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New Quantum Boundaries for q-Simpson's Type Inequalities for Co-Ordinated Convex Functions

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Abstract. The aim of this work is to develop quantum estimates for q-Simpson type integral inequalities for co-ordinated convex functions by using the notion of newly defined q_1q_2-derivatives and integrals. For this, we establish a new identity including quantum integrals and quantum numbers via q_1q_2- differentiable functions. After that, with the help of this equality, we achieved the results we want. The outcomes raised in this paper are extensions and generalizations of the comparable results in the literature on Simpson's inequalities for co-ordinated convex functions.

1. Introduction

Simpson's rules are well-known techniques for the numerical integration and numerical estimation of definite integrals. This method is known to be developed by Thomas Simpson (1710–1761). However, Johannes Kepler used a similar approximation about 100 years ago, so this method is also known as Kepler's rule.

Simpson's quadrature formula (Simpson's 1/3 rule) is stated as:

$$\int_{\alpha}^{\beta} \phi(\varkappa)d\varkappa \approx \frac{\beta-\alpha}{6}\left[\phi(\alpha)+4\phi\left(\frac{\alpha+\beta}{2}\right)+\phi(\beta)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1.1. *Suppose that $\phi:[\alpha,\beta]\to\mathbb{R}$ is a four times continuously differentiable mapping on (α,β), and let $\left\|\phi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\alpha,\beta)}\left|\phi^{(4)}(\varkappa)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\phi(\alpha)+\phi(\beta)}{2}+2\phi\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(\varkappa)d\varkappa\right|\leq\frac{1}{2880}\left\|\phi^{(4)}\right\|_{\infty}(\beta-\alpha)^{4}.$$

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In recent years, many authors have focused on Simpson type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson and Newton type results for convex mappings, because convexity theory is an effective and strong method for solving a great number of problems which arise within different branches of pure and applied mathematics. For example, Dragomir et al. presented new Simpson type results and their applications to quadrature formulas in numerical integration in [14]. What is more, some inequalities of Simpson type for s-convex functions are deduced by Alomari et al. in [5]. Afterwards, Sarikaya et al. observed the variants of Simpson type inequalities based on convexity in [34]. For more recent results, one can read [16, 20, 33].

On the other side, in the field of q-analysis, many studies have recently been carried out, starting with Euler owing to a vast requirement for mathematics that models quantum computing q-calculus occurred for the relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [17–19, 21, 23]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by Jackson [17]. In 1908-1909, the general form of the q-integral and q-difference operator is defined by Jackson [21]. In 1969, for the first time Agarwal [3] defined the q-fractional derivative. In 1966-1967, Al-Salam [6] introduced a q-analog of the q-fractional integral and q-Riemann-Liouville fractional. In 2004, Rajkovic gave a definition of the Riemann-type q-integral which was generalized to Jackson q-integral. In 2013, Tariboon introduced ${}_{\varkappa_1}D_q$-difference operator [7]. Recently, in 2020, Bermudo et al. introduced the notion of ${}^{\beta}D_q$ derivative and integral [9].

Many well-known integral inequalities such as Hölder inequality, Hermite-Hadamard inequality, Simpson's inequality, Newton's inequality, Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz inequality, Gruss inequality, Gruss- Cebysev inequality and other integral inequalities have been studied in the setup of q-calculus using the concept of classical convexity. For more results in this direction, we refer to [1, 2, 7, 8, 11, 13, 15, 17–19, 22, 25, 26, 29–32, 35, 37–42].

Inspired by this ongoing study, we establish some new quantum bounds for q-Simpson's type inequalities for q-differentiable co-ordinated convex functions. This is the primary motivation of this paper. The ideas and strategies of the paper may open new venues for further research in this field.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we present some required definitions and related inequalities about q-calculus. Throughout the paper, we consider that $0 < q, q_1, q_2 < 1$, $\alpha < \beta$, $\gamma < \delta$, and $\Delta = [\alpha, \beta] \times [\gamma, \delta] \subseteq \mathbb{R}^2$.

