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Rotations and Convolutions of Harmonic Convex Mappings

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Abstract. In this paper, we mainly consider the convolutions of slanted half-plane mappings and strip mappings of the unit disk $\mathbb{D}$. If f_1 is a slanted half-plane mapping and f_2 is a slanted half-plane mapping or a strip mapping, then we prove that $f_1 * f_2$ is convex in some direction if $f_1 * f_2$ is locally univalent in $\mathbb{D}$. We also obtain two sufficient conditions for $f_1 * f_2$ to be locally univalent in $\mathbb{D}$. Our results extend many of the recent results in this direction. Moreover, we consider a class of harmonic mappings including slanted half-plane mappings and strip mappings, and as a consequence, we prove that the any convex combination of such locally univalent and sense-preserving mappings is also convex.

1. Introduction and Preliminary results

In this article, we will consider complex-valued harmonic mappings f defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which have the canonical representation of the form $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. This decomposition is unique with the condition $g(0) = 0$. In terms of the canonical decomposition of f, the Jacobian J_f of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the Inverse Mapping Theorem, if the Jacobian of a C^1 mapping from $\mathbb{D}$ to $\mathbb{C}$ is different from zero, then the mapping is locally univalent in $\mathbb{D}$. The classical result of Lewy implies that the converse of this statement also holds for harmonic mappings. Thus, every harmonic mapping f on $\mathbb{D}$ is locally univalent and sense-preserving on $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$, i.e. $|h'(z)| > |g'(z)|$ in $\mathbb{D}$. For locally univalent mappings f, the condition $J_f(z) > 0$ is equivalent to the existence of an analytic function ω_f in $\mathbb{D}$ such that

$$|\omega_f(z)| < 1 \ \text{ for } \ z \in \mathbb{D},$$

where $\omega_f(z) = g'(z)/h'(z)$ is called the dilatation of f. When there is no danger of confusion, it is often convenient to use ω instead ω_f. Let

$$\mathcal{H} = \{f = h + \overline{g} : \ h(0) = g(0) = 0 \text{ and } h'(0) = 1\}.$$

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The class $\mathcal{H}_0$ consists of those functions $f \in \mathcal{H}$ with $g'(0) = 0$.

The family of all sense-preserving univalent harmonic mappings from $\mathcal{H}$ will be denoted by $\mathcal{S}_H$, and let $\mathcal{S}_H^0 = \mathcal{S}_H \cap \mathcal{H}_0$. Clearly, the familiar class $\mathcal{S}$ of normalized analytic univalent functions in $\mathbb{D}$ is contained in $\mathcal{S}_H^0$. The subclass $\mathcal{K}_H$ (resp. $\mathcal{K}_H^0$) consists of functions from $\mathcal{S}_H$ (resp. $\mathcal{S}_H^0$) that map the unit disk $\mathbb{D}$ onto a convex domain. The class $\mathcal{S}_H$ together with its geometric subclasses has been studied extensively by Clunie and Sheil-Small [1] and investigated subsequently by several others (see [4] and the survey article [8]).

1.1. Convolution of harmonic mappings

If $f = h + \overline{g}$ and $F = H + \overline{G}$ are two harmonic mappings of the unit disk $\mathbb{D}$ with power series of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n \quad \text{and} \quad F(z) = \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n,$$

then the harmonic convolution (or Hadamard product) of f and F, denoted by $f * F$, is defined by

$$f * F = h * H + \overline{g * G} = \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n.$$

The space $\mathcal{H}$ is closed under the operation $*$, i.e. $\mathcal{H} * \mathcal{H} \subset \mathcal{H}$, and $f * F = F * f$. Moreover,

$$(h * H)'(z) = \frac{h(z)}{z} * H'(z), \quad \text{i.e.} \quad z(h * H)'(z) = h(z) * zH'(z) = zh'(z) * H(z),$$

which will be used in the proof of our theorems.

In the case of conformal mappings, many important properties of convolution are established (see [11]). For example, convolution of two univalent convex (analytic) functions is convex. On the other hand, most of such results do not carry over to the case of univalent harmonic mappings in $\mathbb{D}$ even in the simplest cases (see [2, 3, 6]). For instance, convolution of two mappings from $\mathcal{K}_H^0$ is not necessarily locally univalent even in restricted cases. In spite of such drawbacks, some properties of the convolution of harmonic univalent mappings have been achieved and these results help to understand how difficult it is to generalize the convolution properties of harmonic mappings (see [2, 3]).

The main aim of this article is to derive convolution results and in particular, we extend and revaluate many of the recent results on slanted half-plane and strip mappings. We begin by recalling some preliminaries in the next subsections.

1.2. Slanted half-plane and strip mappings

A domain D is called convex in the direction α ($0 \leq \alpha < \pi$) if the intersection of D with each line parallel to the line through 0 and $e^{i\alpha}$ is an interval or the empty set. The class of univalent harmonic functions f for which the range $D = f(\mathbb{D})$ is a domain that is convex in the direction α plays a crucial role in deriving properties of convolution of harmonic mappings. Such functions are called convex in the direction α. See, for example, [1–3, 10]. Functions that are convex in the direction $\alpha = 0$ (resp. $\alpha = \pi/2$) is referred to as convex in the real (resp. vertical) direction.

It is known [1] that a harmonic mapping $f = h + \overline{g}$ belongs to $\mathcal{K}_H^0$ if and only if, for each $\alpha \in [0, \pi)$, the analytic function $F = h - e^{2i\alpha} g$ belongs to $\mathcal{S}$ and is convex in the direction α. This result is instrumental in deriving many properties of the class $\mathcal{K}_H^0$ by transferring information from conformal case.

Definition 1.1. *A function $f = h + \overline{g} \in \mathcal{S}_H$ is called a slanted half-plane mapping with γ ($0 \leq \gamma < 2\pi$) and $a \in \mathbb{D}$ if f maps $\mathbb{D}$ onto the half-plane $H_\gamma^a := \left\{w : \operatorname{Re}\left(\frac{e^{i\gamma}}{1+a} w\right) > -\frac{1}{2}\right\}$. The class of all such mappings is denoted by $\mathcal{S}(H_\gamma^a)$.*

In [6, 7], this definition was considered for $a \in (-1, 1)$. In view of the Riemann mapping theorem, standard arguments can be used to obtain the following.

Proposition 1.2. *Each $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$ can be written as*

$$h(z) + e^{-2i(\gamma+\gamma_a)}g(z) = \frac{(1+a')z}{1-e^{i(\gamma+\gamma_a)}z}, \tag{1}$$

where $\gamma_a = \arg(1+\overline{a})$ and $g'(0) = a'e^{2i(\gamma+\gamma_a)}$ with $a' \in \mathbb{D}$ and $a' = |1+a| - 1$.

Proof. In fact, (1) follows from writing

$$\frac{e^{i\gamma}}{1+a}f = \frac{e^{i(\gamma+\gamma_a)}}{|1+a|}(h+\overline{g}) = \frac{e^{i(\gamma+\gamma_a)}h + \overline{e^{-i(\gamma+\gamma_a)}g}}{|1+a|}$$

as $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$ is equivalent to

$$\mathrm{Re}\left(\frac{e^{i\gamma}}{1+a}f(z)\right) = \mathrm{Re}\left(\frac{e^{i(\gamma+\gamma_a)}\left(h(z) + e^{-2i(\gamma+\gamma_a)}g(z)\right)}{|1+a|}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$

Note that $h(0) = g(0) = h'(0) - 1 = 0$ and

$$g'(0) = (|1+a|-1)e^{2i(\gamma+\gamma_a)} = \left((1+\overline{a})e^{-i\gamma_a} - 1\right)e^{2i(\gamma+\gamma_a)} = \left((1+\overline{a}) - e^{i\gamma_a}\right)e^{i(2\gamma+\gamma_a)}.$$

Because $0 < |1+a| < 2$, we remark that $g'(0) \in \mathbb{D}$. The Riemann mapping theorem gives the desired representation (1). □

Obviously, each $f \in \mathcal{S}(H_\gamma^a)$ (resp. $\mathcal{S}(H_\gamma^0)$) belongs to the family $\mathcal{K}_H$ (resp. $\mathcal{K}_H^0$) of convex mappings. We remark that there are infinitely many *slanted half-plane mappings* with a fixed γ and a fixed a.

