Filomat 36:11 (2022), 3899–3917
https://doi.org/10.2298/FIL2211899N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Feng-Liu Type Fixed Point Theorems for w-Distance Spaces and Applications

Hemant Kumar Nashine[a], Rajendra Pant[b]

[a]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, 700000, Vietnam.
[b]Department of Mathematics and Applied Mathematics, University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa

Abstract. In this article, we study Feng-Liu [Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. **317** (2006), 103–112.] type fixed point theorems and present some new results for multi-valued mappings in metric spaces using the concept of ω-distance. We also discuss, some non-trivial examples to illustrate facts. Finally, we present applications of our results to integral inclusions and non-linear matrix equations. An example is given, together with convergence and error analysis, as well as average CPU time analysis and visualization of solution in surface plot.

1. Introduction and Preliminaries

The classical Banach contraction theorem (in short BCT) is an important and fruitful tool in nonlinear analysis. A number of extensions an generalizations of the BCT have been obtained by many mathematicians. Nadler [12] presented a multi-valued version of the BCT. His results was also extended and generalized by many authors. Feng and Liu [7] extended Nadler's result in the following way:

Theorem 1.1. [7]. *Let (Ξ, d) be a complete metric space, $\Im\colon \Xi \to \mathcal{P}_{cl}(\Xi)$ a multi-valued mapping and $f\colon \Xi \to \mathbb{R}$, $f(\nu) = d(\nu, \Im\nu)$ a lower semi-continuous function. If there exist $b, c \in (0, 1)$ with $b < c$ such that for any $\nu \in \Xi$ there is $\vartheta \in \Im\nu$ satisfying*

$$c\, d(\nu, \vartheta) \leq f(\nu) \quad \text{and} \quad f(\vartheta) \leq b\, d(\nu, \vartheta),$$

then $\Im$ has a fixed point in Ξ.

A number of extensions and generalizations of the above theorem appeared in [3, 4, 6, 9, 13, 14] and elsewhere.

On the other hand, in 1996, Kada et al. [8] introduced the concept of w-distance on a metric space and presented a generalized version of Caristi fixed point theorem, Ekeland's ϵ-variational principle and the non-convex minimization theorem (cf. Mizoguchi and Takahashi [11]).

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25, 15A24, 65F45
Keywords. Fixed point, metric space, w-distance spaces, multi-valued mapping.
Received: 31 July 2021; Revised: 31 October 2021; Accepted: 14 November 2021
Communicated by Adrian Petrusel
Corresponding author: Hemant Kumar Nashine
Email addresses: hemantkumarnashine@tdtu.edu.vn (Hemant Kumar Nashine), pant.rajendra@gmail.com (Rajendra Pant)

Definition 1.2. *[8]. Let (Ξ, d) be a metric space. A function $\omega\colon \Xi \times \Xi \to [0,\infty)$ is called a w-distance on Ξ if it satisfies the following properties:*

(W1) $\omega(\vartheta,\mu) \leq \omega(\vartheta,\nu) + \omega(\nu,\mu)$ *for any* $\vartheta,\nu,\mu \in \Xi$;

(W2) ω *is lower semi continuous in its second variable; i.e., if* $\vartheta \in \Xi$ *and* $\nu_n \to \nu \in \Xi$, *then* $\omega(\vartheta,\nu) \leq \liminf_{n\to\infty} \omega(\vartheta,\nu_n)$;

(W3) *for each* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that* $\omega(\mu,\vartheta) \leq \delta$ *and* $\omega(\mu,\nu) \leq \delta$ *imply* $d(\vartheta,\nu) \leq \epsilon$.

The authors in [13], studied Feng-Liu type fixed point theorems and obtained a generalization of Theorem 1.1. Their theorems contain many results as particular cases. In this article, we continue this study and present some new Feng-Liu type fixed point results for multi-valued mappings in metric spaces using the concept of ω-distance. Our results are motivated by Feng and Liu [7], Kada et al. [8] and others.

Now, we recall some notations, definitions and results for the sake of completeness.

Throughout this paper, (Ξ, d) denotes a metric space and $\mathcal{P}_{cl}(\Xi)$ the family of all nonempty closed subsets of Ξ. For any subset $D \neq \emptyset$ of Ξ,

$$d(\nu, D) = \inf_{\vartheta\in D} d(\nu,\vartheta) \text{ and } \omega(\nu, D) = \inf_{\vartheta\in D} \omega(\nu,\vartheta).$$

Definition 1.3. *Let $\mathfrak{I}\colon \Xi \to \mathcal{P}_{cl}(\Xi)$ be a multi-valued mapping. A point $\nu \in \Xi$ is said to be a fixed point of $\mathfrak{I}$ if $\nu \in \mathfrak{I}\nu$.*

Definition 1.4. *[20] A function $f\colon \Xi \to \mathbb{R}$ is called lower semi-continuous (l.s.c., in short) if*

$$f(\nu) \leq \liminf_{n\to\infty} f(\nu_n) \tag{1}$$

for all sequences $\{\nu_n\}$ in Ξ with $\lim_{n\to\infty} \nu_n = \nu \in \Xi$).

Definition 1.5. *Let $\mathbb{F} : (0,\infty) \to \mathbb{R}$ be a function such that*

(F1) $\mathbb{F}$ *is strictly increasing;*

(F2) *for each sequence* $\{\varsigma_s\}$ *of positive numbers,*

$$\lim_{s\to\infty} \varsigma_s = 0 \textit{ if and only if } \lim_{s\to\infty} \mathbb{F}(\varsigma_s) = -\infty;$$

(F3) *there exists* $k \in (0,1)$ *such that* $\lim_{\varsigma\to 0^+} \varsigma^k \mathbb{F}(\varsigma) = 0$;

(F4) $\mathbb{F}(\inf \mathcal{B}) = \inf \mathbb{F}(\mathcal{B})$ *for all* $\mathcal{B} \subseteq (0,1)$ *with* $\inf \mathcal{B} > 0$.

We denote the sets of all functions $\mathbb{F}$ satisfying (F1)–(F3), (F1)–(F4) by $\mathfrak{F}$, $\mathfrak{F}_*$, respectively. It is clear that $\mathfrak{F}_* \subset \mathfrak{F}$ and some examples of functions belonging to $\mathfrak{F}_*$ are $\mathbb{F}_1(\varsigma) = \ln\varsigma$, $\mathbb{F}_2(\varsigma) = \varsigma + \ln\varsigma$, $\mathbb{F}_3(\varsigma) = -1/\sqrt{\varsigma}$, $\mathbb{F}_4(\varsigma) = \ln(\varsigma^2 + \varsigma)$ [20].

Note that, if $\mathbb{F}$ satisfies (F1), then it satisfies (F4) if and only if it is right-continuous.

Definition 1.6. *[20]. A mapping $\mathfrak{I} : \Xi \to \Xi$ is said to be $\mathbb{F}$-contraction if there exist $\mathbb{F} \in \mathfrak{F}$ and $\kappa \in \mathbb{R}^+$ such that*

$$\kappa + \mathbb{F}(d(\mathfrak{I}\nu, \mathfrak{I}\vartheta)) \leq \mathbb{F}(d(\nu,\vartheta)),$$

for all $\nu,\vartheta \in \Xi$ with $d(\mathfrak{I}\nu, \mathfrak{I}\vartheta) > 0$.

It is evident that every contraction mapping is $\mathbb{F}$-contraction (with $\mathbb{F}(\varsigma) = \ln\varsigma$ and $\kappa = -\ln\lambda$) but the converse need not be true. Wardowski [20] showed that each $\mathbb{F}$-contraction on a complete metric space has a fixed point. Afterwards, several researchers obtained various fixed point results using the idea of $\mathbb{F}$-contractions [21].

Definition 1.7. *[18]. A mapping $\Im : \Xi \to 2^{\Xi}$ (= collection of all nonempty subsets of Ξ) is said to be multi-valued $\mathbb{F}$-contraction if there exist $\mathbb{F} \in \mathfrak{F}$ and $\kappa \in \mathbb{R}^{+}$ such that for all $\nu, \vartheta \in \Xi$ with $\vartheta \in \Im\nu$ there exists $\mu \in \Im\vartheta$ for which*

$$\kappa + \mathbb{F}(d(\Im\vartheta, \Im\mu)) \leq \mathbb{F}(\mathcal{N}(\nu, \vartheta)) \tag{2}$$

if $d(\vartheta, \mu) > 0$, where

$$\mathcal{N}(\nu, \vartheta) = \max\left\{d(\nu, \vartheta), d(\nu, \Im\nu), d(\vartheta, \Im\vartheta), \frac{1}{2}[d(\nu, \Im\vartheta) + d(\vartheta, \Im\nu)]\right\}. \tag{3}$$

In [17], Samet et al. defined the α-admissibility of mappings as follows:

Definition 1.8. *[17]. Let $\alpha : \Xi \times \Xi \to [0, \infty)$ be a function. A mapping $\Im : \Xi \to \Xi$ is said to be an α-admissible mapping if, for $\nu, \vartheta \in \Xi$*

$$\alpha(\nu, \vartheta) \geq 1 \Rightarrow \alpha(\Im(\nu), \Im(\vartheta)) \geq 1.$$

Definition 1.9. *[5]. Let $\Im : \Xi \to 2^{\Xi}$ be a multi-valued mappings and $\alpha : \Xi \times \Xi \to [0, \infty)$ a function. The mapping $\Im$ is called α_*-admissible if $\nu_1, \nu_2 \in \Xi$,*

$$\alpha(\nu_1, \nu_2) \geq 1 \Rightarrow \alpha_*(\Im(\nu_1), \Im(\nu_2)) \geq 1$$

where $\alpha_*(\Lambda_1, \Lambda_2) := \inf\{\alpha(\xi_1, \xi_2) : \xi_1 \in \Lambda_1 \text{ and } \xi_2 \in \Lambda_2\}$.

Definition 1.10. *[2]. Let $\alpha, \eta : \Xi \times \Xi \to [0, +\infty)$ be functions. A mapping $\Im : \Xi \to 2^{\Xi}$ is said to be a generalized α_*-admissible mapping with respect to an η if for $\nu_1, \nu_2 \in \Xi$,*

$$\alpha(\nu_1, \nu_2) \geq \eta(\nu_1, \nu_2) \Rightarrow \alpha(\mu_1, \mu_2) \geq \eta(\mu_1, \mu_2) \ \ \forall\, \mu_1 \in \Im\nu_1, \ \forall\, \mu_2 \in \Im\nu_2.$$

If $\eta(\nu_1, \nu_2) = 1$ for all $\nu_1, \nu_2 \in \Xi$, then Definition 1.10 implies Definition 1.9, while if $\alpha(\nu_1, \nu_2) = 1$, $\Im$ is an η_*-subadmissible mapping.

