** *If $ABC = 0, CBC = 0, DCA = 0$ and $DCB = 0$, then $M$ has g-Drazin inverse.*

*Proof.* it is clear by Theorem 3.8 □

**Lemma 3.10.** *If $DCB = 0$ and $CBCB = 0$, then $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ has g-Drazin inverse.*

*Proof.* Write

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = p + q$$

where $p = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ and $q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. In view of [8, Lemma 2.2], $p$ has g-Drazin inverse. According to Lemma 3.6, $q$ has g-Drazin inverse. Also $pq^2 = 0$, $p^2qp = 0$ and $(qp)^2 = 0$. Then by Theorem 2.5, it has g-Drazin inverse. □

**Theorem 3.11.** *If $ABC = 0, ABD = 0, DCB = 0$ and $CBCB = 0$, then $M$ has g-Drazin inverse.*

*Proof.* Write

$$M = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

Clearly, $p$ has g-Drazin inverse. By Lemma 3.10, $q$ has g-Drazin inverse. From $ABC = 0$ and $ABD = 0$ we have $pq^2 = 0$, $p^2qp = 0$ and $(qp)^2 = 0$. Therefore we complete the proof by Theorem 2.5. □

As an immediate consequence, we derive

**Corollary 3.12.** *If $ABC = 0, ABD = 0, BCB = 0$ and $DCB = 0$, then $M$ has g-Drazin inverse.*

## 4. Multiplicative perturbation

Let $M$ be an operator matrix $M$ given by (1.1). It is of interest to consider the g-Drazin inverse of $M$ under generalized Schur condition $D = CA^dB$ (see [14]). We now investigate various perturbation conditions with spectral idempotents under which $M$ has g-Drazin inverse. We now extend [5, Theorem 8] to the g-Drazin inverse of block operator matrices.

**Theorem 4.1.** *Let $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$ and $M$ be given by* (1.1). *If $CA^{\pi}AB = 0, A^{\pi}A^2BC = 0, A^{\pi}BCA^2 = 0, A^{\pi}BCB = 0, ABCA^d = BCAA^d$ and $D = CA^dB$, then $M \in \mathcal{L}(X \oplus Y)^d$.*

*Proof.* Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} AA^{\pi} & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} A^2A^d & B \\ C & CA^dB \end{pmatrix}.$$

By assumption, we verify that $PQP = 0, QPQ = 0, P^2Q^2 = 0$ and $PQ^3 = 0$. Since $AA^{\pi} \in \mathcal{L}(X)^{qnil}$, we easily see that $P$ is quasinilpotent, and then it has g-Drazin inverse. Furthermore, we have

$$Q = Q_1 + Q_2,\ Q_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ Q_2 = \begin{pmatrix} 0 & A^{\pi}B \\ CA^{\pi} & 0 \end{pmatrix}$$

and $Q_2Q_1 = 0$. Since $A^{\pi}BCA^2 = 0, A^{\pi}BCB = 0$, we see that $(A^{\pi}BCA^{\pi})^2 = A^{\pi}BCBCA^{\pi} - A^{\pi}BCA^2(A^d)^2 = 0$ and $(CA^{\pi}B)^2 = CA^{\pi}BC(I - AA^d)B = CA^{\pi}BCB - CA^{\pi}BCA^2(A^d)^2B = 0$. Therefore $Q_2^4 = 0$. Moreover, we have

$$Q_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d.$$

Since $A^{\pi}BCA^2 = 0$, we have $(I - AA^d)BCA^2 = 0$, and so $BCA^2 = AA^dBCA^2$. This implies that $BCA^d = AA^dBCA^d$, and then

$$A^2A^d + AA^dBCA^d = A^2A^d + BCA^d.$$

Since $D = CA^dB$ has g-Drazin inverse, by Cline's formula, $BCA^d$ has g-Drazin inverse. In view of [8, Theorem 2.1], $A^2A^d = A(AA^d)$ has g-Drazin inverse.