We have to give the following notation which will be used many times in the next sections (see, [23]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Definition 2.1. *[36] We consider that a function $\phi : [\alpha, \beta] \to \mathbb{R}$ is continuous. Then, the q_α-derivative of ϕ at $\varkappa \in [\alpha, \beta]$ is defined in the following way;*

$$ {}_{\alpha}d_q\phi(\varkappa) = \frac{\phi(\varkappa) - \phi(q\varkappa + (1-q)\alpha)}{(1-q)(\varkappa - \alpha)}, \ \varkappa \neq \alpha. \tag{1}$$

If we assume $\varkappa = \alpha$ in (1), we define ${}_{\alpha}d_q\phi(\alpha) = \lim_{\varkappa \to \alpha} {}_{\alpha}d_q\phi(\varkappa)$ if it exists and it is finite.

Definition 2.2. *[36] We assume that a function $\phi : [\alpha, \beta] \to \mathbb{R}$ is continuous. Then, the q_α-definite integral on $[\alpha, \beta]$ is defined as:*

$$\int_{\alpha}^{\varkappa} \phi(\tau) \ {}_{\alpha}d_q\tau = (1-q)(\varkappa - \alpha) \sum_{n=0}^{\infty} q^n \phi(q^n\varkappa + (1-q^n)\alpha) \tag{2}$$

for $\varkappa \in [\alpha, \beta]$.

Moreover, we give the succeeding lemma which is necessary to prove the key results of this paper:

Lemma 2.3. *We have the equality*

$$\int_{\alpha}^{\beta} (\varkappa-\alpha)^{a} \ {}_{\alpha}d_{q}\varkappa = \frac{(\beta-\alpha)^{a+1}}{[a+1]_{q}}$$

for $a \in \mathbb{R}\backslash\{-1\}$.

In [7], Alp et al. established the succeeding quantum integral inequality of Hermite-Hadamard type for the convex functions in the developing of q-calculus:

Theorem 2.4. *(q_{α}-Hermite-Hadamard inequality) We assume that a function $\phi : [\alpha,\beta] \rightarrow \mathbb{R}$ is convex differentiable function on $[\alpha,\beta]$ and $0<q<1$. Then, we have the succeeding inequality:*

$$\phi\left(\frac{q\alpha+\beta}{[2]_{q}}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(\varkappa) \ {}_{\alpha}d_{q}\varkappa \leq \frac{q\phi(\alpha)+\phi(\beta)}{[2]_{q}}. \tag{3}$$

In [7] and [29], the authors offered some estimates for the right and left hand sides of the inequality (3).
On the other side, Bermudo et al. gave the following new definitions and related Hermite-Hadamard type inequalities in quantum calculus:

Definition 2.5. *[9] We consider that a function $\phi : [\alpha,\beta] \rightarrow \mathbb{R}$ is continuous. Then, the q^{β}-derivative of ϕ at $\varkappa \in [\alpha,\beta]$ is defined in the following way;*

$${}^{\beta}d_{q}\phi(\varkappa) = \frac{\phi(\varkappa)-\phi(q\varkappa+(1-q)\beta)}{(1-q)(\varkappa-\beta)}, \ \varkappa \neq \beta. \tag{4}$$

If we consider $\varkappa=\beta$ in (4), we define ${}_{\alpha}d_{q}\phi(\beta) = \lim_{\varkappa\rightarrow\beta} \ {}_{\alpha}d_{q}\phi(\varkappa)$ if it exists and it is finite.

Definition 2.6. *[9] We assume that a function $\phi : [\alpha,\beta] \rightarrow \mathbb{R}$ is continuous. Then, the q^{β}-definite integral on $[\alpha,\beta]$ is defined as:*

$$\int_{\varkappa}^{\beta}\phi(\tau) \ {}^{\beta}d_{q}\tau = (1-q)(\beta-\varkappa)\sum_{n=0}^{\infty}q^{n}\phi(q^{n}\varkappa+(1-q^{n})\beta)$$

for $\varkappa \in [\alpha,\beta]$.