We shall use the following lemmas for proving our main results.

Lemma 1.11. [8]. *Let (Ξ, d) be a metric space and let ω be a w-distance on Ξ. Suppose that $\{\vartheta_n\}$, $\{\nu_n\}$ are sequences in Ξ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0, and let $\vartheta, \nu, \mu \in \Xi$. Then the following assertions hold.*

- (i) *If $\omega(\vartheta_n, \nu) \leq \alpha_n$ and $\omega(\vartheta_n, \mu) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\nu = \mu$. In particular, if $\omega(\vartheta, \nu) = \omega(\vartheta, \mu) = 0$, then $\nu = \mu$,*
- (ii) *if $\omega(\vartheta_n, \nu_n) \leq \alpha_n$ and $\omega(\vartheta_n, \nu) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{\nu_n\}$ converges to ν,*
- (iii) *if $\omega(\vartheta_n, \vartheta_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{\vartheta_n\}$ is a Cauchy sequence,*
- (iv) *if $\omega(\nu, \vartheta_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{\vartheta_n\}$ is a Cauchy sequence.*

Lemma 1.12. [8, 19]. *Let ω be a w-distance on a metric space (Ξ, d) and $\{\vartheta_n\}$ be a sequence in Ξ such that for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $m > n > N_\epsilon$ implies $\omega(\vartheta_n, \vartheta_m) < \epsilon$, i.e., $\lim_{m,n\to\infty} \omega(\vartheta_n, \vartheta_m) = 0$. Then $\{\vartheta_n\}$ is a Cauchy sequence.*

Lemma 1.13. [10]. *Let $\mathcal{K}$ be a closed subset of Ξ and ω be a w-distance on Ξ. Assume that there exists $\nu \in \Xi$ such that $\omega(\nu, \nu) = 0$. Then $\omega(\nu, \mathcal{K}) = 0$ if and only if $\nu \in \mathcal{K}$, where $\omega(\nu, \mathcal{K}) = \inf_{\vartheta\in\mathcal{K}} \omega(\nu, \vartheta)$.*

2. F-contraction type Feng-Liu results

Recall that the set $O(\nu_0;\mathfrak{I}) = \{\mathfrak{I}^n\nu_0 : n = 0,1,2,\dots\}$ is called the orbit of the self-mapping $\mathfrak{I}$ at the point $\nu_0 \in \Xi$. If (1) is satisfied for all sequences $\{\nu_n\} \subset O(\nu_0)$, then f is an orbitally l.s.c..

Let $\mathfrak{I} : \Xi \to \mathcal{P}_{cl}(\Xi)$ be a multi-valued mapping, $\mathbb{F} \in \mathfrak{F}$ and $\tau : (0,\infty) \to (0,\infty)$. For $\nu \in \Xi$ with $\omega(\nu,\mathfrak{I}\nu) > 0$, define a set $\mathbb{F}^{\nu}_{\tau} \subseteq \Xi$ as

$$\mathbb{F}^{\nu}_{\tau} = \left\{ \begin{array}{c} \vartheta \in \mathfrak{I}\nu : \mathbb{F}(\omega(\nu,\vartheta)) \leq \mathbb{F}(\max\{\omega(\nu,\mathfrak{I}\nu),\omega(\vartheta,\mathfrak{I}\vartheta)\}) \\ +\tau(\max\{\omega(\nu,\mathfrak{I}\nu),\omega(\vartheta,\mathfrak{I}\vartheta)\}) \end{array} \right\}.$$

Theorem 2.1. *Let (Ξ,d) be a orbitally complete metric space with w-distance ω and $\mathfrak{I}: \Xi \to \mathcal{P}_{cl}(\Xi)$. Assume that*

(a) *the mapping $\nu \mapsto \omega(\nu,\mathfrak{I}\nu)$ is orbitally l.s.c.;*
(b) *$\mathfrak{I}$ is a muti-valued generalized α_*-admissible with respect to an η mapping;*
(c) *there exist functions $\theta,\tau\colon (0,\infty) \to (0,\infty)$ such that*

$$\theta(\varsigma) > \tau(\varsigma), \quad \liminf_{t\to\varsigma^+}\theta(t) > \liminf_{t\to\varsigma^+}\tau(t) \text{ for all } \varsigma \geq 0;$$

(d) *for any $\nu \in \Xi$ with $\omega(\nu,\mathfrak{I}\nu) > 0$, there exists $\vartheta \in \mathbb{F}^{\nu}_{\tau}$ with $\alpha(\nu,\vartheta) \geq \eta(\nu,\vartheta)$ satisfying*

$$\theta(\max\{\omega(\nu,\mathfrak{I}\nu),\omega(\vartheta,\mathfrak{I}\vartheta)\}) + \mathbb{F}(\omega(\vartheta,\mathfrak{I}\vartheta)) \leq \mathbb{F}(\omega(\nu,\vartheta));$$

(e) *if $\{\nu_n\} \subset \Xi$ with $\nu_{n+1} \in \mathfrak{I}\nu_n$, $\nu_n \to \nu \in \Xi$ as $n \to \infty$ and $\alpha(\nu_n,\nu_{n+1}) \geq \eta(\nu_n,\nu_{n+1})$ for all $n \in \mathbb{N}$ then $\alpha(\nu_n,\nu) \geq \eta(\nu_n,\nu)$ for all $n \in \mathbb{N}$.*

Then there exists $\varrho \in \Xi$ such that $\omega(\varrho,\mathfrak{I}\varrho) = 0$. Further, if $\omega(\varrho,\varrho) = 0$ then $\varrho \in \mathfrak{I}\varrho$.

Proof. Suppose that for all $\nu \in \Xi$, $\omega(\nu,\mathfrak{I}\nu) > 0$ and take an arbitrary point $\nu_0 \in \Xi$. From (d), there exists $\nu_1 \in \mathbb{F}^{\nu}_{\tau} \neq \emptyset$. If $\nu_0 \in \Xi$ is any initial point, then there exists $\nu_1 \in \mathbb{F}^{\nu_0}_{\tau}$ with $\alpha(\nu_0,\nu_1) \geq \eta(\nu_0,\nu_1)$ such that

$$\theta(\max\{\omega(\nu_0,\mathfrak{I}\nu_0),\omega(\nu_1,\mathfrak{I}\nu_1)\}) + \mathbb{F}(\omega(\nu_1,\mathfrak{I}\nu_1)) \leq \mathbb{F}(\omega(\nu_0,\nu_1)).$$

For $\nu_1 \in \Xi$ with $\nu_1 \in \mathfrak{I}(\nu_0)$, $\alpha(\nu_0,\nu_1) \geq \eta(\nu_0,\nu_1)$, and there exists $\nu_2 \in \mathbb{F}^{\nu_1}_{\tau}$ with $\nu_2 \in \mathfrak{I}(\nu_1)$. From (b), we have $\alpha(\nu_1,\nu_2) \geq \eta(\nu_1,\nu_2)$ and hence from (d)

$$\theta(\max\{\omega(\nu_1,\mathfrak{I}\nu_1),\omega(\nu_2,\mathfrak{I}\nu_2)\}) + \mathbb{F}(\omega(\nu_2,\mathfrak{I}\nu_2)) \leq \mathbb{F}(\omega(\nu_1,\nu_2)).$$

Continuing this process, we get an iterative sequence $\{\nu_r\}$, where $\nu_{r+1} \in \mathbb{F}^{\nu_r}_{\tau}$, $\nu_{r+1} \in \mathfrak{I}\nu_r$ with $\alpha(\nu_r,\nu_{r+1}) \geq \eta(\nu_r,\nu_{r+1})$ and

$$\theta(\max\{\omega(\nu_r,\mathfrak{I}\nu_r),\omega(\nu_{r+1},\mathfrak{I}\nu_{r+1})\}) + \mathbb{F}(\omega(\nu_{r+1},\mathfrak{I}\nu_{r+1})) \leq \mathbb{F}(\omega(\nu_r,\nu_{r+1})).$$

Therefore for $\nu_{r+2} \in \mathfrak{I}\nu_{r+1}$, we have

$$\theta(\max\{\omega(\nu_r,\nu_{r+1}),\omega(\nu_{r+1},\nu_{r+2})\}) + \mathbb{F}(\omega(\nu_{r+1},\mathfrak{I}\nu_{r+1})) \leq \mathbb{F}(\omega(\nu_r,\nu_{r+1})). \tag{4}$$

We will verify that $\{\nu_r\}$ is a Cauchy sequence. Since $\nu_{r+1} \in \mathbb{F}^{\nu_r}_{\tau}$, then by the definition of $\mathbb{F}^{\nu_r}_{\tau}$, we have

$$\begin{aligned}\mathbb{F}(\omega(\nu_r,\nu_{r+1})) &\leq \mathbb{F}(\max\{\omega(\nu_r,\mathfrak{I}\nu_r),\omega(\nu_{r+1},\mathfrak{I}\nu_{r+1})\}) \\ &\quad + \tau(\max\{\omega(\nu_r,\nu_{r+1}),\omega(\nu_{r+1},\nu_{r+2})\}).\end{aligned} \tag{5}$$

Put $\varrho_r = \omega(\nu_r,\nu_{r+1})$ for $r \in \mathbb{N}$, then $\varrho_r > 0$. From (4) and (5) we have

$$\mathbb{F}(\varrho_{r+1}) \leq \mathbb{F}(\max\{\varrho_r,\varrho_{r+1}\}) + \tau(\max\{\varrho_r,\varrho_{r+1}\}) - \theta(\max\{\varrho_r,\varrho_{r+1}\}). \tag{6}$$

If $\varrho_r \leq \varrho_{r+1}$, then we have

$$\mathbb{F}(\varrho_{r+1}) \leq \mathbb{F}(\varrho_{r+1}) + \tau(\varrho_{r+1}) - \theta(\varrho_{r+1}),$$

a contradiction since from (c), $\theta(\varsigma) > \tau(\varsigma)$. Therefore,

$$\begin{aligned}\mathbb{F}(\varrho_{r+1}) &\leq \mathbb{F}(\varrho_r) + \tau(\varrho_r) - \theta(\varrho_r)\\ &= \mathbb{F}(\varrho_r) - (\theta(\varrho_r) - \tau(\varrho_r)).\end{aligned} \tag{7}$$

From (7), $\{\varrho_r\}$ is decreasing. Therefore, there exists $\delta > 0$ such that $\lim_{r\to\infty} \varrho_r = \delta$. Let $\beta(t) = \theta(t) - \tau(t)$, for all $t > 0$. Then using (7), the following holds:

$$\begin{aligned}\mathbb{F}(\varrho_{r+1}) &\leq \mathbb{F}(\varrho_r) - \beta(\varrho_r)\\ &\leq \mathbb{F}(\varrho_{r-1}) - \beta(\varrho_r) - \beta(\varrho_{r-1})\\ &\vdots\\ &\leq \mathbb{F}(\varrho_0) - \beta(\varrho_r) - \beta(\varrho_{r-1}) - \ldots - \beta(\varrho_0).\end{aligned} \tag{8}$$

Let q_r be the greatest number in $\{0, 1, \ldots, r-1\}$ such that

$$\beta(\varrho_{q_r}) = \min\{\beta(\varrho_0), \beta(\varrho_1), \ldots, \beta(\varrho_r)\}$$

for all $r \in \mathbb{N}$. In this case, $\{q_r\}$ is a nondecreasing sequence. From (8) we get

$$\mathbb{F}(\varrho_r) \leq \mathbb{F}(\varrho_0) - r\beta(\varrho_{q_r}). \tag{9}$$

Now consider the sequence $\{\beta(\varrho_{q_r})\}$. We distinguish two cases.

