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Estimates Concerned with Hankel Determinant for $\mathcal{M}(\alpha)$ Class

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Abstract. In this paper, we give an upper bound of Hankel determinant of ($H_2(1)$) for the classes of $\mathcal{M}(\alpha)$, $\alpha \in \mathbb{C}$. Also, for $\mathcal{M}(\alpha)$, we obtain a sharp estimate for the classical Fekete-Szegö inequality. That is, we will get a sharp upper bound for the Hankel determinant $H_2(1) = c_3 - c_2^2$. Moreover, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2z^2 + c_3z^3 + ...$ which are analytic in $D = \{z : |z| < 1\}$. Also, $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha\right| < 1, \tag{1.1}$$

where $\alpha \in \mathbb{C}$. This class has found many interesting properties [14, 15, 18, 20].

The certain analytic functions which are in the class of $\mathcal{M}(\alpha)$ on the unit disc D are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{M}(\alpha)$ by applying Schwarz lemma. Schwarz lemma has several applications in the field of electrical and electronics engineering. The use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for the analysis of transfer functions in control engineering and multi-notch filter design in signal processing [12, 13].

In this paper, we will give the sharp estimates for the Hankel determinant of the class of analytic function $f \in \mathcal{A}$ will satisfy the condition (1.1). Also, the relationship between the coefficients of the Hankel determinant and the angular derivative of the function f which provides the class $\mathcal{M}(\alpha)$, will be examined. In this examine, the coefficients c_2, c_3 and c_4 will be used.

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Let $f \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [19] as

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \ c_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2.$$

Here, the Hankel determinant $H_2(1) = c_3 - c_2^2$ is well-known as Fekete-Szegö functional [19]. We will get a sharp upper bound for $H_2(1) = c_3 - c_2^2$ for $\mathcal{M}(\alpha)$ in our study.

Let $f(z) \in \mathcal{M}(\alpha)$ and consider the following function

$$\varphi(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha = 1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots.$$

It is an analytic function in D and $\varphi(0) = 1 - \alpha$. Consider the function

$$h(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)}.$$

Here, $h(z)$ is an analytic function in D, $h(0) = 0$ and $|h(z)| < 1$ for $z \in D$. Therefore, the function $h(z)$ satisfies the condition of Schwarz lemma [5]. By the Schwarz lemma, we obtain

$$\begin{aligned} h(z) &= \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} \\ &= \frac{1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots - (1 - \alpha)}{1 - (1 - \overline{\alpha})\left(1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots\right)} \\ &= \frac{\left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots}{1 - (1 - \overline{\alpha})\left(1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots\right)} \end{aligned}$$

and

$$\frac{h(z)}{z^2} = \frac{\left(c_3 - c_2^2\right) + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z + \ldots}{1 - (1 - \overline{\alpha})\left(1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \ldots\right)}.$$

Letting $z \to 0$, then we have

$$\frac{\left|c_3 - c_2^2\right|}{1 - |1 - \alpha|^2} = \frac{|H_2(1)|}{1 - |1 - \alpha|^2} \leq 1$$

and hence

$$|H_2(1)| \leq 1 - |1 - \alpha|^2.$$

Now, let us show the sharpness of this inequality. Let

$$\frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} = z^2$$

and therefore

$$\varphi(z)=\frac{z^2+\varphi(0)}{1+\overline{\varphi(0)}z^2}.$$

From the definition of $\varphi(z)$, we take

$$\left(\frac{z}{f(z)}\right)^2 f'(z)-\alpha=\frac{z^2+\varphi(0)}{1+\overline{\varphi(0)}z^2}, \tag{1.2}$$

$$1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+....=\frac{z^2+\varphi(0)}{1+\overline{\varphi(0)}z^2},$$

$$\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+....=\frac{z^2+\varphi(0)}{1+\overline{\varphi(0)}z^2}+\alpha-1=\frac{\left(1-|1-\alpha|^2\right)z^2}{1+(1-\overline{\alpha})z^2}$$

and therefore

$$\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+....=\frac{\left(1-|1-\alpha|^2\right)}{1+(1-\overline{\alpha})z^2}.$$

