Filomat 36:11 (2022), 3709–3714
https://doi.org/10.2298/FIL2211709L

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Positive Semidefinite Solution to Matrix Completion Problem and Matrix Approximation Problem

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Abstract. In this paper, firstly, we discuss the following matrix completion problem in the spectral norm:

$$\left\| \begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \right\|_2 < 1 \quad \text{subject to} \quad \begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geqslant 0.$$

The feasible condition for the above problem is established, in this case, the general positive semidefinite solution and its minimum rank are presented. Secondly, applying the result of the above problem, we also study the matrix approximation problem:

$$\|A - BXB^*\|_2 < 1 \quad \text{subject to} \quad A - BXB^* \geqslant 0,$$

where $A \in \mathbb{C}_{\geqslant}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $X \in \mathbb{C}_{\geqslant}^{n\times n}$.

1. Introduction

Let $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denote the set of all $m \times n$ matrices over the complex (real) field $\mathbb{C}$ ($\mathbb{R}$), $\mathbb{C}_H^{m\times m}$ denote the set of all $m\times m$ Hermitian matrices, $\mathbb{C}_{\geqslant}^{m\times m}$ denote the set of all $m\times m$ Hermitian positive semidefinite matrices, and I_n denote the identity matrix of order n. For $A \in \mathbb{C}^{m\times n}$, its rank, conjugate transpose and Moore-Penrose inverse are denoted by $r(A)$, A^* and $A^\dagger$ respectively, and $E_A = I_m - AA^\dagger$, $F_A = I_n - A^\dagger A$. The symbols $\|A\|_2$ and $\|A\|_F$ denote the spectral norm and Frobenius norm of $A \in \mathbb{C}^{m\times n}$ respectively. For Hermitian matrix A, its positive and negative indexes of inertia are symbolled by $i_+(A)$ and $i_-(A)$ respectively. If matrix A is positive semidefinite (positive definite), we denote it by $A \geqslant 0$ ($A > 0$) for short.

Let $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ be a closed set. Consider the rank minimization problem:

$$\text{Minimize } \{r(X) : X \in \mathcal{S}\},$$

which has found many applications in system control, matrix completion, machine learning, image reconstruction, quadratic optimization, to name but a few, see [1-6] and the references therein. In many applications, $\mathcal{S}$ is defined by a linear map $\mathscr{A}\colon \mathbb{R}^{m\times n} \to \mathbb{R}^p$. Two typical situations are:

$$\mathcal{S} = \{X \in \mathbb{R}^{m\times n} : \mathscr{A}(X) = b\} \text{ and } \mathcal{S} = \{X \in \mathbb{R}^{n\times n} : \mathscr{A}(X) = b,\ X \geqslant 0\}.$$

2020 *Mathematics Subject Classification*. 15A24, 15A60, 93A10
Keywords. Matrix approximation problem; Matrix completion problem; Positive semidefinite solution; Spectral norm
Received: 10 November 2020; Accepted: 01 March 2022
Communicated by Dragan S. Djordjević
Research supported by the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN202100505), the Natural Science Foundation Project of Chongqing (Grant No. cstc2021jcyj-msxmX0195), and the Program of Chongqing Innovation Research Group Project in University (Grant No. CXQT19018).
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In this paper, we focus our interest on the complex matrices.

In the literatures, rank-constrained matrix approximation problems have been widely studied in the spectral norm and Frobenius norm. In the spectral norm, some of them are as follows:

(1) $\min_X r(X)$ subject to $\|A - BXC\|_2 = \min$, see [7];

(2) $\min_{X=X^*(or\ X=-X^*)} r(X)$ subject to $\|A - BXB^*\|_2 = \min$, see [8];

(3) $\min_{X\geqslant 0} r(X)$ subject to $\|A - BXB^*\|_2 = \min$, see [10];

(4) $\min_X r(X)$ subject to $\|A - BXC\|_2 < 1$, where B has full column rank and C has full row rank, see [6];

(5) $\min_X r(X)$ subject to $\|A - BXC\|_2 < \xi$, see [7];

(6) $\min_{X=X^*(or\ X\geqslant 0)} r(X)$ subject to $\|A - BXB^*\|_2 < 1$, see [9];

(7) $\min_X r(X)$ subject to $\left\|\begin{pmatrix} A & B \\ C & X \end{pmatrix}\right\|_2 < 1$, $\min_{X=X^*} r(X)$ subject to $\left\|\begin{pmatrix} A & B \\ B^* & X \end{pmatrix}\right\|_2 < 1$, see [9].