Case 1: For each $r \in \mathbb{N}$ there is $s > r$ such that $\beta(\varrho_{q_r}) > \beta(\varrho_{q_s})$. Then we obtain a subsequence $\{\varrho_{q_{r_k}}\}$ of $\{\varrho_{q_r}\}$ with $\beta(\varrho_{q_{r_k}}) > \beta(\varrho_{q_{r_{k+1}}})$ for all k. Since $\varrho_{q_{r_k}} \to \delta$ we deduce that

$$\liminf_{k\to\infty} \beta(\varrho_{q_{r_k}}) > 0.$$

Hence

$$\mathbb{F}(\varrho_{r_k}) \leq \mathbb{F}(\varrho_0) - r^k\beta(\varrho_{q_{r_k}}) \text{ for all } k.$$

Consequently, $\lim_{k\to\infty} \mathbb{F}(\varrho_{r_k}) = -\infty$ and by (F2), $\lim_{k\to\infty} \varrho_{r_k} = 0$ which contradicts the fact that $\lim_{k\to\infty} \varrho_{r_k} > 0$.

Case 2: There is $r_0 \in \mathbb{N}$ such that $\beta(\varrho_{q_0}) > \beta(\varrho_{q_s})$ for all $s > r_0$. Then $\mathbb{F}(\varrho_s) \leq \mathbb{F}(\varrho_0) - s\beta(\varrho_{q_{r_0}})$ for all $s > r_0$. Hence, $\lim_{s\to\infty} \mathbb{F}(\varrho_s) = -\infty$ and by (F2), $\lim_{s\to\infty} \varrho_s = 0$, which contradicts the fact that $\lim_{s\to\infty} \varrho_s > 0$.

Therefore in both the cases

$$\lim_{r\to\infty} \varrho_r = 0.$$

Now, from (F3), there exists $k \in (0, 1)$ such that

$$\lim_{r\to\infty} (\varrho_r)^k \mathbb{F}(\varrho_r) = 0.$$

By (9), the following holds for all $r \in \mathbb{N}$:

$$\begin{aligned}(\varrho_r)^k\mathbb{F}(\varrho_r) - (\varrho_r)^k\mathbb{F}(\varrho_0) &\leq (\varrho_r)^k(\mathbb{F}(\varrho_0) - r\beta(\varrho_{q_r})) - (\varrho_r)^k(\mathbb{F}(\varrho_0)\\ &= -r(\varrho_r)^k\beta(\varrho_{q_r}) \leq 0.\end{aligned} \tag{10}$$

Passing to the limit as $r \to \infty$ in (10), we obtain

$$\lim_{r\to\infty} r(\varrho_r)^k\beta(\varrho_{q_r}) = 0.$$

Since $\zeta := \liminf_{r\to\infty} \beta(\varrho_{q_r}) > 0$, there exists $r_0 \in \mathbb{N}$ such that $\beta(\varrho_{q_r}) > \frac{\zeta}{2}$ for all $r \neq r_0$. Thus,

$$r(\varrho_r)^k \frac{\zeta}{2} < r(\varrho_r)^k \beta(\varrho_{q_r}) \tag{11}$$

for all $r \geq r_0$. Letting $r \to \infty$ in (11), we have $0 \leq \lim_{r\to\infty} r(\varrho_r)^k \frac{\zeta}{2} < \lim_{r\to\infty} r(\varrho_r)^k \beta(\varrho_{q_r}) = 0$, that is,

$$\lim_{r\to\infty} r(\varrho_r)^k = 0 \tag{12}$$

From (12), there exits $r_1 \in \mathbb{N}$ such that $r(\varrho_r)^k \leq 1$ for all $r \geq r_1$. So, we have, for all $r \geq r_1$,

$$\varrho_r \leq \frac{1}{r^{1/k}}. \tag{13}$$

In order to show that $\{\nu_r\}$ is a Cauchy sequence consider $s, r \in \mathbb{N}$ such that $s > r \geq r_1$. Using the triangular inequality for ω and from (13), we have

$$\begin{aligned}
\omega(\nu_r, \nu_s) &\leq \omega(\nu_r, \nu_{r+1}) + \omega(\nu_{r+1}, \nu_{r+2}) + \cdots + \omega(\nu_{s-1}, \nu_s) \\
&\leq \varrho_r + \varrho_{r+1} + \cdots + \varrho_{s-1} \\
&= \sum_{i=r}^{s-1} \varrho_i \leq \sum_{i=r}^{\infty} \varrho_i \leq \sum_{r=1}^{\infty} \frac{1}{r^{1/k}}.
\end{aligned}$$

By the convergence of the series $\sum_{r=1}^{\infty} \frac{1}{r^{1/k}}$, passing to the limit as $r \to \infty$, we get $\omega(\nu_r, \nu_s) \to 0$ and by Lemma 1.12, $\{\nu_r\}$ is a Cauchy sequence in Ξ.

Since Ξ is a orbitally complete metric space, there exists $\varrho \in \Xi$ such that $\nu_r \to \varrho$ as $n \to \infty$. Also, $\alpha(\nu_r, \nu_{r+1}) \geq \eta(\nu_r, \nu_{r+1})$. So, using condition (e), we get $\alpha(\nu_r, \varrho) \geq \eta(\nu_r, \varrho)$. Consequentially, from (9) and (F2) we have

$$\lim_{r\to\infty} \omega(\nu_r, \Im\nu_r) = 0.$$

Since $\nu \mapsto \omega(\nu, \Im\nu)$ is orbitally l.s.c.,

$$0 \leq \omega(\varrho, \Im\varrho) \leq \omega(\nu_r, \Im\nu_r) \to 0.$$

This proves $\omega(\varrho, \Im\varrho) = 0$. Since $\omega(\varrho, \varrho) = 0$ and $\Im\varrho$ is closed, by Lemma 1.13, $\varrho \in \Im\varrho$. □

Theorem 2.2. *The conclusion of Theorem 2.1 remains true if the condition* (e) *is replaced by the following one:*

(e') *for every* $\vartheta \in \Xi$ *with* $\vartheta \notin \Im\vartheta$, $\inf\{\omega(\nu, \vartheta) + \omega(\nu, \Im\nu) \mid \nu \in \Xi\} > 0$.

Proof. By Theorem 2.1, we get a sequence $\{\nu_n\}$ converging to $\varrho \in \Xi$. Assume that $\varrho \notin \Im\varrho$. Since for each $\nu \in \Xi$, the mapping $\omega(\nu, \Im\nu) : \Xi \to [0, +\infty)$ is l.s.c, for every $n > n_0$, we get

$$\omega(\nu_n, \varrho) \leq \liminf_{m\to\infty} \omega(\nu_n, \nu_m) \leq \sum_{r=1}^{\infty} \frac{1}{r^{1/k}}.$$

Now, by (e') and the above inequality, we get

$$\begin{aligned}
0 &< \inf\{\omega(\nu, \varrho) + \omega(\nu, \Im(\nu)) : \nu \in \Xi\} \\
&\leq \inf\{\omega(\nu_n, \varrho) + \omega(\nu_n, \Im(\nu_n)) : n > n_0\} \\
&\leq \inf\{2 \sum_{r=1}^{\infty} \frac{1}{r^{1/k}} : n > n_0\} \\
&= \lim_{r\to\infty} 2 \sum_{r=1}^{\infty} \frac{1}{r^{1/k}} = 0.
\end{aligned}$$

which contradicts our assumption. Therefore, $\varrho \in \Im\varrho$. □

If we take $\omega = d$ in Theorem 2.1, we get the following result.

Theorem 2.3. *Let (Ξ, d) be a orbitally complete metric space and $\Im : \Xi \to \mathcal{P}_{cl}(\Xi)$. Assume that*

(a) *the mapping $\nu \mapsto d(\nu, \Im\nu)$ is orbitally l.s.c.;*
(b) *$\Im$ is a muti-valued generalized α_*-admissible with respect to an η mapping;*
(c) *there exist functions $\theta, \tau : (0, \infty) \to (0, \infty)$ such that*

$$\theta(\varsigma) > \tau(\varsigma), \quad \liminf_{t \to \varsigma^+} \theta(t) > \liminf_{t \to \varsigma^+} \tau(t) \text{ for all } \varsigma \geq 0;$$

(d) *for any $\nu \in \Xi$ with $d(\nu, \Im\nu) > 0$, there exists $\vartheta \in \mathbb{G}_\tau^\nu$ with $\alpha(\nu, \vartheta) \geq \eta(\nu, \vartheta)$ satisfying*

$$\theta(\max\{d(\nu, \Im\nu), d(\vartheta, \Im\vartheta)\}) + \mathbb{F}(d(\vartheta, \Im\vartheta)) \leq \mathbb{F}(d(\nu, \vartheta));$$

where $\mathbb{G}_\tau^\nu = \left\{ \begin{array}{c} \vartheta \in \Im\nu : \mathbb{F}(d(\nu, \vartheta)) \leq \mathbb{F}(\max\{d(\nu, \Im\nu), d(\vartheta, \Im\vartheta)\}) \\ +\tau(\max\{d(\nu, \Im\nu), d(\vartheta, \Im\vartheta)\}) \end{array} \right\}.$

(e) *if $\{\nu_n\} \subset \Xi$ with $\nu_{n+1} \in \Im\nu_n$, $\nu_n \to \nu \in \Xi$ as $n \to \infty$ and $\alpha(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$ for all $n \in \mathbb{N}$ then $\alpha(\nu_n, \nu) \geq \eta(\nu_n, \nu)$ for all $n \in \mathbb{N}$.*

Then $\Im$ has a fixed point in Ξ.