Passing to limit ($z\to 0$) in the last equality yields

$$\left|c_3-c_2^2\right|=|H_2(1)|=1-|1-\alpha|^2.$$

In other words, from equality (1.2), we take

$$f(z)=\frac{z}{1-c_2z-z\int\limits_0^z\frac{\left(1-|1-\alpha|^2\right)}{1+(1-\alpha)t^2}dt}.$$

We thus obtain the following lemma.

Lemma 1.1. *If $f(z)\in\mathcal{M}(\alpha)$, then we have the inequality*

$$|H_2(1)|\leq 1-|1-\alpha|^2. \tag{1.3}$$

This result is sharp with equality for the function

$$f(z)=\frac{z}{1-c_2z-z\int\limits_0^z\frac{\left(1-|1-\alpha|^2\right)}{1+(1-\alpha)t^2}dt}.$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [11]:

Lemma 1.2. *Let $w:D\to D$ be an analytic function with $w(z)=c_pz^p+c_{p+1}z^{p+1}+...$, $p\geq 1$. Assume that there is a $c\in\partial D$ so that w extends continuously to c, $|w(c)|)=1$ and $w'(c)$ exists. Then*

$$|w'(c)|\geq p+\frac{1-\left|c_p\right|}{1+\left|c_p\right|} \tag{1.4}$$

and

$$|w'(c)|\geq p. \tag{1.5}$$

Inequalities (1.4) *and* (1.5) *are sharp.*

Inequality (1.4), (1.5) and its generalizations have important applications in the geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 6–11, 16]. Mercer considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtain a new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [17])

Lemma 1.3 (Julia-Wolff lemma). *Let w be an analytic function in D, $w(0) = 0$ and $w(D) \subset D$. If, in addition, the function w has an angular limit $w(c)$ at $c \in \partial D$, $|w(c)| = 1$, then the angular derivative $w'(c)$ exists and $1 \leq |w'(c)| \leq \infty$.*

Corollary 1.4. *The analytic function w has a finite angular derivative $w'(c)$ if and only if w' has the finite angular limit $w'(c)$ at $c \in \partial D$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{M}(\alpha)$ class. Assuming the existence of an angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. In the inequalities obtained, the relationship between the Hankel determinant and the second angular derivative of the $f(z)$ function was established.

Theorem 2.1. *Let $f(z) \in \mathcal{M}(\alpha)$. Suppose that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f(c) = \frac{c}{1+\alpha}$ and $f'(c) = \frac{1}{1+\alpha}$. Then we have the inequality*

$$\left|f''(c)\right| \geq \frac{2|\alpha|^2}{\left(1 - |1-\alpha|^2\right)|1+\alpha|^2}. \tag{2.1}$$

This result is sharp for $\alpha \in \mathbb{R}$.

Proof. Consider the following function

$$h(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)},$$

where

$$\varphi(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha$$

and $\alpha \in \mathbb{C}$. $h(z)$ is an analytic function in D, $h(0) = 0$ and $|h(z)| < 1$ for $z \in D$. Also, since $f(c) = \frac{c}{1+\alpha}$, $f'(c) = \frac{1}{1+\alpha}$ and $c \in \partial D$, we have $|h(c)| = 1$. That is;

$$\varphi(c) = \left(\frac{c}{f(c)}\right)^2 f'(c) - \alpha = \left(\frac{c}{\frac{c}{1+\alpha}}\right)^2 \frac{1}{1+\alpha} - \alpha = 1$$

and

$$|h(c)| = \left|\frac{\varphi(c) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(c)}\right| = \left|\frac{1 - \varphi(0)}{1 - \overline{\varphi(0)}}\right| = \left|\frac{1 - (1-\alpha)}{1 - (1-\overline{\alpha})}\right| = \left|\frac{\alpha}{\overline{\alpha}}\right| = 1.$$