Definition 1.3. *For $0 < \beta < \pi$ and $b \in \mathbb{D}$, let $\mathcal{S}(\Omega_\beta^b)$ denote the class of functions f from $\mathcal{S}_H$ such that f maps $\mathbb{D}$ onto the strip Ω_β^b, where*

$$\Omega_\beta^b := \left\{w : \frac{\beta-\pi}{2\sin\beta} < \mathrm{Re}\left(\frac{1}{1+b}w\right) < \frac{\beta}{2\sin\beta}\right\}.$$

As with Definition 1.1 and Proposition 1.2, it is a simple exercise to obtain the following:

Proposition 1.4. *Each $f = h + \overline{g} \in \mathcal{S}(\Omega_\beta^b)$ has the form*

$$h(z) + e^{-2i\gamma_b}g(z) = \psi(z), \quad \psi(z) = \frac{(1+b')e^{-i\gamma_b}}{2i\sin\beta}\log\left(\frac{1+ze^{i(\beta+\gamma_b)}}{1+ze^{-i(\beta-\gamma_b)}}\right), \tag{2}$$

where $\gamma_b = \arg(1+\overline{b})$ and $b' = |1+b| - 1$.

Note that $h(0) = g(0) = h'(0) - 1 = 0$ and

$$g'(0) = (|1+b|-1)e^{2i\gamma_b} = \left((1+\overline{b})e^{-i\gamma_b} - 1\right)e^{2i\gamma_b} = \left((1+\overline{b}) - e^{i\gamma_b}\right)e^{i\gamma_b}.$$

It follows that each $f \in \mathcal{S}(\Omega_\beta^b)$ (resp. $\mathcal{S}(\Omega_\beta^0)$) belongs to the family $\mathcal{K}_H$ (resp. $\mathcal{K}_H^0$). If $b \in (-1,1)$, then $b' = b$ so that the last relation for $f = h + \overline{g} \in \mathcal{S}(\Omega_\beta^b)$ takes the form

$$h(z) + g(z) = \frac{1+b}{2i\sin\beta}\log\left(\frac{1+ze^{i\beta}}{1+ze^{-i\beta}}\right), \tag{3}$$

where $g'(0) = b$. The class of the strip mappings with $b = 0$ has been considered extensively in the literature [2, 3].

1.3. Some known results on harmonic convolution

In 2001, Dorff in [2] discussed the convolution of functions from $\mathcal{S}(H_0^0)$ with the functions from $\mathcal{S}(\Omega_\beta^0)$ in the following form.

Theorem 1.5. ([2, Theorem 7]) *If $f_1 \in \mathcal{S}(H_0^0)$, $f_2 \in \mathcal{S}(\Omega_\beta^0)$ and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the real direction.*

In 2012, Dorff *et. al.* in [3] proved the following result which concerns the convolution of a function from $\mathcal{S}(H_{\gamma_1}^0)$ with a function from $\mathcal{S}(H_{\gamma_2}^0)$.

Theorem 1.6. ([3, Theorem 2]) *If $f_k \in \mathcal{S}(H_{\gamma_k}^0)$, $k = 1, 2$, and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.*

Using a similar argument as in [3], we generalized Theorem 1.6 to the setting of $\mathcal{S}(H_\gamma^a)$ for $a \in (-1, 1)$. More precisely, we have the following.

Theorem 1.7. ([6, Lemma 2.1]) *Let $a_k \in (-1, 1)$ and $f_k \in \mathcal{S}(H_{\gamma_k}^{a_k})$ for $k = 1, 2$. If $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.*

2. Two Main Theorems

In this paper, we first generalize Theorem 1.5 to the following Theorem 2.1, and generalize Theorems 1.6 and 1.7 to Theorem 2.3.

Theorem 2.1. *Suppose that $f_1 = h_1 + \overline{g_1} \in \mathcal{S}(H_\gamma^a)$ and $f_2 = h_2 + \overline{g_2} \in \mathcal{S}(\Omega_\beta^b)$ for some $a, b \in \mathbb{D}$. If $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-\Gamma := -(\gamma + \gamma_a + \gamma_b)$, where $\gamma_a = \arg(1 + \overline{a})$ and $\gamma_b = \arg(1 + \overline{b})$.*

Corollary 2.2. *Let $a, b \in (-1, 1)$. Suppose that $f_1 = h_1 + \overline{g_1} \in \mathcal{S}(H_\gamma^a)$, $f_2 = h_2 + \overline{g_2} \in \mathcal{S}(\Omega_\beta^b)$ and $f_1 * f_2$ is locally univalent in $\mathbb{D}$. Then $f_1 * f_2$ is convex in the direction $-\gamma$.*

Theorem 2.3. *If $f_k = h_k + \overline{g_k} \in \mathcal{S}(H_{\gamma_k}^{a_k})$ for $k = 1, 2$, and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-\Gamma$, where*

$$\Gamma = (\gamma_1 + \gamma_2 + \gamma_{a_1} + \gamma_{a_2}), \ \ \gamma_{a_1} = \arg(1 + \overline{a_1}) \ \ \text{and} \ \ \gamma_{a_2} = \arg(1 + \overline{a_2}).$$

For the above two theorems, a natural question is to determine conditions on the dilatations of f_1 and f_2 such that $f_1 * f_2$ is locally univalent in $\mathbb{D}$. Partial answer to this question will be given in Section 5. In Section 4, we consider the class $\mathcal{F}_{\lambda,\delta}^a$ which includes both slanted half-plane and strip mappings and discuss the convex combination of functions in $\mathcal{F}_{\lambda,\delta}^a$.

3. Proof Of The Main Results

In order to prove Theorems 2.1 and 2.3, we need the following theorems.

Theorem 3.1. ([11, Lemma 2.7]) *Let $\phi(z)$ and $\Psi(z)$ be analytic in $\mathbb{D}$ with $\phi(0) = \Psi(0) = 0$. If $\phi(z)$ is convex and $\Psi(z)$ is starlike, then for each function $F(z)$ analytic in $\mathbb{D}$ and satisfying $\operatorname{Re} F(z) > 0$, we have*

$$\operatorname{Re}\left\{\frac{(\phi * F\Psi)(z)}{(\phi * \Psi)(z)}\right\} > 0, \ \ \forall z \in \mathbb{D}.$$

Here we say that an analytic function Ψ such that $\Psi(0) = 0$ is starlike in $\mathbb{D}$ if $\Psi(z)$ maps $\mathbb{D}$ univalently onto a domain which is starlike with respect to the origin, i.e., $tw \in \Psi(\mathbb{D})$ whenever $w \in \Psi(\mathbb{D})$ and $t \in [0, 1]$.