In Frobenius norm, there are also many nice results on matrix approximation problems [11-16], we list some of them:

(1) $\min_{r(X)\leqslant k} \|A - BXC\|_F^2$, see [11];

(2) $\min_{r(X)=p, X\geqslant 0} \|A - BXB^*\|_F^2$, see [12];

(3) For an appropriate chosen nonnegative integer k, characterize the set

$$\mathcal{S} = \left\{X \left| \ \left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} X & J \\ K & L \end{pmatrix}\right\|_F = \min \ \text{ subject to } \ r\begin{pmatrix} X & J \\ K & L \end{pmatrix} = k \right.\right\}, \ \text{ see } \ [13];$$

(4) $\min_{r(X)=k} \|A - BXC\|_F^2$, see [14]; $\min_{r(X)=k} \|A - BXB^*\|_F^2$, see [16];

(5) For an appropriate chosen nonnegative integer b, characterize the set

$$\mathcal{S} = \{X \mid \ \|C - AX\|_F = \min \ \text{ subject to } \ r(C - AX) = b\}, \ \text{ see } \ [15].$$

For more information on matrix approximation problems please see the above mentioned papers and the references therein.

As introduced in [1], the matrix approximation problem with respect to positive semidefinite matrix is also significant. So, in this paper, we focus our research interest on the following matrix completion and matrix approximation problems:

Problem I. Given $A \in \mathbb{C}_{\geqslant}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ such that $AA^{\dagger}B = B$, determine the matrix $X \in \mathbb{C}_{\geqslant}^{n\times n}$ such that

$$\left\|\begin{pmatrix} A & B \\ B^* & X \end{pmatrix}\right\|_2 < 1 \quad \text{subject to} \quad \begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geqslant 0.$$

Also, we determine the minimum rank of matrix X.

We emphasize that the condition $AA^{\dagger}B = B$ in Problem I is necessary by Lemma 1.1.

Problem II. Given $A \in \mathbb{C}_{\geqslant}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$, determine the matrix $X \in \mathbb{C}_{\geqslant}^{n\times n}$ (or $X \in \mathbb{C}_{H}^{n\times n}$) such that

$$\|A - BXB^*\|_2 < 1 \quad \text{subject to} \quad A - BXB^* \geqslant 0.$$

For the matrix completion problem

$$\left\|\begin{pmatrix} A & B \\ B^* & X \end{pmatrix}\right\|_2 = \min \quad \text{subject to} \quad \begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geqslant 0.$$

It is easy to know that the minimum spectral norm is $\left\|\begin{pmatrix} A & B \\ B^* & B^*A^{\dagger}B \end{pmatrix}\right\|_2$, and an optimal minimum rank solution is $X = B^*A^{\dagger}B$.

Before proceeding to the next section, we list some useful results which will facilitate the proof of our theorems.

Lemma 1.1. ([17]) *Let* $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$*, where* A *and* D *are Hermitian matrices. Then* $M \geqslant 0$ *if and only if* $A \geqslant 0$*,* $AA^\dagger B = B$ *and* $D - B^*A^\dagger B \geqslant 0$*.*

Remark 1.2. *Under the conditions and notations of Lemma 1.1,* $M \geqslant 0$ *if and only if* M *has the form* $M = \begin{pmatrix} A & AY \\ Y^*A & Y^*AY + M \end{pmatrix}$ *or* $M = \begin{pmatrix} Z^*DZ + N & Z^*D \\ DZ & D \end{pmatrix}$*, where* Y *and* Z *are arbitrary matrices with proper sizes,* M *and* N *are arbitrary positive semidefinite matrices with proper sizes.*