Therefore, $h(z)$ satisfies the assumptions of the Schwarz lemma on the boundary, we obtain

$$\begin{aligned} 2 &\leq |h'(c)| = \frac{1-\left|\varphi(0)\right|^2}{\left|1-\overline{\varphi(0)}\varphi(c)\right|^2}\left|\varphi'(c)\right| = \frac{1-|1-\alpha|^2}{|1-(1-\overline{\alpha})|^2}\left|f''(c)\right||1+\alpha|^2 \\ &= \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(c)\right||1+\alpha|^2 \end{aligned}$$

and hence

$$\left|f''(c)\right| \geq \frac{2\,|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)} = z^2$$

that is,

$$\varphi(z) = \frac{z^2+\varphi(0)}{1+\overline{\varphi(0)}z^2}.$$

If we take the derivative of both sides of the above equation, then

$$\varphi'(z) = \frac{2z\left(1+\overline{\varphi(0)}z^2\right)-2z\overline{\varphi(0)}\left(z^2+\varphi(0)\right)}{\left(1+\overline{\varphi(0)}z^2\right)^2}$$

and, in particular, we have

$$\varphi'(1) = \frac{2\left(1+\overline{\varphi(0)}\right)-2\overline{\varphi(0)}\left(1+\varphi(0)\right)}{\left(1+\overline{\varphi(0)}\right)^2} = \frac{2\left(1-\left|\varphi(0)\right|^2\right)}{\left(1+\overline{\varphi(0)}\right)^2}.$$

Also, $\varphi'(1) = f''(1)\,(1+\alpha)^2$ and $\varphi(0) = 1-\alpha$. Therefore

$$f''(1)\,(1+\alpha)^2 = \frac{2\left(1-|1-\alpha|^2\right)}{(1+(1-\overline{\alpha}))^2}$$

and

$$\left|f''(1)\right| = \frac{2\left(1-|1-\alpha|^2\right)}{|1+\alpha|^2\,|2-\alpha|^2}.$$

For $\alpha \in \mathbb{R}$, we get

$$\left|f''(1)\right| = \frac{2\alpha}{(1+\alpha)^2\,(2-\alpha)}.$$

From (2.1) for $\alpha \in \mathbb{R}$, we have

$$\left|f''(1)\right| \geq \frac{2\alpha^2}{\left(1-(1-\alpha)^2\right)(1+\alpha)^2} = \frac{2\alpha}{(1+\alpha)^2\,(2-\alpha)}.$$

□

The inequality (2.1) can be strengthened as below by taking into account c_2 and c_3 which are second and third coefficients in the expansion of the function $f(z) = z + c_2z^2 + c_3z^3 + ...$ Due to these coefficients, modulus of the Fekete-Szegö functional ($H_2(1)$) is included in the inequality (2.2).

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f''(c)\right| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{2\left(1-|1-\alpha|^2\right)}{1-|1-\alpha|^2+|H_2(1)|}\right). \tag{2.2}$$

The result is sharp for $\alpha \in \mathbb{R}$.

Proof. Let $h(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.4) for $p = 2$, we obtain

$$2+\frac{1-|d_2|}{1+|d_2|} \leq |h'(c)| = \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(c)\right||1+\alpha|^2,$$

where $|d_2| = \frac{|h''(0)|}{2!} = \frac{\left|c_3-c_2^2\right|}{1-|1-\alpha|^2} = \frac{|H_2(1)|}{1-|1-\alpha|^2}$.