Theorem 3.2. ([1]) *A harmonic $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction α $(0 \le \alpha < \pi)$ if and only if $h - e^{2i\alpha}g$ is a univalent analytic function of $\mathbb{D}$ onto a domain convex in the direction α.*

Theorem 3.3. ([10, Theorem 1]) *Let $\varphi(z)$ be a non-constant analytic function in $\mathbb{D}$. The function $\varphi(z)$ maps $\mathbb{D}$ univalently onto a domain convex in the real direction if and only if there are numbers μ and ν, $0 \le \mu < 2\pi$ and $0 \le \nu \le \pi$, such that*

$$\mathrm{Re}\,\{e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})\varphi'(z)\} \ge 0,\ \ z \in \mathbb{D}. \tag{4}$$

Proof of Theorem 2.1. First, we recall that $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$. Next, we consider

$$F_1 = \left(h_1 + e^{-2i(\gamma+\gamma_a)}g_1\right) * \left(h_2 - e^{-2i\gamma_b}g_2\right) \ \text{ and }\ F_2 = \left(h_1 - e^{-2i(\gamma+\gamma_a)}g_1\right) * \left(h_2 + e^{-2i\gamma_b}g_2\right).$$

Then we see that

$$\frac{F_1 + F_2}{2} = h_1 * h_2 - e^{-2i\Gamma}g_1 * g_2,$$

where $\Gamma = \gamma + \gamma_a + \gamma_b$, and without loss of generality, we may assume that $0 \le \Gamma < 2\pi$. To prove that $f_1 * f_2$ is convex in the direction $-\Gamma$, by Theorem 3.2, we only need to prove that the function Φ defined by

$$\Phi = e^{i\Gamma}(F_1 + F_2) \tag{5}$$

is convex in the real direction. To do this, we first consider the following quotients

$$p_1 = \frac{h_1' - e^{-2i(\gamma+\gamma_a)}g_1'}{h_1' + e^{-2i(\gamma+\gamma_a)}g_1'} = \frac{1 - e^{-2i(\gamma+\gamma_a)}\omega_{f_1}}{1 + e^{-2i(\gamma+\gamma_a)}\omega_{f_1}} \tag{6}$$

and

$$p_2 = \frac{h_2' - e^{-2i\gamma_b}g_2'}{h_2' + e^{-2i\gamma_b}g_2'} = \frac{1 - e^{-2i\gamma_b}\omega_{f_2}}{1 + e^{-2i\gamma_b}\omega_{f_2}}, \tag{7}$$

where ω_{f_j} denotes the dilatation of f_j $(j = 1, 2)$. Clearly, p_1 and p_2 are analytic in $\mathbb{D}$ such that

$$\mathrm{Re}\,\{p_1(z)\} > 0 \ \text{ and }\ \mathrm{Re}\,\{p_2(z)\} > 0 \text{ in } \mathbb{D}.$$

We now set $a' = |1 + a| - 1$ and $b' = |1 + b| - 1$. Then, because $f_1 = h_1 + \overline{g_1} \in \mathcal{S}(H_\gamma^a)$, the equation (1) and the relation (6) give

$$h_1'(z) - e^{-2i(\gamma+\gamma_a)}g_1'(z) = p_1(z)\left(h_1'(z) + e^{-2i(\gamma+\gamma_a)}g_1'(z)\right) = \frac{(1 + a')p_1(z)}{\left(1 - e^{i(\gamma+\gamma_a)}z\right)^2}. \tag{8}$$

Similarly, since $f_2 = h_2 + \overline{g_2} \in \mathcal{S}(\Omega_\beta^b)$, by (2) and (7), we have

$$h_2'(z) - e^{-2i\gamma_b}g_2'(z) = p_2(z)\left(h_2'(z) + e^{-2i\gamma_b}g_2'(z)\right) = \frac{(1 + b')p_2(z)}{\left(1 + ze^{i(\beta+\gamma_b)}\right)\left(1 + ze^{-i(\beta-\gamma_b)}\right)}. \tag{9}$$

Now, in view of the relation

$$zF_1'(z) = \left(h_1(z) + e^{-2i(\gamma+\gamma_a)}g_1(z)\right) * z\left(h_2'(z) - e^{-2i\gamma_b}g_2'(z)\right),$$

by (1) and (9), we obtain that

$$\begin{aligned} \frac{F_1'(z)}{(1 + a')(1 + b')} &= \frac{1}{1 - e^{i(\gamma+\gamma_a)}z} * \frac{p_2(z)}{\left(1 + ze^{i(\beta+\gamma_b)}\right)\left(1 + ze^{-i(\beta-\gamma_b)}\right)} \\ &= \frac{p_2(e^{i(\gamma+\gamma_a)}z)}{\left(1 + ze^{i(\beta+\Gamma)}\right)\left(1 + ze^{-i(\beta-\Gamma)}\right)}. \end{aligned}$$

If we set $\mu = 2\pi - \Gamma$ and $\nu = \pi - \beta$, then easily get

$$(1 + ze^{i(\Gamma+\beta)})(1 + ze^{-i(\beta-\Gamma)}) = 1 + 2ze^{i\Gamma}\cos\beta + z^2e^{2i\Gamma} = e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})e^{i\Gamma}$$

which by the last equation implies that

$$\mathrm{Re}\left\{e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})e^{i\Gamma}\frac{F_1'(z)}{(1+a')(1+b')}\right\} = \mathrm{Re}\,\{p_2(e^{i(\gamma+\gamma_a)}z)\} > 0 \tag{10}$$

for $z \in \mathbb{D}$. Again, because

$$zF_2'(z) = z\left(h_1'(z) - e^{-2i(\gamma+\gamma_a)}g_1'(z)\right) * \left(h_2(z) + e^{-2i\gamma_b}g_2(z)\right),$$

we have by (8),

$$zF_2'(z) = (1+a')\left(h_2(z) + e^{-2i\gamma_b}g_2(z)\right) * \frac{zp_1(z)}{(1 - e^{i(\gamma+\gamma_a)}z)^2}.$$

For convenience, we set

$$\phi(z) = h_2(z) + e^{-2i\gamma_b}g_2(z) \ \text{ and } \ \Psi(z) = \frac{e^{i\Gamma}z}{(1 - e^{i(\gamma+\gamma_a)}z)^2}.$$

Then we observe that $\phi(z)$ is convex and $\Psi(z)$ is starlike in the unit disk $\mathbb{D}$. Also, we observe that

$$e^{i\Gamma}\frac{zF_2'(z)}{1+a'} = \phi(z) * p_1(z)\Psi(z)$$

and

$$\begin{aligned}
\phi(z) * \Psi(z) &= e^{i\Gamma}z\left(h_2 + e^{-2i\gamma_b}g_2\right)'(e^{i(\gamma+\gamma_a)}z)\\
&= (1+b')\frac{e^{i\Gamma}z}{(1 + ze^{i(\beta+\Gamma)})(1 + ze^{-i(\beta-\Gamma)})}\\
&= (1+b')\frac{e^{i\Gamma}z}{1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}}.
\end{aligned}$$

Using the last two relations and Theorem 3.1, we find that

$$\begin{aligned}
0 < \mathrm{Re}\left\{\frac{\phi(z) * p_1(z)\Psi(z)}{(\phi * \Psi)(z)}\right\} &= \mathrm{Re}\left\{e^{i\Gamma}z\frac{F_2'(z)}{(1+a')}\cdot\frac{1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}}{(1+b')e^{i\Gamma}z}\right\}\\
&= \mathrm{Re}\left\{e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})e^{i\Gamma}\frac{F_2'(z)}{(1+a')(1+b')}\right\},
\end{aligned}$$

which together with (10) implies that

$$\mathrm{Re}\left\{e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})e^{i\Gamma}\frac{F_1'(z) + F_2'(z)}{(1+a')(1+b')}\right\} > 0 \ \text{ for } \ z \in \mathbb{D}.$$

Using Theorem 3.3, we conclude that Φ defined by (5) is convex in the real direction. The proof is complete.

Proof of Theorem 2.3. We consider

$$F_1 = \left(h_1 + e^{-2i(\gamma_1+\gamma_{a_1})}g_1\right) * \left(h_2 - e^{-2i(\gamma_2+\gamma_{a_2})}g_2\right),$$

and

$$F_2 = \left(h_1 - e^{-2i(\gamma_1+\gamma_{a_1})}g_1\right) * \left(h_2 + e^{-2i(\gamma_2+\gamma_{a_2})}g_2\right).$$

Then we see that

$$\frac{F_1 + F_2}{2} = h_1 * h_2 - e^{-2i\Gamma} g_1 * g_2,$$

which shows that we only need to prove that $e^{i\Gamma}(F_1 + F_2)$ is convex in the real direction by Theorem 3.2.