Lemma 1.3. *Let* $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$*, where* A *and* D *are Hermitian matrices. Then* $M > 0$ *if and only if* $A > 0$ *and* $D - B^*A^{-1}B > 0$*.*

Lemma 1.4. ([17]) *Let* $A \in \mathbb{C}_H^{m\times m}$*,* $B \in \mathbb{C}^{m\times n}$ *and* $D \in \mathbb{C}_H^{n\times n}$*. Then*

$$i_\pm(D - B^*A^\dagger B) = i_\pm \begin{pmatrix} A^3 & AB \\ (AB)^* & D \end{pmatrix} - i_\pm(A).$$

Note that, when the matrix A in Lemma 1.4 is nonsingular, the sub-block A^3 can be replaced by A.

2. Main results

In this section, firstly, we establish the feasible condition for Problem I, and give a general positive semidefinite solution to this problem. Secondly, we apply the results of Problem I to study Problem II.

Theorem 2.1. *Let* $A \in \mathbb{C}_\geqslant^{m\times m}$ *and* $B \in \mathbb{C}^{m\times n}$ *be given, such that* $AA^\dagger B = B$*, and* $X \in \mathbb{C}_\geqslant^{n\times n}$ *be unknown.*
(i) *Problem I is feasible if and only if*

$$\|A\|_2 < 1. \tag{1}$$

In this case, the general positive semidefinite solution to this problem can be expressed by

$$X = B^*A^\dagger B + Y, \tag{2}$$

where Y *satisfies the following inequality*

$$0 \leqslant Y < I - B^*A^\dagger B - B^*(I - A)^{-1}B = I - B^*(I - A)^{-1}A^\dagger B. \tag{3}$$

(ii) *The minimum rank of the solution to Problem I is* $r(B)$*, and an optimal minimum rank solution to this problem can be expressed by* $X^\star = B^*A^\dagger B$*.*

Proof. (i) It follows form Lemma 1.1 and $\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geqslant 0$ that X can be written as the form given by (2) with $Y \geqslant 0$ unknown. And $\left\| \begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \right\|_2 < 1$ is equivalent to $\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} < I$. Then

$$\begin{pmatrix} I - A & -B \\ -B^* & I - B^*A^\dagger B - Y \end{pmatrix} > 0. \tag{4}$$

According to Lemma 1.3, (4) is equivalent to

$$I - A > 0, \qquad I - B^*A^\dagger B - B^*(I - A)^{-1}B - Y > 0. \tag{5}$$

The first inequality in (5) is equivalent to $\|A\|_2 < 1$. The second inequality in (5) is solvable for Y if and only if

$$I - B^*A^\dagger B - B^*(I-A)^{-1}B > 0 \quad \text{or} \quad i_+[I - B^*A^\dagger B - B^*(I-A)^{-1}B] = n. \tag{6}$$

Applying Lemma 1.4 to (6), we have

$$\begin{aligned}
& i_+[I - B^*A^\dagger B - B^*(I-A)^{-1}B] \\
= \;& i_+\begin{pmatrix} A^3 & AB \\ (AB)^* & I - B^*(I-A)^{-1}B \end{pmatrix} - i_+(A) \\
= \;& i_+\left\{\begin{pmatrix} A^3 & AB \\ (AB)^* & I \end{pmatrix} - \begin{pmatrix} 0 \\ B^* \end{pmatrix}(I-A)^{-1}\begin{pmatrix} 0 & B \end{pmatrix}\right\} - i_+(A) \\
= \;& i_+\begin{pmatrix} I-A & 0 & (I-A)B \\ 0 & A^3 & AB \\ B^*(I-A) & B^*A & I \end{pmatrix} - i_+(I-A) - i_+(A) \\
= \;& i_+\begin{pmatrix} I-A & -A & (I-A)B \\ -A & A^3 + A(I-A)^{-1}A & 0 \\ B^*(I-A) & 0 & I \end{pmatrix} - i_+(I-A) - i_+(A) \\
= \;& i_+\begin{pmatrix} I-A & -A & 0 \\ -A & A^3 + A(I-A)^{-1}A & 0 \\ 0 & 0 & I \end{pmatrix} - i_+(I-A) - i_+(A) \\
= \;& i_+\begin{pmatrix} I-A & -A \\ -A & A^3 + A(I-A)^{-1}A \end{pmatrix} + n - i_+(I-A) - i_+(A) \\
= \;& i_+(I-A) + i_+(A^3) + n - i_+(I-A) - i_+(A) \\
= \;& n,
\end{aligned}$$