Theorem 2.7. *(q^{β}-Hermite-Hadamard inequality, [9])We assume that a function $\phi : [\alpha,\beta] \rightarrow \mathbb{R}$ is convex differentiable function on $[\alpha,\beta]$. Then, we obtain the succeeding inequality*

$$\phi\left(\frac{\alpha+q\beta}{[2]_{q}}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(\varkappa) \ {}^{\beta}d_{q}\varkappa \leq \frac{\phi(\alpha)+q\phi(\beta)}{[2]_{q}}. \tag{5}$$

In [10], Budak offered some estimates for the right and left hand sides of the inequality (5).
In [28], Latif defined $q_{\alpha\gamma}$-integral and partial q-derivatives for two variables functions as follows:

Definition 2.8. *We suppose that a two variables function $\phi:\Delta\to\mathbb{R}$ is continuous. Then, the definite $q_{\alpha\gamma}$-integral on Δ is defined by*

$$\int_{\alpha}^{\varkappa}\int_{\gamma}^{y}\phi(\tau,\sigma)\ {}_{\gamma}d_{q_2}\sigma\ \ {}_{\alpha}d_{q_1}\tau = (1-q_1)(1-q_2)(\varkappa-\alpha)(y-\gamma)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m\phi\left(q_1^n\varkappa+\left(1-q_1^n\right)\alpha, q_2^m y+\left(1-q_2^m\right)\gamma\right)$$

for $(\varkappa,y)\in\Delta$.

Lemma 2.9. *[24] If the assumptions of Definition 2.8 hold, then*

$$\begin{aligned}\int_{y_1}^{y}\int_{\varkappa_1}^{\varkappa}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma &= \int_{y_1}^{y}\int_{\alpha}^{\varkappa}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma-\int_{y_1}^{y}\int_{\alpha}^{\varkappa_1}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma\\ &= \int_{\gamma}^{y}\int_{\alpha}^{\varkappa}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma-\int_{\gamma}^{y_1}\int_{\alpha}^{\varkappa}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma\\ &\quad-\int_{\gamma}^{y}\int_{\alpha}^{\varkappa_1}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma+\int_{\gamma}^{y_1}\int_{\alpha}^{\varkappa_1}\phi(\tau,\sigma)\ {}_{\alpha}d_{q_1}\tau\ {}_{\gamma}d_{q_2}\sigma.\end{aligned}$$

Definition 2.10. *[28] We consider that a two variables function $\phi:\Delta\to\mathbb{R}$ is continuous. Then, the partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(\varkappa,y)\in\Delta$ can be given as follows:*

$$\begin{aligned}\frac{{}_{\alpha}\partial_{q_1}\phi(\varkappa,y)}{{}_{\alpha}\partial_{q_1}\varkappa} &= \frac{\phi(\varkappa,y)-\phi(q_1\varkappa+(1-q_1)\alpha,y)}{(1-q_1)(\varkappa-\alpha)},\ \varkappa\neq\beta\\ \frac{{}_{\gamma}\partial_{q_2}\phi(\varkappa,y)}{{}_{\gamma}\partial_{q_2}y} &= \frac{\phi(\varkappa,y)-\phi(\varkappa,q_2y+(1-q_2)\gamma)}{(1-q_2)(y-\gamma)},\ y\neq\gamma\\ \frac{{}_{\alpha,\gamma}\partial^2_{q_1,q_2}\phi(\varkappa,y)}{{}_{\alpha}\partial_{q_1}\varkappa\ {}_{\gamma}\partial_{q_2}y} &= \frac{1}{(\varkappa-\alpha)(y-\gamma)(1-q_1)(1-q_2)}\Big[\phi(q_1\varkappa+(1-q_1)\alpha,q_2y+(1-q_2)\gamma)\\ &\quad+\phi(\varkappa,y)-\phi(q_1\varkappa+(1-q_1)\alpha,y)-\phi(\varkappa,q_2y+(1-q_2)\gamma)\Big],\ \varkappa\neq\alpha,\ y\neq\gamma.\end{aligned}$$