Without loss of generality, we may assume that $0 \le \Gamma < 2\pi$. Also, let ω_k denote the dilatation of f_k for $k = 1, 2$, and consider the following quotients

$$q_k = \frac{h_k' - e^{-2i(\gamma_k+\gamma_{a_k})} g_k'}{h_k' + e^{-2i(\gamma_k+\gamma_{a_k})} g_k'} = \frac{1 - e^{-2i(\gamma_k+\gamma_{a_k})}\omega_{f_k}}{1 + e^{-2i(\gamma_k+\gamma_{a_k})}\omega_{f_k}}, \quad k = 1, 2, \tag{11}$$

from which we observe that

$$\mathrm{Re}\,\{q_k(z)\} > 0 \quad \text{for } z \in \mathbb{D} \text{ and for } k = 1, 2.$$

Because $f_k = h_k + \overline{g_k} \in \mathcal{S}(H^{a_k}_{\gamma_k})$, (1) and the relation (11) for $k = 1, 2$, give

$$h_k'(z) - e^{-2i(\gamma_k+\gamma_{a_k})} g_k'(z) = q_k(z)\left(h_k'(z) + e^{-2i(\gamma_k+\gamma_{a_k})} g_k'(z)\right) = \frac{|1 + a_k| q_k(z)}{(1 - e^{i(\gamma_k+\gamma_{a_k})} z)^2}. \tag{12}$$

Since

$$zF_1'(z) = \left(h_1(z) + e^{-2i(\gamma_1+\gamma_{a_1})} g_1(z)\right) * z\left(h_2'(z) - e^{-2i(\gamma_2+\gamma_{a_2})} g_2'(z)\right),$$

by (1) and the relation (12), we deduce that

$$zF_1'(z) = |1 + a_1|\,|1 + a_2| \frac{z q_2(e^{i(\gamma_1+\gamma_{a_1})} z)}{(1 - e^{i\Gamma} z)^2}.$$

Similarly, we obtain that

$$zF_2'(z) = |1 + a_1|\,|1 + a_2| \frac{z q_1(e^{i(\gamma_2+\gamma_{a_2})} z)}{(1 - e^{i\Gamma} z)^2}.$$

Let $\mu = -\Gamma$ and $\nu = 0$. Then, we have

$$(1 - e^{i\Gamma} z)^2 = 1 - 2ze^{i\Gamma} + z^2 e^{2i\Gamma} = 1 - 2ze^{-i\mu} + z^2 e^{-2i\mu}.$$

Then the last three equations imply that

$$\mathrm{Re}\left\{e^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + z^2 e^{-2i\mu}) e^{i\Gamma}\left(F_1'(z) + F_2'(z)\right)\right\} > 0 \quad \text{for } z \in \mathbb{D}.$$

Using Theorem 3.3, the proof is complete.

4. Convex combination of mappings in $\mathcal{F}^a_{\lambda,\delta}$

Let $\mathcal{F}^a_{\lambda,\delta}$ consist of functions $F = H + \overline{G} \in \mathcal{H}$ satisfying

$$H'(z) + \overline{\delta^2} e^{-2i\gamma_a} G'(z) = \frac{1 + a'}{(1 + \lambda\delta e^{i\gamma_a} z)(1 + \overline{\lambda}\delta e^{i\gamma_a} z)}, \tag{13}$$

where $\lambda,\ \delta \in \partial\mathbb{D}$, $a \in \mathbb{D}$ with $a' = |1 + a| - 1$, and $\gamma_a = \arg(1 + \overline{a})$. If $a \in (-1, 1)$, then (13) reduces to

$$H'(z) + \overline{\delta^2} G'(z) = \frac{1 + a}{(1 + \lambda\delta z)(1 + \overline{\lambda}\delta z)}.$$

Moreover, it is obvious that $\mathcal{S}(H^a_\gamma) \subseteq \mathcal{F}^a_{-1,e^{i\gamma}}$ and $\mathcal{S}(\Omega^b_\beta) \subseteq \mathcal{F}^b_{e^{i\beta},1}$.

4.1. Rotations

For $f = h + \overline{g} \in \mathcal{H}$ and $\mu \in \partial\mathbb{D}$, motivated by the work of Ferrada-Salas *et. al.* in [5], we consider the standard rotation f^{μ} of f by the relation

$$f^{\mu}(z) = \overline{\mu} f(\mu z) \tag{14}$$

and thus, has the canonical decomposition

$$f^{\mu} = h^{\mu} + \overline{g^{\mu}}, \text{ where } h^{\mu}(z) = \overline{\mu} h(\mu z) \text{ and } g^{\mu}(z) = \mu g(\mu z). \tag{15}$$

Since $(h^{\mu})'(z) = h'(\mu z)$ and $(g^{\mu})'(z) = \mu^2 g'(\mu z)$, it follows easily that

$$\omega_{f^{\mu}}(z) = \mu^2 \frac{g'(\mu z)}{h'(\mu z)} = \mu^2 \omega_f(\mu z). \tag{16}$$

Thus, we may formulate the above discussion as

Lemma 4.1. *Let $\mu \in \partial\mathbb{D}$ and $f = h + \overline{g} \in \mathcal{H}$. Then f^{μ} is locally univalent and sense-preserving on $\mathbb{D}$ if and only if f is locally univalent and sense-preserving on $\mathbb{D}$.*

Lemma 4.2. *Let $F = H + \overline{G} \in \mathcal{F}^{a}_{\lambda,\delta}$ for given $\lambda,\ \delta \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. Also, let $\gamma_a = \arg(1+\overline{a})$. Then $F^{\overline{\delta}e^{-i\gamma_a}} = h + \overline{g}$ satisfies that*

$$h'(z) + g'(z) = \frac{|1+a|}{(1+\lambda z)(1+\overline{\lambda} z)},$$

that is, $F^{\overline{\delta}e^{-i\gamma_a}} \in \mathcal{F}^{a'}_{\lambda,1}$, where $a' = |1+a| - 1$.

Proof. It follows from the definition of rotation (see (14)) that

$$h(z) = \delta e^{i\gamma_a} H(\overline{\delta} e^{-i\gamma_a} z) \text{ and } g(z) = \overline{\delta} e^{-i\gamma_a} G(\overline{\delta} e^{-i\gamma_a} z),$$

which yield that

$$h'(z) = H'(\overline{\delta} e^{-i\gamma_a} z) \text{ and } g'(z) = \overline{\delta^2} e^{-2i\gamma_a} G'(\overline{\delta} e^{-i\gamma_a} z),$$

respectively. The last two relations and (13) imply that

$$h'(z) + g'(z) = H'(\overline{\delta} e^{-i\gamma_a} z) + \overline{\delta^2} e^{-2i\gamma_a} G'(\overline{\delta} e^{-i\gamma_a} z) = \frac{|1+a|}{(1+\lambda z)(1+\overline{\lambda} z)}$$

and the proof is complete. □

Theorem 4.3. *Let $F = H + \overline{G} \in \mathcal{F}^{a}_{\lambda,\delta}$ for given $\lambda,\ \delta \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. If F is locally univalent and sense-preserving on $\mathbb{D}$, then F is a convex harmonic mapping.*

Proof. Let $\gamma_a = \arg(1+\overline{a})$. Since $F^{\mu}(z) = \overline{\mu} F(\mu z)$ for $\mu \in \partial\mathbb{D}$, it is obvious that F is convex if and only if $F^{\overline{\delta}e^{-i\gamma_a}}$ is convex. Again, since F is locally univalent and sense-preserving on $\mathbb{D}$, Lemma 4.1 implies that $F^{\overline{\delta}e^{-i\gamma_a}}$ is locally one-to-one and sense-preserving on $\mathbb{D}$. Let $F^{\overline{\delta}e^{-i\gamma_a}} = h + \overline{g}$. By Theorem 3.2, it suffices to prove that $\varphi_{\theta} =: e^{i\theta} h - e^{-i\theta} g$ is convex in the real direction for all $\theta \in [0, 2\pi)$.