which shows that the second inequality in (5) is solvable. Moreover, it follows from $AA^\dagger B = B$ that there exists a matrix $Q \in \mathbb{C}^{m\times n}$ such that $B = AQ$, hence,

$$\begin{aligned}
I - B^*A^\dagger B - B^*(I-A)^{-1}B &= I - Q^*AQ - Q^*A(I-A)^{-1}AQ \\
&= I - Q^*AQ - Q^*[I - (I-A)](I-A)^{-1}AQ \\
&= I - Q^*(I-A)^{-1}AQ = I - Q^*(I-A)^{-1}AA^\dagger AQ \\
&= I - Q^*A(I-A)^{-1}A^\dagger AQ = I - B^*(I-A)^{-1}A^\dagger B.
\end{aligned}$$

So, (3) is evident.

(*ii*) Since $X = B^*A^\dagger B + Y \geqslant B^*A^\dagger B \geqslant 0$, hence

$$r(X) \geqslant r(B^*A^\dagger B) = r[(A^\dagger)^{\frac{1}{2}}B] = r(A^\dagger B) = r(B).$$

The lower bound is obtained when we take $Y = 0$. □

The following corollary can also be proved by Lemma 1.3 similarly.

Corollary 2.2. *If the condition* $\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geqslant 0$ *in Problem I is replaced by* $\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} > 0$, *then the feasible condition to this problem is also* $\|A\|_2 < 1$ *(or* $0 < A < I$*), and the general positive semidefinite solution* X *can be expressed by* $X = B^*A^{-1}B + Y$, *where* Y *satisfies the following inequality*

$$0 < Y < I - B^*A^{-1}B - B^*(I-A)^{-1}B = I - B^*(I-A)^{-1}A^{-1}B.$$

Next, we apply Theorem 2.1 to investigate Problem II. Without loss of generality, in the following contents, we always assume that matrix B has the singular value decomposition $B = U\begin{pmatrix} 0 & 0 \\ 0 & \Sigma \end{pmatrix}V^*$. Correspondingly, partition matrices A and X as

$$A = U\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}U^*, \quad X = V\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}V^*.$$

Then

$$A - BXB^* = U\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} - \Sigma X_{22}\Sigma \end{pmatrix}U^*. \tag{7}$$

Theorem 2.3. *Let $A \in \mathbb{C}_{\geqslant}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and $X \in \mathbb{C}_{\geqslant}^{n\times n}$ be unknown. Then, Problem II is feasible if and only if*

$$\|A_{11}\|_2 = \|E_B A E_B\|_2 < 1.$$

In this case, a general positive semidefinite solution to this problem can be expressed by

$$X = V\begin{pmatrix} Z^*X_{22}Z + N & Z^*X_{22} \\ X_{22}Z & X_{22} \end{pmatrix}V^*, \tag{8}$$

in which Z is arbitrary with proper size, and N is arbitrary positive semidefinite matrix, $X_{22} = \Sigma^{-1}(A_{22} - A_{12}^*A_{11}^\dagger A_{12} - Y)\Sigma^{-1}$, where Y satisfies the following inequalities