Therefore, we take

$$2+\frac{1-\frac{|H_2(1)|}{1-|1-\alpha|^2}}{1+\frac{|H_2(1)|}{1-|1-\alpha|^2}} \leq \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(c)\right||1+\alpha|^2,$$

$$1+\frac{2}{1+\frac{|H_2(1)|}{1-|1-\alpha|^2}} \leq \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(c)\right||1+\alpha|^2$$

and so

$$\left|f''(c)\right| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{2\left(1-|1-\alpha|^2\right)}{1-|1-\alpha|^2+|H_2(1)|}\right).$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)} = z^2\frac{z+a}{1+az},$$

where $a = \frac{|H_2(1)|}{1-|1-\alpha|^2} \leq 1$ and which is equivalent to

$$\varphi(z) = \frac{z^3+az^2+\varphi(0)(1+az)}{1+az+\overline{\varphi(0)}(z^3+az^2)}.$$

Then

$$\begin{aligned}\varphi'(z) \quad = \quad & \frac{\left(3z^2+2az+a\varphi(0)\right)\left(1+az+\overline{\varphi(0)}\left(z^3+az^2\right)\right)}{\left(1+az+\overline{\varphi(0)}(z^3+az^2)\right)^2} \\ & -\frac{\left(a+\overline{\varphi(0)}\left(3z^2+2az\right)\right)\left(z^3+az^2+\varphi(0)(1+az)\right)}{\left(1+az+\overline{\varphi(0)}(z^3+az^2)\right)^2},\end{aligned}$$

in particular,

$$\varphi'(1) = \frac{\alpha\,(3+a)}{(1+a)\,(2-\alpha)} = \frac{\alpha\left(3+\frac{|H_2(1)|}{1-|1-\alpha|^2}\right)}{\left(1+\frac{|H_2(1)|}{1-|1-\alpha|^2}\right)(2-\alpha)} = \frac{\alpha}{2-\alpha}\,\frac{3\left(1-|1-\alpha|^2\right)+|H_2(1)|}{1-|1-\alpha|^2+|H_2(1)|}$$

Also, for $\alpha \in \mathbb{R}$, we have $f''(1)\,(1+\alpha)^2 = \varphi'(1)$, and $\varphi(0) = 1-\alpha$. Therefore

$$f''(1)\,(1+\alpha)^2 = \frac{\alpha}{2-\alpha}\,\frac{3\alpha\,(2-\alpha)+|H_2(1)|}{\alpha\,(2-\alpha)+|H_2(1)|}$$

and so

$$f''(1) = \frac{\alpha}{(1+\alpha)^2\,(2-\alpha)}\,\frac{3\alpha\,(2-\alpha)+|H_2(1)|}{\alpha\,(2-\alpha)+|H_2(1)|}.$$

From (2.2) for $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\left|f''(c)\right| &\geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{2\left(1-|1-\alpha|^2\right)}{1-|1-\alpha|^2+|H_2(1)|}\right)\\ &= \frac{\alpha}{(1+\alpha)^2\,(2-\alpha)}\,\frac{3\alpha\,(2-\alpha)+|H_2(1)|}{\alpha\,(2-\alpha)+|H_2(1)|}.\end{aligned}$$

□

This completes the proof.

In the following theorem, inequality (2.2) has been strengthened by adding the consecutive terms c_2, c_3 and c_4 of $f(z)$ function.

Theorem 2.3. *Let $f(z) \in \mathcal{M}(\alpha)$. Suppose that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f(c) = \frac{c}{1+\alpha}$ and $f'(c) = \frac{1}{1+\alpha}$. Then we have the inequality*

$$\left|f''(c)\right| \geq \frac{2|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{\left(1-|1-\alpha|^2-|H_2(1)|\right)^2}{\left(1-|1-\alpha|^2\right)^2-|H_2(1)|^2+2\left(1-|1-\alpha|^2\right)\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right). \tag{2.3}$$

The result is sharp for $\alpha \in \mathbb{R}$.