For more details related to q-integrals and derivatives for the functions of two variables (see, [28]).
On the other side, Budak et al. gave the following definitions of q_{α}^{δ}, q_{β}^{γ} and $q^{\beta\delta}$ integrals:

Definition 2.11. *[12] We suppose that a two variables function $\phi:\Delta\to\mathbb{R}$ is continuous. Then, the following q_{α}^{δ}, q_{γ}^{β}, and $q^{\beta\delta}$ integrals on Δ are defined by*

$$\begin{aligned}\int_{\alpha}^{\varkappa}\int_{y}^{\delta}\phi(\tau,\sigma)\ {}^{\delta}d_{q_2}\sigma\ \ {}_{\alpha}d_{q_1}\tau &= (1-q_2)(\delta-y)\sum_{m=0}^{\infty}q_2^m\int_{\alpha}^{\varkappa}\phi\left(\tau,q_2^my+\left(1-q_2^m\right)\delta\right)\ {}_{\alpha}d_{q_1}\tau\\ &= (1-q_1)(1-q_2)(\varkappa-\alpha)(\delta-y)\\ &\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\phi\left(q_1^n\varkappa+\left(1-q_1^n\right)\alpha,q_2^my+\left(1-q_2^m\right)\delta\right),\end{aligned}$$

$$\int_{\varkappa}^{\beta}\int_{\gamma}^{y}\phi(\tau,\sigma)\ {}_{\gamma}d_{q_2}\sigma\ \ {}^{\beta}d_{q_1}\tau = (1-q_1)(1-q_2)(\beta-\varkappa)(y-\gamma) \tag{6}$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m\phi\left(q_1^n\varkappa+\left(1-q_1^n\right)\beta,q_2^m y+\left(1-q_2^m\right)\gamma\right)$$

and

$$\int_{\varkappa}^{\beta}\int_{y}^{\delta}\phi(\tau,\sigma)\ {}^{\delta}d_{q_2}\sigma\ \ {}^{\beta}d_{q_1}\tau = (1-q_1)(1-q_2)(\beta-\varkappa)(\delta-y) \tag{7}$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m\phi\left(q_1^n\varkappa+\left(1-q_1^n\right)\beta,q_2^m y+\left(1-q_2^m\right)\delta\right)$$

respectively, for $(\varkappa,y)\in\Delta$.

Theorem 2.12. *(q_1q_2-Hölder's inequality for two variables functions, [28]) Let $\varkappa,y>0,p_1>1$ such that $\frac{1}{p_1}+\frac{1}{r_1}=1$. Then*

$$\int_0^{\varkappa}\int_0^{y}\left|\phi(\varkappa,y)G(\varkappa,y)\right|d_{q_1}\varkappa d_{q_2}y\le\left(\int_0^{\varkappa}\int_0^{y}\left|\phi(\varkappa,y)\right|^{p_1}d_{q_1}\varkappa d_{q_2}y\right)^{\frac{1}{p_1}}\left(\int_0^{\varkappa}\int_0^{y}\left|G(\varkappa,y)\right|^{r_1}d_{q_1}\varkappa d_{q_2}y\right)^{\frac{1}{r_1}}.$$

In [4], Ali et al. gave new definitions of partial q_1q_2-derivatives in the following way:

Definition 2.13. *[4] Let a two variables function $\phi:\Delta\to\mathbb{R}$ be continuous. Then the partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(\varkappa,y)\in\Delta$ can be given as follows:*

$$\frac{{}^{\beta}\partial_{q_1}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_1}\varkappa} = \frac{\phi(q_1\varkappa+(1-q_1)\beta,y)-\phi(\varkappa,y)}{(1-q_1)(\beta-\varkappa)},\ \varkappa\neq\beta$$

$$\frac{{}^{\delta}\partial_{q_2}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_2}y} = \frac{\phi(\varkappa,q_2y+(1-q_2)\delta)-\phi(\varkappa,y)}{(1-q_2)(\delta-y)},\ \delta\neq y$$