The following result is an application of the above theorem.

Theorem 2.4. *Let (Ξ, d) be a orbitally complete metric space and $\Im : \Xi \to C(\Xi)$ a continuous mapping. Assume that*

(a) *$\Im$ is a muti-valued generalized α_*-admissible with respect to an η mapping;*
(b) *there exist functions $\theta, \tau : (0, \infty) \to (0, \infty)$ such that*

$$\theta(\varsigma) > \tau(\varsigma), \quad \liminf_{t \to \varsigma^+} \theta(t) > \liminf_{t \to \varsigma^+} \tau(t) \text{ for all } \varsigma \geq 0;$$

(c) *for any $\nu \in \Xi$ there exists $\vartheta \in \Xi$ with $\mathcal{H}(\Im\nu, \Im\vartheta) > 0$ and $\alpha(\nu, \vartheta) \geq \eta(\nu, \vartheta)$ satisfying*

$$\theta(\max\{d(\nu, \Im\nu), d(\vartheta, \Im\vartheta)\}) + \mathbb{F}(\mathcal{H}(\Im\nu, \Im\vartheta)) \leq \mathbb{F}(d(\nu, \vartheta)),$$

where $\mathcal{H}$ is generalized Pompeiu Hausdorff metric, i.e.,

$$\mathcal{H}(A, B) = \max\left\{\sup_{\nu \in A} d(\nu, B), \sup_{\vartheta \in B} d(\vartheta, A)\right\};$$

(d) *if $\{\nu_n\} \subset \Xi$ with $\nu_{n+1} \in \Im\nu_n$, $\nu_n \to \nu \in \Xi$ as $n \to \infty$ and $\alpha(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$ for all $n \in \mathbb{N}$ then $\alpha(\nu_n, \nu) \geq \eta(\nu_n, \nu)$ for all $n \in \mathbb{N}$.*

Then $\Im$ has a fixed point in Ξ.

Proof. Since $\Im$ is continuous it is l.s.c. Therefore $d(\nu, \Im\nu)$ is l.s.c. Also,

$$\begin{aligned} \theta(m(x, y)) + d(\vartheta, \Im\vartheta) &\leq \theta(m(x, y)) + \mathbb{F}(\mathcal{H}(\Im\nu, \Im\vartheta)) \\ &\leq \mathbb{F}(d(\nu, \vartheta)), \end{aligned}$$

where $m(x, y) = \max\{d(\nu, \Im\nu), d(\vartheta, \Im\vartheta)\}$. Thus all the conditions of Theorem 2.1 are satisfied. Therefore $\Im$ has a fixed point in Ξ. □

If Ξ is complete, $\theta(s) = k > 0$ (a constant) and $\alpha(\nu, \vartheta) = \eta(\nu, \vartheta) = 1$ in the above theorem then we get the following result.

Theorem 2.5. *Let (Ξ, d) be a complete metric space and $\Im : \Xi \to C(\Xi)$ a continuous mapping. Assume that for any $\nu \in \Xi$ there exists $\vartheta \in \Xi$ with $\mathcal{H}(\Im\nu, \Im\vartheta) > 0$*

$$k + \mathbb{F}(\mathcal{H}(\Im\nu, \Im\vartheta)) \leq \mathbb{F}(d(\nu, \vartheta)).$$

Then $\Im$ has a fixed point in Ξ.

3. Implicit type Feng-Liu results

Denote $\Phi := \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$ satisfying the following conditions:

(a) φ is increasing and $\varphi(0) = 0$;
(b) $\sum_{n=1}^{\infty} \varphi^n(\zeta) < \infty$, for $\zeta > 0$; where φ^n is the n-th iterate.

It should be noted that $\varphi(\zeta) < \zeta$ and the family $\Phi \neq \emptyset$.

Example 3.1. *Consider $\Xi = [0,1]$ with usual distance. Define the mapping $\varphi(\zeta) = \frac{3\lambda\zeta}{7}$, where $0 < \lambda < 1$. Then we have $\varphi^n(\zeta) = \frac{3^n\lambda^n\zeta}{7^n}$. Therefore, $\sum_{n=1}^{\infty} \varphi^n(\zeta) = \sum_{n=1}^{\infty} \frac{3^n\lambda^n\zeta}{7^n} < \infty$ and hence $\Phi \neq \emptyset$.*

We consider a family of functions $\Lambda := \{\psi : \mathbb{R}^5 \to \mathbb{R}\}$ satisfying the properties:

(ψ_1) ψ is non-decreasing in the fourth variable;
(ψ_2) if $\vartheta, \nu, \mu \in \mathbb{R}_+$ satisfy $\vartheta \le \psi(\nu, \nu, \vartheta, \nu + \vartheta, \mu)$, then there exists $\varphi \in \Phi$ such that $\vartheta \le \varphi(\nu)$.

Example 3.2. *Let $\psi(q_1, q_2, q_3, q_4, q_5) = (aq_1^2 - b\frac{q_2^2+q_3^2}{q_4+q_5+1})^{1/2}$, $1/2 < a < 1$ and $0 < b < 1/2$.*

(ψ_1) ψ is non-decreasing in the fourth variable.
(ψ_2) For $\vartheta, \nu, \mu \in \mathbb{R}_+$, we have

$$\vartheta \le \psi(\nu, \nu, \vartheta, \vartheta + \nu, \mu) = \left(a\nu^2 - b\frac{\vartheta^2 + \nu^2}{1 + \vartheta + \nu + \mu}\right)^{1/2}.$$

It is clear that $\vartheta \le \varphi(\nu)$, where $\varphi(\nu) = h\nu$ and $h = \sqrt{a} < 1$.

Example 3.3. *Let $\psi(q_1, q_2, q_3, q_4, q_5) = (aq_1^2 - b\frac{q_2^2+q_3^2}{q_4^2+q_5^2+1})^{1/2}$, $1/2 < a < 1$ and $0 < b < 1/2$.*

(ψ_1) ψ is non-decreasing in the fourth variable.
(ψ_2) For $\vartheta, \nu, \mu \in \mathbb{R}_+$, we have

$$\vartheta \le \psi(\nu, \nu, \vartheta, \vartheta + \nu, \mu) = \left(a\nu^2 - b\frac{\vartheta^2 + \nu^2}{1 + (\vartheta + \nu)^2 + \mu^2}\right)^{1/2}.$$

It is clear that $\vartheta \le \varphi(\nu)$, where $\varphi(\nu) = h\nu$ and $h = \sqrt{a} < 1$.

Example 3.4. *Let $\psi(q_1, q_2, q_3, q_4, q_5) = hq_2$ where $h \in [0, 1)$. Then*

(ψ_1) ψ is non-decreasing in the fourth variable.
(ψ_2) If $\vartheta \le \psi(\nu, \nu, \vartheta, \nu + \vartheta, \mu)$ for some $\vartheta, \nu, \mu \in \mathbb{R}_+$ then $\vartheta \le \varphi(\nu)$ where $\varphi(\nu) = h\nu$.

Example 3.5. *Let $\psi(q_1, q_2, q_3, q_4, q_5) = a\max\{q_1, q_2, q_3\} + bq_4$ with $a, b \ge 0$ and $a + 2b < 1$.*

(ψ_1) ψ is non-decreasing in the fourth variable.
(ψ_2) Let $\vartheta \le \psi(\nu, \nu, \vartheta, \zeta + \nu, \mu)$ for some $\vartheta, \nu, \mu \in \mathbb{R}_+$.
If $\vartheta > \nu$, we get

$$\vartheta \le \left(\frac{b}{1 - a - b}\right)\nu$$

a contradiction. If $\vartheta \le \nu$, we get

$$\vartheta \le \left(\frac{a + b}{1 - b}\right)\nu.$$

Now, there exists a $\varphi \in \Phi$ defined by $\varphi(\nu) = \left(\frac{a+b}{1-b}\right)\nu$ such that $\vartheta \le \varphi(\nu)$.

Let $\mathfrak{I} : \Xi \to \mathcal{P}_{cl}(\Xi)$ be a multi-valued mapping, $\psi \in \Lambda$. We define the set $\Upsilon(\varrho) \subseteq \Xi$ for $\varrho \in \Xi$ with $f(\varrho) = \omega(\varrho, \mathfrak{I}\varrho) > 0$ as

$$\Upsilon(\varrho) = \{\, \vartheta \in \mathfrak{I}\varrho : \omega(\varrho, \vartheta) \leq \max\{\omega(\varrho, \mathfrak{I}\varrho), \omega(\vartheta, \mathfrak{I}\vartheta)\}\,\}.$$

Theorem 3.6. *Let (Ξ, d) be a metric space with w-distance ω and $\mathfrak{I} : \Xi \to \mathcal{P}_{cl}(\Xi)$. Assume that*

(I_1) *the mapping $\varrho \mapsto f(\varrho)$ is orbitally l.s.c.;*
(I_2) *there exists $\nu_0 \in \Xi$ and $\nu_1 \in \mathfrak{I}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq \eta(\nu_0, \nu_1)$;*
(I_3) *$\mathfrak{I}$ is a muti-valued generalized α_*-admissible with respect to an η mapping;*
(I_4) *(Ξ, d) is $\mathfrak{I}$-orbitally complete at ν_0;*
(I_5) *for any $\varrho \in \Xi$ with $f(\varrho) > 0$, there exist $\vartheta \in \Upsilon(\varrho)$ and $\psi \in \Lambda$ satisfying*

$$\omega(\vartheta, \mathfrak{I}\vartheta) \leq \psi\left(\begin{array}{c} \omega(\varrho, \vartheta), \omega(\varrho, \mathfrak{I}\varrho), \omega(\vartheta, \mathfrak{I}\vartheta), \\ \omega(\varrho, \mathfrak{I}\vartheta), \omega(\vartheta, \mathfrak{I}\varrho) \end{array} \right);$$

(I_6) *if $\{\nu_n\} \subset \Xi$ with $\nu_{n+1} \in \mathfrak{I}\nu_n$, $\nu_n \to \nu \in \Xi$ as $n \to \infty$ and $\alpha(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(\nu_n, \nu) \geq \eta(\nu_n, \nu)$ for all $n \in \mathbb{N}$.*

Then there exists $\varrho \in \Xi$ such that $\omega(\varrho, \mathfrak{I}\varrho) = 0$. Further, if $\omega(\varrho, \varrho) = 0$ then $\varrho \in \mathfrak{I}\varrho$.