Moreover, Lemma 4.2 shows that

$$\begin{aligned}
\varphi_{\theta}'(z) &= e^{i\theta} h'(z) - e^{-i\theta} g'(z) \\
&= (h'(z) + g'(z)) \left(i \sin\theta + \frac{h'(z) - g'(z)}{h'(z) + g'(z)} \cos\theta \right) \\
&= \frac{|1+a|}{(1+\lambda z)(1+\overline{\lambda} z)} \left(i \sin\theta + \frac{h'(z) - g'(z)}{h'(z) + g'(z)} \cos\theta \right).
\end{aligned}$$

Set $\operatorname{Re}\lambda = \cos\alpha$. If $\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi)$, then we let $\mu = 0$ and $\nu = \pi - \alpha$. This yields that

$$(1 + \lambda z)(1 + \overline{\lambda} z) = 1 + 2z\cos\alpha + z^2 = e^{i\mu}\left(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}\right)$$

and, because $F^{\overline{\delta}e^{-i\gamma_a}}$ is sense-preserving, it follows that

$$\operatorname{Re}\left\{e^{i\mu}\left(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}\right)\varphi_\theta'(z)\right\} = |1 + a|\operatorname{Re}\left\{\frac{h'(z) - g'(z)}{h'(z) + g'(z)}\right\}\cos\theta > 0$$

for $z \in \mathbb{D}$. If $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, then we let $\mu = \pi$ and $\nu = \alpha$, which yields

$$-(1 + \lambda z)(1 + \overline{\lambda} z) = e^{i\mu}\left(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}\right)$$

and thus, we find that

$$\operatorname{Re}\left\{e^{i\mu}\left(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}\right)\varphi_\theta'(z)\right\} = -|1 + a|\operatorname{Re}\left\{\frac{h'(z) - g'(z)}{h'(z) + g'(z)}\right\}\cos\theta > 0$$

for $z \in \mathbb{D}$. Using Theorem 3.3, we conclude that for each $\theta \in [0, 2\pi)$, the analytic function φ_θ is convex in the real direction. The proof is complete. □

In [5], the following result was proved.

Theorem 4.4. ([5, Theorem 1.2]) *Let $\lambda,\ \delta \in \partial\mathbb{D}$. If $F_j \in \mathcal{F}^0_{\lambda,\delta}$, $j = 1, 2, \ldots, n$, then any convex combination of the F_j is a convex harmonic mapping.*

In order to prove Theorem 4.6 below, as a generalization of Theorem 4.4, we need the following lemma.

Lemma 4.5. (See [12] and also [5, Lemma 2.1]) *Let ω_1 and ω_2 be two analytic functions in the unit disk that map $\mathbb{D}$ to itself. Then for any real number θ and all $z \in \mathbb{D}$,*

$$\operatorname{Re}\left\{\frac{1 - \omega_1(z)\overline{\omega_2(z)}}{(1 + e^{-2i\theta}\omega_1(z))\left(1 + e^{2i\theta}\overline{\omega_2(z)}\right)}\right\} > 0.$$

Theorem 4.6. *Let $\lambda,\ \delta \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. If $F_j = H_j + \overline{G_j} \in \mathcal{F}^a_{\lambda,\delta}$ is locally univalent and sense-preserving on $\mathbb{D}$, $j = 1, 2, \ldots, n$, then any convex combination of the F_j is a convex harmonic mapping.*

Proof. Let $\omega_j = G_j'/H_j'$ denote the dilatation of F_j for $j = 1, 2, \ldots, n$, and $F = H + \overline{G} = \sum_{j=1}^n t_jF_j$, where $t_1, t_2, \ldots,\ t_n$ are nonnegative real numbers with $\sum_{j=1}^n t_j = 1$. Then

$$H(z) = \sum_{j=1}^n t_jH_j \ \text{ and } \ G(z) = \sum_{j=1}^n t_jG_j,$$

which together with (13) imply that $F \in \mathcal{F}^a_{\lambda,\delta}$. Moreover,

$$H_j'(z) = \frac{|1 + a|}{(1 + \lambda\delta e^{i\gamma_a}z)(1 + \overline{\lambda}\delta e^{i\gamma_a}z)\left(1 + \overline{\delta}^2e^{-2i\gamma_a}\omega_j(z)\right)},$$

$$G_j'(z) = \frac{|1 + a|\omega_j(z)}{(1 + \lambda\delta e^{i\gamma_a}z)(1 + \overline{\lambda}\delta e^{i\gamma_a}z)\left(1 + \overline{\delta}^2e^{-2i\gamma_a}\omega_j(z)\right)},$$

$$H'(z) = \frac{|1+a|}{(1+\lambda\delta e^{i\gamma_a}z)(1+\overline{\lambda}\delta e^{i\gamma_a}z)} \sum_{j=1}^{n} \frac{t_j}{\left(1+\overline{\delta^2}e^{-2i\gamma_a}\omega_j(z)\right)},$$

and

$$G'(z) = \frac{|1+a|}{(1+\lambda\delta e^{i\gamma_a}z)(1+\overline{\lambda}\delta e^{i\gamma_a}z)} \sum_{j=1}^{n} \frac{t_j\omega_j(z)}{\left(1+\overline{\delta^2}e^{-2i\gamma_a}\omega_j(z)\right)}.$$

where $\gamma_a = \arg(1+\overline{a})$. Consider the function

$$\Phi(z) = \left|\sum_{j=1}^{n} \frac{t_j}{\left(1+\overline{\delta^2}e^{-2i\gamma_a}\omega_j(z)\right)}\right|^2 - \left|\sum_{j=1}^{n} \frac{t_j\omega_j(z)}{\left(1+\overline{\delta^2}e^{-2i\gamma_a}\omega_j(z)\right)}\right|^2, \quad z \in \mathbb{D}.$$

Finally, since

$$J_F(z) = \frac{|1+a|^2}{\left|1+\lambda\delta e^{i\gamma_a}z\right|^2\left|1+\overline{\lambda}\delta e^{i\gamma_a}z\right|^2}\Phi(z),$$

Theorem 4.3 shows that we only need to prove $\Phi(z) > 0$ in the unit disk. By a similar reasoning as in the proof of Theorem 4.4, we find that $\Phi(z) > 0$ in the unit disk. The proof is complete . □

5. Further Results on convolution

For notational consistency, it is appropriate to consider harmonic mappings $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$ with the dilatation

$$\omega(z) = -e^{2i(\gamma+\gamma_a)}\frac{e^{i(\gamma+\gamma_a)}z - a'}{1-a'e^{i(\gamma+\gamma_a)}z}, \text{ and } h(z) + e^{-2i(\gamma+\gamma_a)}g(z) = \frac{(1+a')z}{1-e^{i(\gamma+\gamma_a)}z}.$$

Solving these two equations gives slanted half-plane mappings with γ and a as

$$f_\gamma^a = h_\gamma^a + \overline{g_\gamma^a} \tag{17}$$

where

$$h_\gamma^a(z) = \frac{z - \frac{1+a'}{2}e^{i(\gamma+\gamma_a)}z^2}{(1-e^{i(\gamma+\gamma_a)}z)^2} = \frac{(1+a')I(z) + (1-a')zI'(z)}{2}, \quad I(z) = \frac{z}{1-e^{i(\gamma+\gamma_a)}z},$$

and

$$e^{-2i(\gamma+\gamma_a)}g_\gamma^a(z) = \frac{a'z - \frac{1+a'}{2}e^{i(\gamma+\gamma_a)}z^2}{(1-e^{i(\gamma+\gamma_a)}z)^2} = \frac{(1+a')I(z) - (1-a')zI'(z)}{2}.$$

Obviously, when a is real and $a \in (-1,1)$, we have $\gamma_a = 0$ and $a' = a$ so that f_γ^a coincides with the slanted half-plane mappings considered in [6].