Since $ABCA^d = BCAA^d$, we check that

$$\begin{aligned} (A^2A^d)(BCA^d) &= A(AA^dBCA^d) \\ &= ABCA^d \\ &= BCAA^d \\ &= (BCA^d)(A^2A^d). \end{aligned}$$

By virtue of [8, Theorem 2.1], $A^2A^d + BCA^d$ has g-Drazin inverse. By using Cline's formula again, $Q_1$ has g-Drazin inverse. Therefore $Q$ has g-Drazin inverse. According to Theorem 2.2, $M$ has g-Drazin inverse, as asserted. □

**Corollary 4.2.** *Let $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$ and $M$ be given by* (1.1). *If $CA^{\pi}AB = 0, A^{\pi}A^2BC = 0, A^{\pi}BCA^2 = 0, A^{\pi}BCB = 0, A^2BCA = ABCA^2$ and $D = CA^dB$, then $M \in \mathcal{L}(X \oplus Y)^d$.*

*Proof.* As in the proof of Theorem 4.1, $BCA^d = AA^dBCA^d$. Since $A^2BCA = ABCA^2$, we have

$$\begin{array}{rcl} ABCA^d & = & A(AA^dBCA^d) \\ & = & A^d(A^2BCA)(A^d)^2 \\ & = & A^d(ABCA^d)(A^2A^d) \\ & = & BCA^d(A^2A^d) \\ & = & BCAA^d. \end{array}$$

Therefore we complete the proof by Theorem 4.1. □

Regarding a complex matrix as the operator matrix on $\mathbb{C}\times\cdots\times\mathbb{C}$, we now present a numerical example to demonstrate Theorem 4.1.

**Example 4.3.** *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

*be complex matrices and set*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

*Then*

$$A^d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*We easily check that*

$$CA^\pi AB = 0, A^\pi A^2BC = 0, A^\pi BCA^2 = 0,$$
$$A^\pi BCB = 0, ABCA^d = BCAA^d, D = CA^dB.$$

*In this case, $A, D$ and $M$ have Drazin inverses, and so they have g-Drazin inverses.*

By the other splitting approach, we derive

**Theorem 4.4.** *Let $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$ and $M$ be given by* (1.1). *If $A^\pi A^2BC = 0, A^\pi BCBC = 0, A^\pi CABC = 0, ABCA^d = BCAA^d$ and $D = CA^dB$, then $M \in \mathcal{L}(X \oplus Y)^d$.*

*Proof.* We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

Then we check that $P^2QP = 0, (QP)^2 = 0, Q^2 = 0$. Clearly, $Q$ has g-Drazin inverse. Moreover, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix},$$

$P_2P_1 = 0$ and $P_2$ is quasinilpotent. Since $A^d = A(A^d)^2$, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d.$$

As in the proof of Theorem 4.1, we easily check that $A^2A^d + AA^dBCA^d$ has g-Drazin inverse. Therefore $P_1$ has g-Drazin inverse. By Lemma 2.1, $P$ has g-Drazin inverse. According to Theorem 2.5, $M$ has g-Drazin inverse. □

**Corollary 4.5.** *Let $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$ and $M$ be given by* (1.1). *If $A^{\pi}A^2BC = 0, A^{\pi}BCBC = 0, A^{\pi}CABC = 0, A^2BCA = ABCA^2$ and $D = CA^dB$, then $M \in \mathcal{L}(X \oplus Y)^d$.*

*Proof.* As in the proof of Corollary 4.2, we prove that $ABCA^d = BCAA^d$. This completes the proof by Theorem 4.4. □

**Corollary 4.6.** *Let $A \in \mathcal{L}(X)^d, D \in \mathcal{L}(Y)^d$ and $M$ be given by* (1.1). *If $A^{\pi}BC = 0, A^2BCA = ABCA^2$ and $D = CA^dB$, then $M \in \mathcal{L}(X \oplus Y)^d$.*

*Proof.* This is obvious by Corollary 4.5. □

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