Proof. Suppose that for all $\nu \in \Xi$, we have $\omega(\nu, \mathfrak{I}\nu) > 0$. By ($I_2$) there exist $\nu_0 \in \Xi$ and $\nu_1 \in \Upsilon(\nu_0)$ with $\alpha(\nu_0, \nu_1) \geq \eta(\nu_0, \nu_1)$ such that

$$\omega(\nu_1, \mathfrak{I}\nu_1) \quad \leq \quad \psi\left(\begin{array}{c} \omega(\nu_0, \nu_1), \omega(\nu_0, \mathfrak{I}\nu_0), \omega(\nu_1, \mathfrak{I}\nu_1), \\ \omega(\nu_0, \mathfrak{I}\nu_1), \omega(\nu_1, \mathfrak{I}\nu_0) \end{array} \right).$$

For $\nu_1 \in \Xi$ with $\nu_1 \in \mathfrak{I}(\nu_0)$, $\alpha(\nu_0, \nu_1) \geq \eta(\nu_0, \nu_1)$, and there exists $\nu_2 \in \Upsilon(\nu_1)$ with $\nu_2 \in \mathfrak{I}(\nu_1)$. From ($I_3$), we have $\alpha(\nu_1, \nu_2) \geq \eta(\nu_1, \nu_2)$ and hence from (I_5)

$$\omega(\nu_2, \mathfrak{I}\nu_2) \leq \psi\left(\begin{array}{c} \omega(\nu_1, \nu_2), \omega(\nu_1, \mathfrak{I}\nu_1), \omega(\nu_2, \mathfrak{I}\nu_2), \\ \omega(\nu_1, \mathfrak{I}\nu_2), \omega(\nu_2, \mathfrak{I}\nu_1) \end{array} \right).$$

Continuing this process, we get an iterative sequence $\{\nu_r\}$, where $\nu_{r+1} \in \Upsilon(\nu_r)$, $\nu_{r+1} \notin \mathfrak{I}\nu_{r+1}$ with $\alpha(\nu_r, \nu_{r+1}) \geq \eta(\nu_r, \nu_{r+1})$ and

$$\omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}) \leq \psi\left(\begin{array}{c} \omega(\nu_r, \nu_{r+1}), \omega(\nu_r, \mathfrak{I}\nu_r), \omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}), \\ \omega(\nu_r, \mathfrak{I}\nu_{r+1}), \omega(\nu_{r+1}, \mathfrak{I}\nu_r) \end{array} \right).$$

Using (ψ_1) we obtain

$$\omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}) \leq \psi\left(\begin{array}{c} \omega(\nu_r, \nu_{r+1}), \omega(\nu_r, \mathfrak{I}\nu_r), \omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}), \\ \omega(\nu_r, \nu_{r+1}) + (\nu_{r+1}, \mathfrak{I}\nu_{r+1}), \omega(\nu_{r+1}, \mathfrak{I}\nu_r) \end{array} \right).$$

It follows from (ψ_2) that there is $\varphi \in \Phi$ such that

$$\omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}) \leq \varphi(\omega(\nu_r, \nu_{r+1})). \tag{14}$$

We now show that the sequence $\{\nu_r\}$ is a Cauchy. Since $\nu_{r+1} \in \Upsilon(\nu_r)$, by the definition of $\Upsilon(\nu_r)$,

$$\omega(\nu_r, \nu_{r+1}) \leq \max\{\omega(\nu_r, \mathfrak{I}\nu_r), \omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1})\} \tag{15}$$

Put $\sigma_r = \omega(\nu_r, \nu_{r+1})$ for $r \in \mathbb{N}$. Then $\sigma_r > 0$. From (14) and (15) we have

$$\omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1}) \leq \varphi(\max\{\omega(\nu_r, \mathfrak{I}\nu_r), \omega(\nu_{r+1}, \mathfrak{I}\nu_{r+1})\})$$

i.e.,

$$\sigma_{r+1} \le \varphi(\max\{\sigma_r, \sigma_{r+1}\}).$$

If $\sigma_r \le \sigma_{r+1}$, then we have

$$\sigma_{r+1} \le \varphi(\sigma_{r+1}) < \sigma_{r+1},$$

a contradiction. Thus $\sigma_r > \sigma_{r+1}$ for all $r \in \mathbb{N}$ and

$$\sigma_{r+1} \le \varphi(\sigma_r). \tag{16}$$

From (16) and using the triangular inequality, for all $r, s \in \mathbb{N}$ with $s > r$,

$$\begin{aligned} \omega(\nu_r, \nu_{r+s}) &\le \omega(\nu_r, \nu_{r+1}) + \omega(\nu_{r+1}, \nu_{n+2}) + \ldots + \omega(\nu_{s-1}, \nu_s) \\ &\le \sum_{k=r}^{s} \varphi^k(\omega(\nu_0, \nu_1) \\ &\le \sum_{k\ge r} \varphi^k(\omega(\nu_0, \nu_1) \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Therefore, $\{\nu_r\}$ is a Cauchy sequence in $O(\nu_0, \mathfrak{I})$.

Since Ξ is a $\mathfrak{I}$-orbitally complete, there exists an $\varrho \in \Xi$ such that $\nu_r \to \varrho$ as $r \to \infty$. Consequentially, from (16), $\lim_{r\to\infty} \omega(\nu_r, \mathfrak{I}\nu_r) = 0$. Since $\nu \mapsto f(\nu)$ is orbitally l.s.c.,

$$0 \le \omega(\varrho, \mathfrak{I}\varrho) \le \liminf_{r\to\infty} \omega(\nu_r, \mathfrak{I}\nu_r) = 0.$$

This proves $\omega(\varrho, \mathfrak{I}\varrho) = 0$. Since $\omega(\varrho, \varrho) = 0$ and $\mathfrak{I}\varrho$ is closed, by Lemma 1.13, $\varrho \in \mathfrak{I}\varrho$. □

Our second result is related to multi-valued mappings $\mathfrak{I}$ on the metric space Ξ, where $\mathfrak{I}\nu$ is compact for all $\nu \in \Xi$.

Theorem 3.7. *The conclusion of Theorem 3.6 remains true if* $\mathfrak{I} : \Xi \to C(\Xi)$.

Another result is as follows.

Theorem 3.8. *The conclusion of Theorem 3.6 (or Theorem 3.7) remains true if the condition* (I_6) *is replaced by the* (e').

Proof. We refer the proof of Theorem 3.6. □

4. Ordered version of Feng-Liu results

We shall now consider spaces equipped with a partial order. We say $(\Xi, d, \sqsubseteq)$ an ordered metric space if:

(i) (Ξ, d) is a metric space,

(ii) $(\Xi, \sqsubseteq)$ is a partially ordered set.

Elements $\nu, \vartheta \in \Xi$ are called comparable if $\nu \sqsubseteq \vartheta$ or $\vartheta \sqsubseteq \nu$ holds.

A multi-valued mapping $\mathfrak{I} : (\Xi, d, \sqsubseteq) \to 2^{\Xi}$ is said to be $\sqsubseteq$-weakly comparative if, for each $\nu \in \Xi$ and $\vartheta \in \mathfrak{I}\nu$ with $\nu \sqsubseteq \vartheta$, we have $\vartheta \sqsubseteq \zeta$ for all $\zeta \in \mathfrak{I}\vartheta$.

We define the set $\Upsilon(\varrho, \sqsubseteq) \subseteq \Xi$ for $\varrho \in \Xi$ with $f(\varrho) > 0$ as

$$\Upsilon(\varrho, \sqsubseteq) = \{\,\vartheta \in \mathfrak{I}\varrho : \omega(\varrho, \vartheta) \le \max\{\omega(\varrho, \mathfrak{I}\varrho), \omega(\vartheta, \mathfrak{I}\vartheta)\},\ \varrho \sqsubseteq \vartheta\}.$$

Theorem 4.1. *Let $(\Xi, d, \sqsubseteq)$ be a ordered metric space with w-distance ω and $\Im : \Xi \to \mathcal{P}_{cl}(\Xi)$. Assume that*

(i) *the mapping $\varrho \mapsto f(\varrho)$ is ordered orbitally l.s.c.;*
(ii) *there exist $\nu_0 \in \Xi$ and $\nu_1 \in \Im\nu_0$ such that $\nu_0 \sqsubseteq \nu_1$;*
(iii) *$\Im$ is $\sqsubseteq$-weakly comparative;*
(iv) *(Ξ, d) is $\Im$-orbitally complete at ν_0;*
(v) *for any $\varrho \in \Xi$ with $f(\varrho) > 0$, there exist $\vartheta \in \Upsilon(\varrho)$ and $\psi \in \Lambda$ satisfying*

$$\omega(\vartheta, \Im\vartheta) \leq \psi\begin{pmatrix} \omega(\varrho, \vartheta), \omega(\varrho, \Im\varrho), \omega(\vartheta, \Im\vartheta), \\ \omega(\varrho, \Im\vartheta), \omega(\vartheta, \Im\varrho) \end{pmatrix}.$$

If the condition

$$\begin{cases} \text{if } \{\nu_n\} \subset \Xi \text{ with } \nu_{n+1} \in \Im\nu_n, \nu_n \to \zeta \text{ in } \Xi \\ \text{as } n \to \infty, \text{ then } \nu_n \sqsubseteq \zeta \text{ for all } n \end{cases} \tag{17}$$

holds. Then there exists $\varrho \in \Xi$ such that $\omega(\varrho, \Im\varrho) = 0$. Further, if $\omega(\varrho, \varrho) = 0$ then $\varrho \in \Im\varrho$.

Proof. Following proof of Theorem 3.6 and the fact that $\Upsilon(\nu, \sqsubseteq) \subseteq \Xi$, we can show that $\{\nu_n\}$ is a Cauchy sequence in $(\Xi, d, \sqsubseteq)$ with $\nu_{n-1} \sqsubseteq \nu_n$ for $n \in \mathbb{N}$. From the completeness of Ξ, there exist a $\zeta \in \Xi$ such that $\nu_n \to \zeta$ as $n \to +\infty$. By assumption (17), $\nu_n \sqsubseteq \zeta$, for all n. The rest of the proof follows in the same way as the proof of Theorem 3.6. □

5. Binary relation version of Feng-Liu results

Let $(\Xi, d, \Re)$ be a binary metric space, where $\Re$ is a binary relation over Ξ. Define $\mathbb{S} := \Re \cup \Re^{-1}$. It is easy to see that, for all $\nu, \vartheta \in \Xi$, $(\nu, \vartheta) \in \mathbb{S} \Leftrightarrow (\nu, \vartheta) \in \Re$ or $(\vartheta, \nu) \in \Re$.
Let Ξ be a nonempty set and $\Re$ be a binary relation over Ξ. A multi-valued mapping $\Im : \Xi \to 2^{\Xi}$ is said to be $\Re$-weakly comparative if, for each $\nu \in \Xi$ and $\vartheta \in \Im\nu$ with $(\nu, \vartheta) \in \mathbb{S}$, we have $(\vartheta, \zeta) \in \mathbb{S}$ for all $\zeta \in \Im\vartheta$.
A function $f : (\Xi, d, \Re) \to \mathbb{R}$ is called binary orbitally l.s.c. if $f(\nu) \leq \liminf_{n\to\infty} f(\nu_n)$ for all sequences $\{\nu_n\}$ in Ξ with $(\Im\nu_n, \Im\nu_{n+1}) \in \mathbb{S}$ for all $n \geq 1$ and $\lim_{n\to\infty} \nu_n = \nu \in \Xi$.