$$0 \leqslant Y < I - A_{12}^*A_{11}^\dagger A_{12} - A_{12}^*(I - A_{11})^{-1}A_{12} = I - A_{12}^*(I - A_{11})^{-1}A_{11}^\dagger A_{12}, \tag{9}$$

$$0 \leqslant Y \leqslant A_{22} - A_{12}^*A_{11}^\dagger A_{12}. \tag{10}$$

Proof. By Lemma 1.1, it follows from $A \geqslant 0$ that $A_{11} \geqslant 0$ and $A_{11}A_{11}^\dagger A_{12} = A_{12}$. In view of Theorem 2.1, Problem II is feasible if and only if $1 > \|A_{11}\|_2 = \|E_B A E_B\|_2$. Moreover

$$A_{22} - \Sigma X_{22}\Sigma = A_{12}^*A_{11}^\dagger A_{12} + Y.$$

Solving the above equation produces $X_{22} = \Sigma^{-1}(A_{22} - A_{12}^*A_{11}^\dagger A_{12} - Y)\Sigma^{-1}$, where Y satisfies the condition (9). Moreover, note that X is positive semidefinite, so is $X_{22} \geqslant 0$. Therefore, (10) is evident. Since X_{22} is positive semidefinite, by Remark 1.2, X has the form (8). □

According to Theorem 2.3, the Hermitian solution to Problem II can be stated as follows.

Corollary 2.4. *Let $A \in \mathbb{C}_{\geqslant}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and $X \in \mathbb{C}_{H}^{n\times n}$ be unknown. Then, Problem II is feasible if and only if*

$$\|A_{11}\|_2 = \|E_B A E_B\|_2 < 1.$$

In this case, a general Hermitian solution to this problem can be expressed by

$$X = V\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & \Sigma^{-1}(A_{22} - A_{12}^*A_{11}^\dagger A_{12} - Y)\Sigma^{-1} \end{pmatrix}V^*,$$

where $X_{11} = X_{11}^*$ and X_{12} are arbitrary matrices with proper sizes, and Y satisfies the following inequality

$$0 \leqslant Y < I - A_{12}^*A_{11}^\dagger A_{12} - A_{12}^*(I - A_{11})^{-1}A_{12} = I - A_{12}^*(I - A_{11})^{-1}A_{11}^\dagger A_{12}.$$

Corollary 2.5. *Let $A \in \mathbb{C}_{\geqslant}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and $X \in \mathbb{C}_{\geqslant}^{n\times n}$ be unknown. If $\|A\|_2 < 1$, then Problem II is feasible, and a general positive semidefinite solution to this problem can be expressed by* (8) *with Y satisfying*

$$0 \leqslant Y \leqslant A_{22} - A_{12}^*A_{11}^\dagger A_{12}. \tag{11}$$

Proof. It follows from $\|A\|_2 < 1$ that $\|E_B A E_B\|_2 \leqslant \|A\|_2 < 1$, so, Problem II is feasible. Moreover, $0 \leqslant A < I$, i.e., $I - A > 0$. Therefore

$$I - A = U \begin{pmatrix} I - A_{11} & -A_{12} \\ -A_{12}^* & I - A_{22} \end{pmatrix} U^* > 0,$$

which means that $I - A_{11} > 0$ and $I - A_{22} - A_{12}^*(I - A_{11})^{-1}A_{12} > 0$. Hence, the following inequality holds

$$I - A_{12}^* A_{11}^\dagger A_{12} - A_{12}^*(I - A_{11})^{-1}A_{12} > A_{22} - A_{12}^* A_{11}^\dagger A_{12}.$$

Combining (9) and (10) yields (11). □

3. Conclusion

In this paper, we have discuss a matrix completion problem (Problem I), its feasible condition, general expression for the positive semidefinite solution and the minimum rank are established. Furthermore, based on the results of Problem I, we derive the feasible condition and the expression for the positive semidefinite (Hermitian) solution to a matrix approximation problem (Problem II).

4. Acknowledgements

The author would like to thank the Editor and the referees for their very detailed comments and valuable suggestions which greatly improved the presentation of this paper.

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