Moreover, if $a \in (-1,1)$, then $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$ satisfies the relation

$$h(z) + e^{-2i\gamma}g(z) = \frac{(1+a)z}{1-e^{i\gamma}z}, \quad z \in \mathbb{D}.$$

It is worth recalling that functions $f \in \mathcal{S}(H_\gamma^a)$ with $\gamma = 0$ and $a \in (-1,1)$ are usually referred to as *the right half-plane mappings*, especially when $a = 0$. In the latter case, we have $f_0 = h_0 + \overline{g_0} \in \mathcal{S}(H_0^0)$ with the dilatation $\omega_0(z) = -z$, where

$$h_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) \tag{18}$$

and

$$g_0(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right), \tag{19}$$

so that

$$h_0(z) + g_0(z) = \frac{z}{1-z}.$$

The function f_0 is central for many extremal problems for the family $\mathcal{K}_H^0$, and f_0 maps the unit disk onto the half-plane $\{w : \operatorname{Re} w > -1/2\}$. In [6], we considered the convolution of f_0 with functions from $\mathcal{S}(H_\gamma^a)$, where $a \in (-1, 1)$, and proved the following theorem.

Theorem 5.1. ([6, Theorem 2.1]) *Let $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$ with*

$$h(z) + e^{-2i\gamma} g(z) = \frac{(1+a)z}{1-e^{i\gamma}z} \quad \text{and} \quad \omega(z) = e^{2i\gamma}\frac{ze^{i\theta}+a}{1+aze^{i\theta}},$$

where $\theta \in \mathbb{R}$ and $a \in (-1, 1)$. If one of the following conditions holds, then $f_0 * f \in \mathcal{S}_H$ and is convex in the direction $-\gamma$.

(i) $\cos(\theta - \gamma) = -1$ *and* $-1/3 \le a < 1$.
(ii) $-1 < \cos(\theta - \gamma) \le 1$ *and* $a^2 < \frac{1}{5-4\cos(\theta-\gamma)}$.

The following remark is worth of noting. After that we recall [6, Theorems 2.2 and 2.3].

Remark 5.2. *Let $a \in (-1, 1)$. For any $f = h + \overline{g} \in \mathcal{H}$, the representation for f_0^a given by* (17) *quickly gives that*

$$\left(h_0^a * h\right)(z) = \frac{(1+a)h(z) + (1-a)zh'(z)}{2},$$

and

$$\left(g_0^a * g\right)(z) = \frac{(1+a)g(z) - (1-a)zg'(z)}{2}.$$

Then by a computation, we see that the dilatation $\widetilde{\omega}$ of $f_0^a * f$ is given by

$$\widetilde{\omega(z)} = \frac{2ag'(z) - (1-a)zg''(z)}{2h'(z) + (1-a)zh''(z)}. \tag{20}$$

Theorem 5.3. *[6, Theorems 2.2 and 2.3] Let $f \in \mathcal{S}_H^0$ with the dilatation $\omega(z) = e^{i\theta}z^n$, where n is a positive integer and $\theta \in \mathbb{R}$. Suppose that $a \in [\frac{n-2}{n+2}, 1)$ and $f_\gamma^a \in \mathcal{S}(H_\gamma^a)$ with the dilatation*

$$\omega_{f_\gamma^a}(z) = e^{2i\gamma}\frac{a - e^{i\gamma}z}{1 - ae^{i\gamma}z}.$$

(i) *If $f \in \mathcal{S}(H_{\gamma_1}^0)$, then $f * f_\gamma^a$ is convex in the direction $-(\gamma_1 + \gamma)$.*
(ii) *If $f \in \mathcal{S}(\Omega_\beta^0)$, where $0 < \beta < \pi$, then $f * f_\gamma^a$ is convex in the direction $-\gamma$.*

Using Theorem 5.3, we have the following.

Theorem 5.4. *Let $f \in \mathcal{S}_H^0$ with the dilatation $\omega(z) = e^{i\theta}z^n$, where n is a positive integer and $\theta \in \mathbb{R}$. Let $a \in \mathbb{D}$, $a' = |1 + a| - 1$, $0 \le \gamma$, $\gamma_1 < 2\pi$, and $f_\gamma^a \in \mathcal{S}(H_\gamma^a)$ with the dilatation*

$$\omega_{f_\gamma^a}(z) = e^{2i(\gamma+\gamma_a)}\frac{a' - ze^{i(\gamma+\gamma_a)}}{1 - a'ze^{i(\gamma+\gamma_a)}},$$

where $\gamma_a = \arg(1 + \overline{a})$. Suppose that $|a + 1| \in [\frac{2n}{n+2}, 2)$. Then we have the following:

(i) *If* $f \in \mathcal{S}(H^0_{\gamma_1})$, *then* $f * f^a_\gamma$ *is convex in the direction* $-(\gamma + \gamma_a + \gamma_1)$.

(ii) *If* $f \in \mathcal{S}(\Omega^0_\beta)$ *for some* β *with* $0 < \beta < \pi$, *then* $f * f^a_\gamma$ *is convex in the direction* $-(\gamma + \gamma_a)$.

Proof. The assumption implies that $a' \in [\frac{n-2}{n+2}, 1)$ and thus, Theorem 5.3(i) shows that $f * f^a_\gamma$ is locally univalent in $\mathbb{D}$. The conclusion follows from Theorem 2.3.

The conclusion of the second part follows from Theorem 5.3(ii) and a similar reasoning as in the proof of the first part. □

We now give an example of Case (i) of Theorem 5.4.

Example 5.5. *Let* $f^a_\gamma = h^a_\gamma + \overline{g^a_\gamma} \in \mathcal{S}(H^a_\gamma)$ *with the dilatation*

$$\omega_{f^a_\gamma}(z) = e^{2i(\gamma+\gamma_a)} \frac{a' - e^{i(\gamma+\gamma_a)}z}{1 - a' e^{i(\gamma+\gamma_a)}z}, \ \text{and} \ h^a_\gamma(z) + e^{-2i(\gamma+\gamma_a)} g^a_\gamma(z) = \frac{(1+a')z}{1 - e^{i(\gamma+\gamma_a)}z}.$$

We recall that

$$h^a_\gamma(z) = \frac{(1+a')I(z) + (1-a')zI'(z)}{2}, \quad I(z) = \frac{z}{1 - e^{i(\gamma+\gamma_a)}z},$$

and

$$e^{-2i(\gamma+\gamma_a)} g^a_\gamma(z) = \frac{(1+a')I(z) - (1-a')zI'(z)}{2}.$$

Let $f = h + \overline{g} \in \mathcal{S}(H^0_{\gamma_1})$ *with the dilatation* $\omega(z) = -e^{3i\gamma_1}z$. *Then*

$$h(z) = \frac{J(z) + zJ'(z)}{2}, \quad e^{-2i\gamma_1} g(z) = \frac{J(z) - zJ'(z)}{2}, \ J(z) = \frac{z}{1 - e^{i\gamma_1}z}.$$

Let $F = f * f^a_\gamma = H + \overline{G}$. *Then*

$$\begin{aligned} zJ'(z) * zI'(z) &= e^{-i(\gamma+\gamma_a+\gamma_1)} \frac{e^{i\gamma_1}z}{(1 - e^{i\gamma_1}z)^2} * \frac{e^{i(\gamma+\gamma_a)}z}{(1 - e^{i(\gamma+\gamma_a)}z)^2} \\ &= z \frac{1 + e^{i(\gamma+\gamma_a+\gamma_1)}z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^3}. \end{aligned}$$

$$\begin{aligned} H &= h * h^a_\gamma = \frac{J(z) + zJ'(z)}{2} * \frac{(1+a')I(z) + (1-a')zI'(z)}{2} \\ &= \frac{(1+a')}{4} e^{-i\gamma_1} I(e^{i\gamma_1}z) + \frac{(1+a')}{4} zJ'(e^{i(\gamma+\gamma_a)}z) \\ &\quad + \frac{(1-a')}{4} zI'(e^{i\gamma_1}z) + \frac{(1-a')z}{4} \frac{1 + e^{i(\gamma+\gamma_a+\gamma_1)}z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^3} \\ &= \frac{(1+a')}{4} \frac{z}{1 - e^{i(\gamma+\gamma_a+\gamma_1)}z} + \frac{1}{2} \frac{z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^2} + \frac{(1-a')z}{4} \frac{1 + e^{i(\gamma+\gamma_a+\gamma_1)}z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^3}. \end{aligned}$$

$$\begin{aligned} G &= g * g^a_\gamma = e^{2i(\gamma+\gamma_a+\gamma_1)} \frac{J(z) - zJ'(z)}{2} * \frac{(1+a')I(z) - (1-a')zI'(z)}{2} \\ e^{-2i(\gamma+\gamma_a+\gamma_1)} G &= \frac{(1+a')}{4} \frac{z}{1 - e^{i(\gamma+\gamma_a+\gamma_1)}z} - \frac{1}{2} \frac{z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^2} + \frac{(1-a')z}{4} \frac{1 + e^{i(\gamma+\gamma_a+\gamma_1)}z}{(1 - e^{i(\gamma+\gamma_a+\gamma_1)}z)^3}. \end{aligned}$$