We define the set $\Upsilon(\varrho, \sqsubseteq) \subseteq \Xi$ for $\varrho \in \Xi$ with $f(\varrho) > 0$ and a binary relation $\Re$, as

$$\Upsilon(\vartheta, \Re) = \{\, \vartheta \in \Im\varrho : \omega(\varrho, \vartheta) \leq \max\{\omega(\varrho, \Im\varrho), \omega(\vartheta, \Im\vartheta)\},\ (\varrho, \vartheta) \in \mathbb{S}\}.$$

Theorem 5.1. *Let $(\Xi, d, \Re)$ be a binary metric space with w-distance ω and $\Im : \Xi \to \mathcal{P}_{cl}(\Xi)$. Assume that*

(i) *the mapping $\varrho \mapsto f(\varrho)$ is binary orbitally l.s.c.;*
(ii) *there exist $\nu_0 \in \Xi$ and $\nu_1 \in \Im\nu_0$ such that $(\nu_0, \nu_1) \in \mathbb{S}$;*
(iii) *$\Im$ is an $\Re$-weakly comparative mapping;*
(iv) *(Ξ, d) is $\Im$-orbitally complete at ν_0;*
(v) *for any $\varrho \in \Xi$ with $f(\varrho) > 0$, there exist $\vartheta \in \Upsilon(\varrho)$ and $\psi \in \Lambda$ satisfying*

$$\omega(\vartheta, \Im\vartheta) \leq \psi\begin{pmatrix} \omega(\varrho, \vartheta), \omega(\varrho, \Im\varrho), \omega(\vartheta, \Im\vartheta), \\ \omega(\varrho, \Im\vartheta), \omega(\vartheta, \Im\varrho) \end{pmatrix}.$$

If the condition

$$\begin{cases} \text{if } \{\nu_n\} \subset \Xi \text{ with } \nu_{n+1} \in \Im\nu_n, \nu_n \to \zeta \text{ in } \Xi \\ \text{as } n \to +\infty, \text{ then } (\nu_n, \zeta) \in \mathbb{S} \text{ for all } n \end{cases} \tag{18}$$

holds. Then there exists $\varrho \in \Xi$ such that $\omega(\varrho, \Im\varrho) = 0$. Further, if $\omega(\varrho, \varrho) = 0$ then $\varrho \in \Im\varrho$.

If $\mathfrak{I}$ is single valued, Ξ is complete in the above theorem then we get the following result.

Theorem 5.2. *Let $(\Xi, d, \mathfrak{R})$ be a binary complete metric space and $\mathfrak{I}: \Xi \to \Xi$ a continuous mapping such that*

(a) *there exist $\nu_0 \in \Xi$ such that $(\nu_0, \mathfrak{I}\nu_0) \in \mathbb{S}$;*
(b) *$\mathfrak{I}$ is an $\mathfrak{R}$-weakly comparative mapping, that is, for $\nu, \vartheta \in \Xi$ with $(\nu, \vartheta) \in \mathbb{S}$, we have $(\mathfrak{I}\nu, \mathfrak{I}\vartheta) \in \mathbb{S}$;*
(c) *for any $\nu \in \Xi$ there exists $\vartheta \in \Xi$ with $(\nu, \vartheta) \in \mathbb{S}$ and $d(\mathfrak{I}\nu, \mathfrak{I}\vartheta) > 0$ satisfying*

$$d(\mathfrak{I}\nu, \mathfrak{I}\vartheta) \leq \psi\begin{pmatrix} d(\nu, \vartheta), d(\nu, \mathfrak{I}\nu), d(\vartheta, \mathfrak{I}\vartheta), \\ d(\nu, \mathfrak{I}\vartheta), d(\vartheta, \mathfrak{I}\nu) \end{pmatrix}.$$

Then $\mathfrak{I}$ has a fixed point in Ξ.

6. Examples

In this section, we present some illustrative examples.

Example 6.1. *Let $\Xi = [0, \infty)$ be equipped with the usual metric d and ω a w-distance on Ξ defined by $\omega(\nu, \vartheta) = \max\{\nu, \vartheta\}$. Define $\mathbb{F}(t) = \ln t$, $\theta(t) = k$, $\tau(t) = \frac{9k}{10}$ with $k \in (0, \ln 2]$ and*

$$\mathfrak{I}\nu = \begin{cases} [\frac{\nu^2}{2}, \frac{\nu}{2}], & \text{if } \nu \in [0,1) \\ [\frac{1}{9}, \frac{1}{4}], & \text{otherwise,} \end{cases}$$

and

$$\alpha(\nu, \vartheta) = 3 \text{ and } \eta(\nu, \vartheta) = 2 \text{ for all } \nu, \vartheta \in \Xi.$$

Then $\omega(\nu, \mathfrak{I}\nu) = \nu$ is continuous on Ξ and hence orbitally l.s.c. on Ξ. So, condition (a) of Theorem 2.1 is satisfied. It is trivial to verify that conditions (b), (c) and (e) also hold.

To verify condition (d), we consider following two cases:

Case 1 $\nu \in [0, 1)$. *Take $\vartheta = \frac{\nu}{2} \in \mathfrak{I}\nu$. Then $\vartheta \in \mathbb{F}_\tau^\nu$, since*

$$\begin{aligned} \mathbb{F}(\omega(\nu, \vartheta)) &= \mathbb{F}(\nu) \leq \mathbb{F}(\nu) + \frac{9k}{10} \\ &= \mathbb{F}(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}) + \tau(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}). \end{aligned}$$

Also

$$\begin{aligned} \theta(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}) + \mathbb{F}(\omega(\vartheta, \mathfrak{I}\vartheta)) &= k + \mathbb{F}\left(\frac{\nu}{2}\right) \\ &\leq \mathbb{F}(\nu) = \mathbb{F}(\omega(\nu, \vartheta)). \end{aligned}$$

Case 2 $\nu \in [1, \infty)$. *Take $\vartheta = \frac{1}{4} \in \mathfrak{I}\nu$. Then $\vartheta \in \mathbb{F}_\tau^\nu$, since*

$$\begin{aligned} \mathbb{F}(\omega(\nu, \vartheta)) &= \mathbb{F}(\nu) \leq \mathbb{F}(\nu) + \frac{9k}{10} \\ &= \mathbb{F}(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}) + \tau(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}). \end{aligned}$$

Also

$$\begin{aligned} \theta(\max\{\omega(\nu, \mathfrak{I}\nu), \omega(\vartheta, \mathfrak{I}\vartheta)\}) + \mathbb{F}(\omega(\vartheta, \mathfrak{I}\vartheta)) &= k + \mathbb{F}\left(\frac{1}{4}\right) \\ &\leq \mathbb{F}(\nu) = \mathbb{F}(\omega(\nu, \vartheta)). \end{aligned}$$

Finally, there exists $0 \in \Xi$ *such that* $\omega(0,0) = 0$. *Therefore all the conditions of Theorem 2.1 are satisfied and* $0 \in \Im 0$.

Example 6.2. *Let* $\Xi, d, \mathbb{F}, \theta, \tau, \alpha, \eta$ *and* $\Im$ *be as in Example 6.1. Let* ω *be a w-distance on* Ξ *defined by* $\omega(\nu, \vartheta) = \vartheta$. *Then* $\omega(\nu, \Im\nu) = \begin{cases} \frac{\nu^2}{2}, & \text{if } \nu \in [0,1) \\ \frac{1}{9}, & \text{otherwise.} \end{cases}$ *is orbitally l.s.c. on* Ξ. *So, condition (a) of Theorem 2.1 is satisfied. It is trivial to verify that conditions (b), (c) and (e) also hold.*

To verify condition (d), we consider following two cases:

Case 1 $\nu \in [0,1)$. *Take* $\vartheta = \frac{\nu^2}{2} \in \Im\nu$. *Then* $\vartheta \in \mathbb{F}_\tau^\nu$, *since*

$$
\begin{aligned}
\mathbb{F}(\omega(\nu,\vartheta)) &= \mathbb{F}\left(\frac{\nu^2}{2}\right) \le \mathbb{F}\left(\frac{\nu^2}{2}\right) + \frac{9k}{10} \\
&= \mathbb{F}(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}) + \tau(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\theta(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}) + \mathbb{F}(\omega(\vartheta,\Im\vartheta)) &= k + \mathbb{F}\left(\frac{\nu^2}{8}\right) \le \mathbb{F}\left(\frac{\nu^4}{2}\right) \\
&= \mathbb{F}(\omega(\nu,\vartheta))
\end{aligned}
$$

Case 2 $\nu \in [1,\infty)$. *Take* $\vartheta = \frac{1}{4} \in \Im\nu$. *Then* $\vartheta \in \mathbb{F}_\tau^\nu$, *since*

$$
\begin{aligned}
\mathbb{F}(\omega(\nu,\vartheta)) &= \mathbb{F}\left(\frac{1}{4}\right) \le \mathbb{F}\left(\frac{1}{4}\right) + \frac{9k}{10} \\
&= \mathbb{F}(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}) + \tau(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\theta(\max\{\omega(\nu,\Im\nu), \omega(\vartheta,\Im\vartheta)\}) + \mathbb{F}(\omega(\vartheta,\Im\vartheta)) &= k + \mathbb{F}\left(\frac{1}{8}\right) \\
&\le \mathbb{F}\left(\frac{1}{4}\right) = \mathbb{F}(\omega(\nu,\vartheta)).
\end{aligned}
$$

Finally, there exists $0 \in \Xi$ *such that* $\omega(0,0) = 0$. *Therefore all the conditions of Theorem 2.1 are satisfied and* $0 \in \Im 0$.

7. Applications

In this section we present two applications of our results.