Proof. Let $h(z)$ be the same as in the proof of Theorem 2.1 and $\lambda(z) = z^2$. By the maximum principle, for each $z \in D$, we have the inequality $|h(z)| \leq |\lambda(z)|$. Therefore

$$\begin{aligned}\vartheta(z) &= \frac{h(z)}{\lambda(z)} = \frac{\varphi(z)-\varphi(0)}{\left(1-\overline{\varphi(0)}\varphi(z)\right)z^2}\\ &= \frac{\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\ldots}{\left[1-(1-\overline{\alpha})\left(1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\ldots\right)\right]z^2}\\ &= \frac{\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\ldots}{\left[1-(1-\overline{\alpha})\left(1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\ldots\right)\right]}\end{aligned}$$

is analytic function in D and $|\vartheta(z)| \leq 1$ for $|z| < 1$. In particular, we have

$$|\vartheta(0)| = \frac{\left|c_3-c_2^2\right|}{1-|1-\alpha|^2} = \frac{|H_2(1)|}{1-|1-\alpha|^2} \tag{2.4}$$

and

$$|\vartheta'(0)| = \frac{\left|2c_4 - 4c_2c_3 + 2c_2^3\right|}{1 - |1-\alpha|^2} = \frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{1 - |1-\alpha|^2}.$$

Furthermore, the geometric meaning of the derivative and the inequality $|h(z)| \leq |\lambda(z)|$ imply the inequality

$$\frac{ch'(c)}{h(c)} = |h'(c)| \geq |\lambda'(c)| = \frac{c\lambda'(c)}{\lambda(c)}.$$

The composite function

$$g(z) = \frac{\vartheta(z) - \vartheta(0)}{1 - \overline{\vartheta(0)}\vartheta(z)}$$

is analytic in D, $g(0) = 0$, $\left|g(z)\right| < 1$ for $|z| < 1$ and $\left|g(c)\right| = 1$ for $c \in \partial D$. For $p = 1$, from (1.4), we obtain

$$\begin{aligned}
\frac{2}{1 + \left|g'(0)\right|} &\leq \left|g'(c)\right| = \frac{1 - |\vartheta(0)|^2}{\left|1 - \overline{\vartheta(0)}\vartheta(c)\right|^2} |\vartheta'(c)| \\
&\leq \frac{1 + |\vartheta(0)|}{1 - |\vartheta(0)|} \{|h'(c)| - |\lambda'(c)|\} \\
&= \frac{1 + \frac{|H_2(1)|}{1-|1-\alpha|^2}}{1 - \frac{|H_2(1)|}{1-|1-\alpha|^2}} \left\{\frac{1 - |1-\alpha|^2}{|\alpha|^2} \left|f''(c)\right| |1+\alpha|^2 - 2\right\}.
\end{aligned}$$

Since

$$g'(z) = \frac{1 - |\vartheta(0)|^2}{\left(1 - \overline{\vartheta(0)}\vartheta(z)\right)^2} \vartheta'(z)$$

and, in particular, we have

$$\begin{aligned}
\left|g'(0)\right| &= \frac{|\vartheta'(0)|}{1 - |\vartheta(0)|^2} = \frac{\frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{1-|1-\alpha|^2}}{1 - \left(\frac{|H_2(1)|}{1-|1-\alpha|^2}\right)^2} \\
&= \left(1 - |1-\alpha|^2\right) \frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{\left(1 - |1-\alpha|^2\right)^2 - |H_2(1)|^2}.
\end{aligned}$$

We observe that

$$\frac{2}{1 + \left(1-|1-\alpha|^2\right)\frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{\left(1-|1-\alpha|^2\right)^2 - |H_2(1)|^2}} \leq \frac{1-|1-\alpha|^2+|H_2(1)|}{1-|1-\alpha|^2-|H_2(1)|} \left\{\frac{1-|1-\alpha|^2}{|\alpha|^2} \left|f''(c)\right| |1+\alpha|^2 - 2\right\}$$

which is equivalent to

$$\left|f''(c)\right| \geq \frac{2|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2} \left(1 + \frac{\left(1-|1-\alpha|^2-|H_2(1)|\right)^2}{\left(1-|1-\alpha|^2\right)^2 - |H_2(1)|^2 + 2\left(1-|1-\alpha|^2\right)\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}\right).$$