Let $a = -\frac{2}{3} + \frac{\sqrt{3}}{3}i$, *then*

$$a' = |1+a| - 1 = -\frac{1}{3}, \ \gamma_a = \arg(1+\overline{a}) = \arg(\frac{1}{3} - \frac{\sqrt{3}}{3}i) = \frac{5\pi}{3}.$$

The images of $\mathbb{D}$ *under* $f * f^a_\gamma$ *are shown in Figure 1, where the left is for the case* $\gamma + \gamma_1 = \frac{\pi}{3}$ *and the right is for the case* $\gamma + \gamma_1 = \frac{5\pi}{6}$.

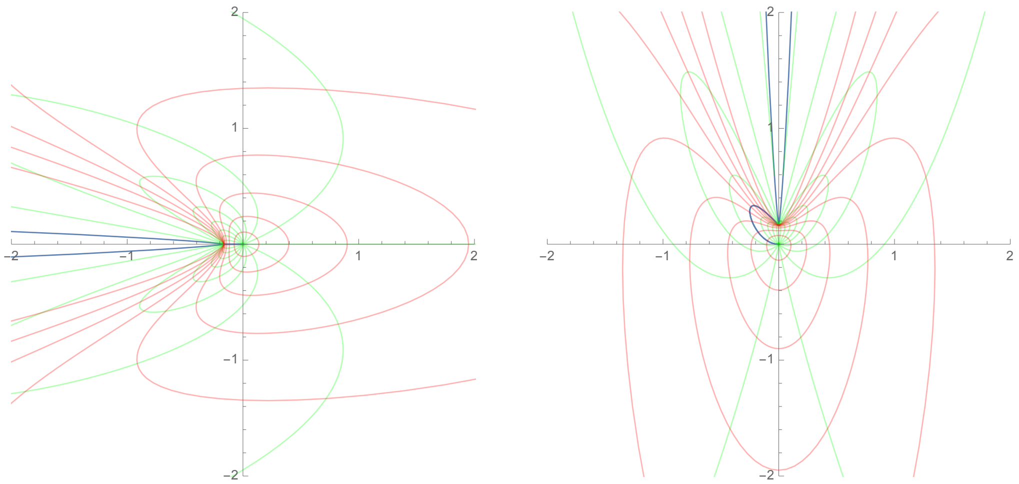

Figure 1: Images of $\mathbb{D}$ under $f * f_\gamma^a$

Finally, we state several lemmas concerning rotations of functions involved and use them to prove Theorem 5.8.

Lemma 5.6. *Let $f_j \in \mathcal{H}$, and $\mu_j \in \partial\mathbb{D}$, for $j = 1, 2$. Then*

$$(f_1^{\mu_1} * f_2^{\mu_2})(z) = \overline{\mu_1\mu_2}\,(f_1 * f_2)(\mu_1\mu_2 z) = (f_1 * f_2)^{\mu_1\mu_2}(z).$$

Proof. Follows from the definitions of f^μ and the convolution. □

The relations (15) and (16) and the simple fact that

$$\frac{e^{i\gamma}}{1+a} f(e^{i\xi}z) = \frac{e^{i\gamma}e^{i\xi}}{1+a}[e^{-i\xi}h(e^{i\xi}z) + \overline{e^{i\xi}g(e^{i\xi}z)}] = \frac{e^{i(\gamma+\xi)}}{1+a} f^{e^{i\xi}}(z)$$

give the following lemma.

Lemma 5.7. *Let $a \in \mathbb{D}$, $a' = |1+a| - 1$, $\gamma_a = \arg(1+\overline{a})$ and $f = h + \overline{g} \in \mathcal{S}(H_\gamma^a)$. Then $f^{e^{-i(\gamma+\gamma_a)}} = H + \overline{G} \in \mathcal{S}(H_0^{a'})$ with*

$$H(z) + G(z) = \frac{(1+a')z}{1-z}. \tag{21}$$

Moreover,

(i) *if $\omega_f(z) = e^{2i(\gamma+\gamma_a)} \frac{a' - ze^{i(\gamma+\gamma_a)}}{1 - a'ze^{i(\gamma+\gamma_a)}}$, then $\omega_{f^{e^{-i(\gamma+\gamma_a)}}}(z) = \frac{a'-z}{1-a'z}$;*

(ii) *if $\omega_f(z) = e^{2i(\gamma+\gamma_a)} \frac{a'+ze^{i\theta}}{1+a'ze^{i\theta}}$, then $\omega_{f^{e^{-i(\gamma+\gamma_a)}}}(z) = \frac{a'+ze^{i(\theta-\gamma-\gamma_a)}}{1+a'ze^{i(\theta-\gamma-\gamma_a)}}$.*

Theorem 5.8. *For $j = 1, 2$, let $a_j \in \mathbb{D}$, $a_j' = |1+a_j| - 1$, $\gamma_{a_j} = \arg(1+\overline{a_j})$, $0 \le \gamma_1, \ \gamma_2 < 2\pi$, $\Gamma = (\gamma_1+\gamma_2+\gamma_{a_1}+\gamma_{a_2})$ and $f_j = h_j + \overline{g_j} \in \mathcal{S}(H_{\gamma_j}^{a_j})$ with the dilatations*

$$\omega_{f_1}(z) = e^{2i(\gamma_1+\gamma_{a_1})} \frac{a_1' - ze^{i(\gamma_1+\gamma_{a_1})}}{1 - a_1'ze^{i(\gamma_1+\gamma_{a_1})}} \ \ and \ \ \omega_{f_2}(z) = e^{2i(\gamma_2+\gamma_{a_2})} \frac{a_2' + ze^{i\theta}}{1 + a_2'ze^{i\theta}}.$$

If one of the following holds, then $f_1 * f_2$ is convex in the direction $-\Gamma$.

(i) $\cos(\theta - \gamma_2 - \gamma_{a_2}) = -1$ *and* $1 + 3a_1' + 3a_2' + a_1'a_2' \ge 0$.