7.1. Application to integral inclusions

Consider the integral inclusion

$$
\vartheta(t) \in \gamma(t) + \int_a^b M(t,s,\vartheta(s))\,ds, \quad t \in J = [a,b], \tag{19}
$$

where $\gamma \in \Xi = C[a,b]$ is a given function, $M\colon J \times J \times \mathbb{R} \to C(\mathbb{R})$ is a given set-valued mapping and $\vartheta \in \Xi$ is the unknown function. Here, $\Xi = C[a,b]$ is the standard Banach space of continuous real functions with the supremum norm.

Consider the following assumptions:

(I) For each $\vartheta \in \Xi$, the mapping $M_\vartheta : J^2 \to C(\mathbb{R})$ given by $M_{\vartheta(t,s)} = M(t, s, \vartheta(s))$, is continuous;

(II) for every $\vartheta \in \Xi$ there is a function m_ϑ in $M_{\vartheta(t,s)}$ such that

$$\vartheta(t) \leq \gamma(t) + \int_a^b m_\vartheta(t,s)\, ds, \quad t, s \in J;$$

(III) for all $m_\nu(t,s) \in M_\nu(t,s)$ and $m_\vartheta(t,s) \in M_\vartheta(t,s)$

$$|m_\nu(t,s) - m_\vartheta(t,s)| \leq \frac{e^{-k}}{b-a} |\nu(t) - \vartheta(t)|$$

for all $t, s \in J$.

Theorem 7.1. *Let the assumptions* (I)–(III) *hold. Then the integral inclusion* (7.5) *has a solution in X.*

Proof. Let $\mathfrak{I} : \Xi \to C(\Xi)$ be the operator given by

$$\mathfrak{I}\vartheta = \left\{ \nu \in X : \nu(t) \in \gamma(t) + \int_a^b M(t, s, \vartheta(s))\, ds, \quad t \in [a, b] \right\}.$$

Obviously, $\vartheta \in \Xi$ is a solution of the inclusion (7.5) if and only if ϑ is a fixed point of operator $\mathfrak{I}$.

We first check that the operator $\mathfrak{I}$ is well-defined. Indeed, let $\vartheta \in \Xi$ be arbitrary. By (I), the set-valued operator $M_\vartheta : J^2 \to C(\mathbb{R})$ is continuous (w.r.t. Pompeiu-Hausdorff metric on $C(\mathbb{R})$. From the Michael's selection theorem, it follows that there exists a continuous function $m_\vartheta : J^2 \to \mathbb{R}$ such that $m_{\vartheta(t,s)} \in M_{\vartheta(t,s)}$ for each $(t, s) \in J^2$. Hence, the function $\nu(t) = \gamma(t) + \int_a^b m_{\vartheta(t,s)}\, ds$ belongs to $\mathfrak{I}\vartheta$, i.e., $\mathfrak{I}\vartheta \neq \emptyset$. Since γ and M_ϑ are continuous on J, resp. J^2, their ranges are bounded and hence $\mathfrak{I}\vartheta$ is bounded.

Also,

$$\begin{aligned}
\sup_{\varphi \in \mathfrak{I}\nu} d(\varphi, \mathfrak{I}\vartheta) &= \sup_{\varphi \in \mathfrak{I}\nu} \inf_{\chi \in \mathfrak{I}\vartheta} d(\varphi, \chi) \\
&= \sup_{\varphi \in \mathfrak{I}\nu} \inf_{\chi \in \mathfrak{I}v} \max_{t \in J} |\varphi(t) - \chi(t)| \\
&= \sup_{m_\nu \in M_\nu} \inf_{m_\vartheta \in M_\vartheta} \max_{t \in J} \left| \int_a^b [m_\nu(t,s) - m_\vartheta(t,s)]\, ds \right| \\
&\leq \sup_{m_\nu \in M_\nu} \inf_{m_\vartheta \in M_\vartheta} \max_{t \in J} \int_a^b |m_\nu(t,s) - m_\vartheta(t,s)|\, ds \\
&\leq \frac{e^{-k}}{b-a} \max_{t \in J} \int_a^b |\nu(t) - \vartheta(t)|\, ds \\
&\leq \frac{e^{-k}}{b-a} \max_{t \in J} |\nu(t) - \vartheta(t)| \int_a^b ds \\
&= e^{-k} d(\nu, \vartheta).
\end{aligned}$$

Similarly, one can see that

$$\sup_{\chi \in \Im\vartheta} d(\chi, \Im u) \leq e^{-k} d(\nu, \vartheta).$$

Therefore, we have

$$\mathcal{H}(\Im\nu, \Im\vartheta) \leq e^{-k} d(\nu, \vartheta).$$

Taking logarithm on both sides.

$$\ln(\mathcal{H}(\Im\nu, \Im\vartheta)) \leq \ln(e^{-k} d(\nu, \vartheta)),$$

and hence

$$k + \ln(\mathcal{H}(\Im\nu, \Im\vartheta)) \leq \ln(d(\nu, \vartheta)),$$

Taking $\mathbb{F}(\xi) = \ln(\xi)$, and $\theta(s) = k$. Then $\Im$ satisfies all the conditions of Theorem 2.5, and so $\Im$ has a fixed point, that is, the integral inclusion (7.5) has a solution in $\Xi = C[a, b]$. □

7.2. *Application to nonlinear matrix equations*

Let $\mathcal{H}(n)$ stand for the set of all $n \times n$ Hermitian matrices over $\mathbb{C}$, $\mathcal{K}(n) \Big(\subset \mathcal{H}(n) \Big)$ stand for the set of all $n \times n$ positive semi-definite matrices, $\mathcal{P}(n) \Big(\subset \mathcal{K}(n) \Big)$ stand for the set of $n \times n$ positive definite matrices, $\mathcal{M}(n)$ stand for the set of all $n \times n$ matrices over $\mathbb{C}$.

For a matrix $\mathcal{B} \in \mathcal{H}(n)$, we will denote by $s(\mathcal{B})$ any of its singular values and by $s^+(\mathcal{B})$ the sum of all of its singular values, that is, the trace norm $\|\mathcal{B}\| = s^+(\mathcal{B})$. For $C, \mathcal{D} \in \mathcal{H}(n), C \geq \mathcal{D}$ (resp. $C \succ \mathcal{D}$) will mean that the matrix $C - \mathcal{D}$ is positive semi-definite (resp. positive definite).

The following lemmas are needed in the subsequent discussion.

Lemma 7.2. [15]. *If $A \geq O$ and $B \geq O$ are $n \times n$ matrices, then*

$$0 \leq tr(AB) \leq \|A\| tr(B).$$

Lemma 7.3. [15]. *If $A \in \mathcal{H}(n)$ such that $A \prec I_n$, then $\|A\| < 1$.*

Consider the NME

$$\mathcal{Z} = Q + \sum_{i=1}^{k} \mathcal{B}_i^* F(\mathcal{Z}) \mathcal{B}_i, \tag{20}$$

where $Q \in \mathcal{P}(n)$, $\mathcal{B}_i \in \mathcal{M}(n)$, $i = 1, \ldots, k$, and the operators $F \colon \mathcal{P}(n) \to \mathcal{P}(n)$ is continuous in the trace norm.

Theorem 7.4. *Consider the problem described by (20). Assume that:*

(H_1) *there exists $Q \in \mathcal{P}(n)$, such that $\sum_{i=1}^m \mathcal{B}_i^* F(Q) \mathcal{B}_i \succ 0$;*
(H_2) $\sum_{i=1}^m \mathcal{B}_i \mathcal{B}_i^* \prec \eta I_n$;
(H_3) *there exists $\mathcal{Z}_0 \in \mathcal{P}(n)$ such that*

$$\mathcal{Z}_0 \leq Q + \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{Z}_0) \mathcal{B}_i;$$

(H_4) *for every $\mathcal{K}$, $\mathcal{L} \in \mathcal{P}(n)$ with $\mathcal{K} \leq \mathcal{L}$ implies*

$$\sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{K}) \mathcal{B}_i \leq \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{L}) \mathcal{B}_i;$$

(H_5) *for every* $\mathcal{K},\ \mathcal{L} \in \mathcal{P}(n)$ *such that* $\mathcal{K} \le \mathcal{L}$ *with* $\sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{K})\mathcal{B}_i \ne \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{L})\mathcal{B}_i$, *then for* $a, b \ge 0$ *and* $a + 2b < 1$.

$$tr(F(\mathcal{K}) - F(\mathcal{L})) \le \frac{a}{\eta} \max\left\{ \begin{array}{c} tr(\mathcal{K} - \mathcal{L}), tr\left(\mathcal{K} - Q - \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{K})\mathcal{B}_i\right), \\ tr\left(\mathcal{L} - Q - \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{L})\mathcal{B}_i\right) \end{array} \right\}$$
$$+ \frac{b}{\eta}\left[\ tr\left(\mathcal{K} - Q - \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{L})\mathcal{B}_i\right)\ \right].$$

Then the matrix equation (20) has a unique solution.

Proof. Let us consider the set $\Delta = \{\mathcal{Z} \in \mathcal{P}(n) : \|\mathcal{Z}\| \le M\}$, which is a closed subset of $\mathcal{P}(n)$.
Define the operators $\mathcal{T} : \Delta \to \Delta$ by

$$\mathcal{T}(\mathcal{Z}) = Q + \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{Z})\mathcal{B}_i,$$

for $\mathcal{Z} \in \Delta$. It is clear that finding positive definite solution(s) of the system (20) is equivalent to finding fixed point(s) of $\mathcal{T}$.
Define a binary relation

$$\Re = \{(\mathcal{X},\ \mathcal{Y}) \in \mathcal{P}(n) \times \mathcal{P}(n) : \mathcal{X} \le \mathcal{Y}\}.$$

Notice that $\mathcal{T}$ is well defined and continuous. From assumption (H_3), $(\mathcal{Z}_0, \mathcal{T}\mathcal{Z}_0) \in \Re$, and from (H_4), $\mathcal{T}$ is $\Re$-weakly comparative.