To prove the sharpness of the inequality (2.3), let

$$\frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} = z^3$$

which is equivalent to

$$\varphi(z) = \frac{z^3 + \varphi(0)}{1 + \overline{\varphi(0)} z^3}.$$

If we take the derivative of both sides of the above equation, then

$$\varphi'(z) = \frac{3z^2 \left(1 + \overline{\varphi(0)} z^3\right) - 3z^2 \overline{\varphi(0)} \left(z^3 + \varphi(0)\right)}{\left(1 + \overline{\varphi(0)} z^3\right)^2}$$

and, in particular, we have

$$\varphi'(z) = \frac{3\left(1 - \left|\varphi(0)\right|^2\right)}{\left(1 + \overline{\varphi(0)}\right)^2}.$$

Since $\varphi'(1) = f''(1)(1+\alpha)^2$ and $\varphi(0) = 1 - \alpha$, therefore

$$f''(1)(1+\alpha)^2 = \frac{3\left(1 - |1-\alpha|^2\right)}{\left(1 + \overline{1-\alpha}\right)^2}$$

and

$$f''(1) = \frac{3\left(1 - |1-\alpha|^2\right)}{(2-\overline{\alpha})^2 (1+\alpha)^2}.$$

For $\alpha \in \mathbb{R}$, we have

$$f''(1) = \frac{3\left(1 - |1-\alpha|^2\right)}{(2-\alpha)^2 (1+\alpha)^2} = \frac{3\alpha}{(2-\alpha)(1+\alpha)^2}.$$

On the other hand, we obtain

$$\varphi(z) = \frac{z^3 + \varphi(0)}{1 + \overline{\varphi(0)} z^3},$$

$$1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + ... = \frac{z^3 + 1 - \alpha}{1 + (1-\overline{\alpha}) z^3}$$

and so

$$\begin{aligned} \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + ... &= \frac{z^3 + 1 - \alpha}{1 + (1-\overline{\alpha}) z^3} - (1-\alpha) \\ &= \frac{\left(1 - |1-\alpha|^2\right) z^3}{1 + (1-\overline{\alpha}) z^3}. \end{aligned}$$

If we divide both sides of the equation by z^2, we have

$$\left(c_3 - c_2^2\right) + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z + ... = \frac{\left(1 - |1-\alpha|^2\right) z}{1 + (1-\overline{\alpha}) z^3}.$$

Passing to limit ($z \to 0$) in the last equality yields $c_3 - c_2^2 = H_2(1) = 0$. Similarly, using straightforward calculations, we take $2c_4 - 4c_2c_3 + 2c_2^3 = 1 - |1-\alpha|^2$. Therefore, we obtain

$$\frac{2|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{\left(1-|1-\alpha|^2-|H_2(1)|\right)^2}{\left(1-|1-\alpha|^2\right)^2-|H_2(1)|^2+2\left(1-|1-\alpha|^2\right)\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right)$$
$$=\frac{2|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{\left(1-|1-\alpha|^2\right)^2}{\left(1-|1-\alpha|^2\right)^2+\left(1-|1-\alpha|^2\right)\left(1-|1-\alpha|^2\right)}\right)$$
$$=\frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+\frac{2\left(1-|1-\alpha|^2\right)^2}{2\left(1-|1-\alpha|^2\right)^2}\right)$$
$$=\frac{3|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}.$$

Thus, for $\alpha \in \mathbb{R}$, we obtain

$$\frac{2|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(1+\frac{\left(1-|1-\alpha|^2-|H_2(1)|\right)^2}{\left(1-|1-\alpha|^2\right)^2-|H_2(1)|^2+2\left(1-|1-\alpha|^2\right)\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right)$$
$$=\frac{3\alpha}{(2-\alpha)(1+\alpha)^2}. \quad \square$$

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