(ii) $\cos(\theta-\gamma_2-\gamma_{a_2})=1$ *and* $1+3a_1'+3a_1'a_2'+(a_1')^2a_2'>0.$

Proof. Set $F_j=H_j+\overline{G_j}=f_j^{e^{-i(\gamma_j+\gamma_{a_j})}}$ for $j=1,2$. Then Lemma 5.7 implies that $F_j\in\mathcal{S}(H_0^{a_j'})$ with

$$H_j(z)+G_j(z)=\frac{(1+a_j')z}{1-z},$$

and,

$$\omega_{F_1}(z)=\frac{a_1'-z}{1-a_1'z}\ \text{ and }\ \omega_{F_2}(z)=\frac{a_2'+ze^{i(\theta-\gamma_2-\gamma_{a_2})}}{1+a_2'ze^{i(\theta-\gamma_2-\gamma_{a_2})}}.$$

The representation given by (20) implies that the dilatation $\widetilde{\omega}$ of F_1*F_2 becomes

$$\widetilde{\omega}=\frac{2a_1'G_2'-(1-a_1')zG_2''}{2H_2'+(1-a_1')zH_2''}.$$

Since

$$G_2'=\omega_{F_2}H_2'\ \text{ and }\ G_2''=\omega_{F_2}'H_2'+\omega_{F_2}H_2'',$$

the dilatation $\widetilde{\omega}$ takes the form

$$\widetilde{\omega}=\frac{[2a_1'\omega_{F_2}-(1-a_1')z\omega_{F_2}']H_2'-(1-a_1')\omega_{F_2}zH_2''}{2H_2'+(1-a_1')zH_2''}. \tag{22}$$

On the other hand, we have

$$\begin{aligned}
H_2'(z) &= \frac{1+a_2'}{(1-z)^2(1+\omega_{F_2}(z))},\\
H_2''(z) &= \frac{(1+a_2')[2(1+\omega_{F_2}(z))-(1-z)\omega_{F_2}'(z)]}{(1-z)^3(1+\omega_{F_2}(z))^2},\\
\frac{zH_2''(z)}{H_2'(z)} &= \frac{z[2(1+\omega_{F_2}(z))-(1-z)\omega_{F_2}'(z)]}{(1-z)(1+\omega_{F_2}(z))},\ \text{ and}\\
\omega_{F_2}'(z) &= \frac{(1-a_2'^2)e^{i(\theta-\gamma_2-\gamma_{a_2})}}{\left(1+a_2'ze^{i(\theta-\gamma_2-\gamma_{a_2})}\right)^2}.
\end{aligned}$$

A tedious calculation shows that

$$\begin{aligned}
\widetilde{\omega} &= \frac{[2a_1'\omega_{F_2}-(1-a_1')z\omega_{F_2}'](1-z)(1+\omega_{F_2})-(1-a_1')\omega_{F_2}z[2(1+\omega_{F_2})-(1-z)\omega_{F_2}']}{2(1-z)(1+\omega_{F_2})+(1-a_1')z[2(1+\omega_{F_2})-(1-z)\omega_{F_2}']}\\
&= \frac{2(a_1'-z)\omega_{F_2}(1+\omega_{F_2})-(1-a_1')z(1-z)\omega_{F_2}'}{2(1-a_1'z)(1+\omega_{F_2})-(1-a_1')z(1-z)\omega_{F_2}'}\\
&= \frac{2(a_1'-z)(a_2'+ze^{i(\theta-\gamma_2-\gamma_{a_2})})(1+ze^{i(\theta-\gamma_2-\gamma_{a_2})})-(1-a_1')(1-a_2')(z-z^2)e^{i(\theta-\gamma_2-\gamma_{a_2})}}{2(1-a_1'z)(1+ze^{i(\theta-\gamma_2-\gamma_{a_2})})(1+a_2'ze^{i(\theta-\gamma_2-\gamma_{a_2})})-(1-a_1')(1-a_2')(z-z^2)e^{i(\theta-\gamma_2-\gamma_{a_2})}}\\
&= -e^{2i(\theta-\gamma_2-\gamma_{a_2})}\frac{t(z)}{t^*(z)},
\end{aligned}$$

where

$$t(z)=z^3+c_2z^2+c_1z+c_0\ \text{ and }\ t^*(z)=1+\overline{c_2}z+\overline{c_1}z^2+\overline{c_0}z^3,$$

with

$$\begin{aligned} c_2 &= \frac{a_1' + 3a_2' - a_1'a_2' + 1}{2} e^{-i(\theta-\gamma_2-\gamma_{a_2})} - a_1', \\ c_1 &= a_2' e^{-2i(\theta-\gamma_2-\gamma_{a_2})} - \frac{3a_1' + a_2' + a_1'a_2' - 1}{2} e^{-i(\theta-\gamma_2-\gamma_{a_2})}, \text{ and} \\ c_0 &= -a_1'a_2' e^{-2i(\theta-\gamma_2-\gamma_{a_2})}. \end{aligned}$$

Case 1. $\cos(\theta - \gamma_2 - \gamma_{a_2}) = -1$ and $1 + 3a_1' + 3a_2' + a_1'a_2' \geq 0$.

In this case, since $\cos(\theta - \gamma_2 - \gamma_{a_2}) = -1$, the constants c_0, c_1, c_2 take the form

$$c_2 = -\frac{3a_1' + 3a_2' - a_1'a_2' + 1}{2}, \; c_1 = \frac{3a_1' + 3a_2' + a_1'a_2' - 1}{2}, \; c_0 = -a_1'a_2',$$

and thus, we have

$$t(z) = (z-1)\left(z^2 - \frac{3a_1' + 3a_2' - a_1'a_2' - 1}{2} z + a_1'a_2'\right).$$

Cohn's Lemma (see [9]) shows that $|\widetilde{\omega}(z)| < 1$ and the conclusion for this case holds.

Case 2. $\cos(\theta - \gamma_2 - \gamma_{a_2}) = 1$ and $1 + 3a_1' + 3a_1'a_2' + (a_1')^2 a_2' > 0$.

In this case, since $\cos(\theta - \gamma_2 - \gamma_{a_2}) = 1$, we find that

$$c_2 = \frac{-a_1' + 3a_2' - a_1'a_2' + 1}{2}, \; c_1 = \frac{-3a_1' + a_2' - a_1'a_2' + 1}{2}, \; c_0 = -a_1'a_2',$$

so that

$$t(z) = z^3 - \frac{a_1' - 3a_2' + a_1'a_2' - 1}{2} z^2 - \frac{3a_1' - a_2' + a_1'a_2' - 1}{2} z - a_1'a_2'.$$

Now, we let

$$t_1(z) =: \frac{t(z) - c_0 t^*(z)}{z} = b_2 z^2 + b_1 z + b_0 \;\text{ and }\; t_1^*(z) =: \overline{b_2} + \overline{b_1} z + \overline{b_0} z^2.$$

A calculation yields that

$$\begin{aligned} b_2 &= 1 - (a_1'a_2')^2, \\ b_1 &= \frac{(1-a_1')\left(1 + 3a_2' + 3a_1'a_2' + a_1'(a_2')^2\right)}{2}, \\ b_0 &= \frac{(1+a_2')\left(1 - 3a_1' + 3a_1'a_2' - (a_1')^2 a_2'\right)}{2}, \end{aligned}$$

and therefore,

$$\begin{aligned} b_2^2 - b_0^2 &= \frac{(1-a_1')(1-a_2')}{4}\left(3 + a_2' + a_1'a_2' + 3a_1'(a_2')^2\right)\left(1 + 3a_1' + 3a_1'a_2' + (a_1')^2 a_2'\right) \\ &= \frac{(1-a_1')(1-a_2')}{4}(3 + a_2')\left(1 + \frac{a_1'a_2'(1+3a_2')}{3+a_2'}\right)\left(1 + 3a_1' + 3a_1'a_2' + (a_1')^2 a_2'\right), \end{aligned}$$

which shows that $|b_2| > |b_0|$ if and only if

$$1 + 3a_1' + 3a_1'a_2' + (a_1')^2 a_2' > 0.$$

Again, we let

$$t_2(z) = \frac{\overline{b_2} t_1(z) - b_0 t_1^*(z)}{z}.$$

Since b_2, b_1 and b_0 are real numbers, the calculation shows that the zero point z_0 of $t_2(z)$ is given by

$$z_0 = \frac{-b_1}{b_2 + b_0} =: \frac{u}{v} = -\frac{1 + 3a_2' + 3a_1'a_2' + a_1'(a_2')^2}{3 + a_2' + a_1'a_2' + 3a_1'(a_2')^2}.$$

We obtain that

$$u^2 - v^2 = -8\left(1 - (a_2')^2\right)\left(1 - (a_1'a_2')^2\right) < 0,$$

which implies that $z_0 \in \mathbb{D}$. Cohn's Lemma shows that $|\widetilde{\omega}(z)| < 1$ and the conclusion for this case holds.

Finally, the desired conclusion follows from Theorem 2.3. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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