Now, for $(\mathcal{K},\ \mathcal{L}) \in \Re$, from assumption (H_5), we have

$$\begin{aligned}
&\|\mathcal{T}(\mathcal{K}) - \mathcal{T}(\mathcal{L})\|_{tr} = tr(\mathcal{T}(\mathcal{K}) - \mathcal{T}(\mathcal{L})) \\
&= tr(\sum_{i=1}^m \mathcal{B}_i^*(F(\mathcal{K}) - F(\mathcal{L}))\mathcal{B}_i) \\
&= \sum_{i=1}^m tr(\mathcal{B}_i^*(F(\mathcal{K}) - F(\mathcal{L}))\mathcal{B}_i) \\
&= \sum_{i=1}^m tr(\mathcal{B}_i\mathcal{B}_i^*(F(\mathcal{K}) - F(\mathcal{L}))) \\
&= tr((\sum_{i=1}^m \mathcal{B}_i\mathcal{B}_i^*)(F(\mathcal{K}) - F(\mathcal{L}))) \\
&\le \|\sum_{i=1}^m \mathcal{B}_i\mathcal{B}_i^*\| \times \|(F(\mathcal{K}) - F(\mathcal{L}))\|_{tr} \\
&\le \frac{\|\sum_{i=1}^m \mathcal{B}_i\mathcal{B}_i^*\|}{\eta} \times \left[\begin{array}{c} a\ \max\left\{\ \|\mathcal{K} - \mathcal{L}\|_{tr}, \|\mathcal{K} - \mathcal{T}\mathcal{K}\|_{tr}, \|\mathcal{L} - \mathcal{T}\mathcal{L}\|_{tr}\ \right\} \\ +b\|\mathcal{K} - \mathcal{T}\mathcal{L}\|_{tr} \end{array} \right] \\
&\le a\ \max\left\{\ \|\mathcal{K} - \mathcal{L}\|_{tr}, \|\mathcal{K} - \mathcal{T}\mathcal{K}\|_{tr}, \|\mathcal{L} - \mathcal{T}\mathcal{L}\|_{tr}\ \right\} + b\|\mathcal{K} - \mathcal{T}\mathcal{L}\|_{tr}.
\end{aligned}$$

Consider $\psi \in \Lambda$ given by $\psi(r_1, r_2, r_3, r_4, r_5) = a\max\{r_1, r_2, r_3\} + b[r_4]$ where $a, b \ge 0$ and $a + 2b < 1$. Thus all the hypotheses of Theorem 5.2 are satisfied, therefore there exists $\widehat{\mathcal{Z}} \in \mathcal{P}(n)$ such that $\mathcal{T}(\widehat{\mathcal{Z}}) = \widehat{\mathcal{Z}}$, and hence the matrix equation (20) has a solution in $\mathcal{P}(n)$. □

Example 7.5. *Consider the following non-linear equation:*

$$\mathcal{T}(\mathcal{Z}) \quad = \quad Q + \mathcal{B}_1^* F(\mathcal{Z})\mathcal{B}_1 + \mathcal{B}_2^* F(\mathcal{Z})\mathcal{B}_2$$

Consider matrices $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{Q}$, $\mathcal{Z}_0$ $\mathcal{K}$, $\mathcal{L}$ *as*

$$\mathcal{B}_1 = \begin{bmatrix} 2.1572 & 0.0758 & 0.0105 & 0.5680 \\ 0.7487 & 1.5457 & 0.4859 & 0.7734 \\ 0.1209 & 0.2918 & 1.4497 & 0.8768 \\ 0.0544 & 0.2715 & 0.1318 & 1.7113 \end{bmatrix}, \mathcal{B}_2 = \begin{bmatrix} 1.4059 & 0.5581 & 0.4653 & 0.6253 \\ 0.4085 & 1.9346 & 0.5010 & 0.3839 \\ 0.3851 & 0.6482 & 1.3886 & 0.7527 \\ 0.8695 & 0.3122 & 0.5059 & 2.2562 \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} 4.2052 & 2.3634 & 2.2443 & 3.3809 \\ 2.3634 & 6.5039 & 2.6620 & 2.3096 \\ 2.2443 & 2.6620 & 4.4006 & 3.2642 \\ 3.3809 & 2.3096 & 3.2642 & 8.2542 \end{bmatrix}, \mathcal{Z}_0 = \begin{bmatrix} 0.9507 & 0 & 0 & 0 \\ 0 & 1.0373 & 0 & 0 \\ 0 & 0 & 0.9176 & 0 \\ 0 & 0 & 0 & 0.9176 \end{bmatrix},$$

$$\mathcal{K} = \begin{bmatrix} 7.1848 & 2.4186 & 0.8847 & 1.2350 \\ 2.4186 & 5.6600 & 2.1379 & 2.0534 \\ 0.8847 & 2.1379 & 4.6685 & 1.9751 \\ 1.2350 & 2.0534 & 1.9751 & 5.0362 \end{bmatrix}, \mathcal{L} = \begin{bmatrix} 7.2027 & 2.4246 & 0.8869 & 1.2381 \\ 2.4246 & 5.6742 & 2.1432 & 2.0586 \\ 0.8869 & 2.1432 & 4.6801 & 1.9800 \\ 1.2381 & 2.0586 & 1.9800 & 5.0488 \end{bmatrix}.$$

The initial matrices are

$$\mathcal{U}_0 = \begin{bmatrix} 7.1848 & 2.4186 & 0.8847 & 1.2350 \\ 2.4186 & 5.6600 & 2.1379 & 2.0534 \\ 0.8847 & 2.1379 & 4.6685 & 1.9751 \\ 1.2350 & 2.0534 & 1.9751 & 5.0362 \end{bmatrix}, \mathcal{V}_0 = 10^4 \times \begin{bmatrix} 1.1275 & 0.9852 & 0.6370 & 0.7097 \\ 0.9852 & 1.0179 & 0.7118 & 0.7708 \\ 0.6370 & 0.7118 & 0.5399 & 0.5696 \\ 0.7097 & 0.7708 & 0.5696 & 0.6321 \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 558.2799 & 428.9370 & 256.3169 & 292.2718 \\ 428.9370 & 470.3649 & 320.8055 & 342.4425 \\ 256.3169 & 320.8055 & 270.0951 & 265.8767 \\ 292.2718 & 342.4425 & 265.8767 & 311.8194 \end{bmatrix}.$$

We take $r = 4$, $\eta = 1.1356e + 03$, $a = 0.99, b = 0.01$, *tolerance: tol=1e-14 and* $F(\mathcal{X}) = \mathcal{X}^{0.0001}$ *to test our algorithm. The numerical results are given in Table 1.*

Table 1. Three initial value analysis

Initial. Mat	$F(\mathcal{X})$	*Iter no.*	*CPU*	*Error*
U_0	$U_0^{0.0001}$	5	0.054401	0
V_0	$V_0^{0.0001}$	6	0.025553	0
W_0	$W_0^{0.0001}$	6	0.033240	0

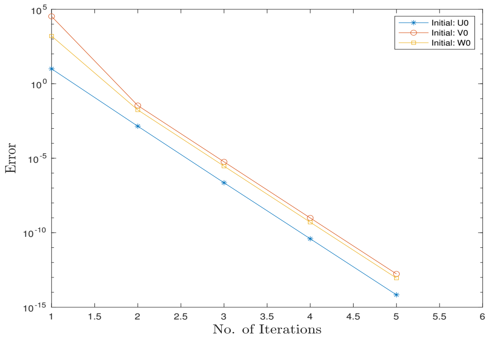
After 6 *successive iterations, we obtain the following positive-definite solution*

$$\widehat{\mathcal{Z}} = \begin{bmatrix} 4.2140 & 2.3676 & 2.2474 & 3.3873 \\ 2.3676 & 6.5119 & 2.6661 & 2.3147 \\ 2.2474 & 2.6661 & 4.4061 & 3.2696 \\ 3.3873 & 2.3147 & 3.2696 & 8.2667 \end{bmatrix}.$$

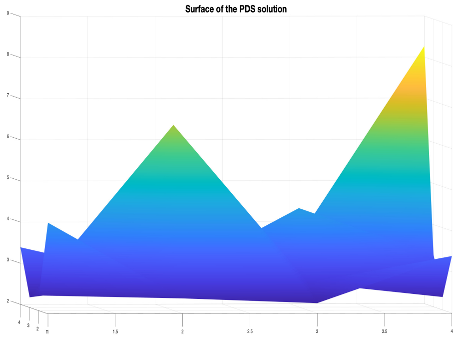
The graphical view of convergence and solution plots are shown in Figure 7.5 and Figure 7.5 below:

Author's Contributions: All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

Conflict of interest: The authors declare that there is no conflict of interest.



Convergence behavior



Surface plot

Acknowledgement

We are very thankful to the reviewers for their constructive comments and suggestions that have been useful for the improvement of this paper. This research is funded by the Foundation for Science and Technology Development of Ton Duc Thang University (FOSTECT), website: http:// fostect.tdtu.edu.vn, under Grant FOSTECT.2019.14.

References

[1] A. Aghajani, M. Mursaleen, A. Shole Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta. Math. Sci. **35**(3)(2015) 552–566.

[2] M. U. Ali, T. Kamran, W. Sintunavarat, P. Katchang, *Mizoguchi-Takahashi's fixed point theorem with α, η functions*, Abstract and Applied Analysis, Volume **2013**, Article ID 418798, 4 pages.

[3] I. Altun, G. Minak, H. Dağ, *Multivalued F-contractions on complete metric space*, J. Nonlinear Convex Anal. **16**(4) (2015), 659–666.

[4] I. Altun, M. Olgun, G. Minak, *On a new class of multivalued weakly Picard operators on complete metric spaces*, Taiwanese J. Math. **19** (3) (2015), 659–672.

[5] J. H. Asl, S. Rezapour, N. Shahzad, *On fixed points of $\alpha - \psi$-contractive multifunctions*, Fixed Point Theory Appl. **2012**, 2012, 212.

[6] Lj. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal. **71**(7-8) (2009), 2716–2723.

[7] Y. Feng, S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl. **317** (2006), 103–112.

[8] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japon. **44** (1996) 381–391.

[9] D. Klim, and D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl. **334**(1) (2007), 132–139.

[10] L. J. Lin and W. S. Du, *Some equivalent formulations of the generalized Ekeland's variational principle and their applications*, Nonlinear Analysis **67** (2007), 187–199.

[11] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141**(1) (1989), 177–188.

[12] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475-488.

[13] H. K. Nashine, R. W. Ibrahim, B. E. Rhoades and R. Pant, *Unified Feng-Liu type fixed point theorems solving control problems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115**(1) (2021), Paper No. 5, 17 pp.
[14] A. Nicolae, *Fixed point theorems for multi-valued mappings of Feng-Liu type*, Fixed Point Theory **12** (1) (2011), 145–154.
[15] A. C. M. Ran, M. C. B. Reurings, *On the matrix equation $X + A^*F(X)A = Q$: solutions and perturbation theory*, Linear Algebr. Appl. 346(2002), 15–26.
[16] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc. **132** (2004), 1435–1443.
[17] B. Samet, C. Vetro, P. Vetro, *Fixed-point theorems for $\alpha - \psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154-2165.
[18] M. Sgroi, C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat **27** (7) (2013), 1259-1268.
[19] T. Suzuki, *Several fixed point theorems in complete metric space*, Yokohama Math. J. **44** (1997), 61–72.
[20] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012**:94 (2012).
[21] D. Wardowski, N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math. **47**(1) (2014), 146–155.