$$\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\phi(\varkappa,y)}{{}_{\alpha}\partial_{q_1}\varkappa\ {}^{\delta}\partial_{q_2}y} = \frac{1}{(\varkappa-\alpha)(\delta-y)(1-q_1)(1-q_2)}\Big[\phi(q_1\varkappa+(1-q_1)\alpha,q_2y+(1-q_2)\delta)$$
$$-\phi(q_1\varkappa+(1-q_1)\alpha,y)-\phi(\varkappa,q_2y+(1-q_2)\delta)+\phi(\varkappa,y)\Big],\ \varkappa\neq\alpha,\ y\neq\delta,$$

$$\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_1}\varkappa\ {}_{\gamma}\partial_{q_2}y} = \frac{1}{(\beta-\varkappa)(y-\gamma)(1-q_1)(1-q_2)}\Big[\phi(q_1\varkappa+(1-q_1)\beta,q_2y+(1-q_2)\gamma)$$
$$-\phi(q_1\varkappa+(1-q_1)\beta,y)-\phi(\varkappa,q_2y+(1-q_2)\gamma)+\phi(\varkappa,y)\Big],\ \varkappa\neq\beta,\ y\neq\gamma,$$

$$\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_1}\varkappa\ {}^{\delta}\partial_{q_2}y} = \frac{1}{(\beta-\varkappa)(\delta-y)(1-q_1)(1-q_2)}\Big[\phi(q_1\varkappa+(1-q_1)\beta,q_2y+(1-q_2)\delta)$$
$$-\phi(q_1\varkappa+(1-q_1)\beta,y)-\phi(\varkappa,q_2y+(1-q_2)\delta)+\phi(\varkappa,y)\Big],\ \varkappa\neq\beta,\ y\neq\delta,$$

$$\frac{{}^{\beta}\partial_{q_1}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_1}\varkappa} = \frac{\phi(\varkappa,y)-\phi(q_1\varkappa+(1-q_1)\beta,y)}{(1-q_1)(\varkappa-\beta)},\ \varkappa\neq\beta,$$

$$\frac{{}^{\delta}\partial_{q_1}\phi(\varkappa,y)}{{}^{\beta}\partial_{q_2}y} = \frac{\phi(\varkappa,y)-\phi(\varkappa,q_2y+(1-q_2)\delta)}{(1-q_2)(y-\delta)},\ y\neq\gamma,$$

$$\frac{{}_{\alpha,\ \gamma}\partial^2_{q_1,q_2}\phi(\varkappa,y)}{{}_{\alpha}\partial_{q_1}\varkappa\ {}_{\gamma}\partial_{q_2}y} = \frac{1}{(\varkappa-\alpha)(y-\gamma)(1-q_1)(1-q_2)}\Big[\phi(q_1\varkappa+(1-q_1)\alpha,q_2y+(1-q_2)\gamma)$$
$$-\phi(q_1\varkappa+(1-q_1)\alpha,y)-\phi(\varkappa,q_2y+(1-q_2)\gamma)+\phi(\varkappa,y)\Big],\ \varkappa\neq\alpha,\ y\neq\gamma.$$

3. Crucial Lemmas

In this section, we offer a new identity involving the quantum derivatives, quantum integrals, and quantum numbers. Moreover, we give some calculated quantum integrals.

Let us start with the following useful lemma.

Lemma 3.1. *Let $\phi : \Delta \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}$ is continuous and integrable on Δ. Then the following identity holds for q_1q_2-integrals:*

$$\begin{aligned} & {}^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi) \\ = \;& (\beta-\alpha)(\delta-\gamma)\int_0^1\int_0^1 \Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma, \end{aligned} \tag{8}$$

where

$$\begin{aligned} & {}^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi) \\ = \;& \frac{1}{[6]_{q_1}[6]_{q_2}}\left\{\frac{q_1[4]_{q_1}}{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)+q_1[4]_{q_1}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)\right. \\ & \left.+q_1q_2[4]_{q_1}[4]_{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+\frac{q_2[4]_{q_2}}{q_1}\phi\left(\alpha,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+q_2[4]_{q_2}\phi\left(\beta,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right\} \\ & +\frac{\phi(\alpha,\gamma)+q_1\phi(\beta,\gamma)+q_2\phi(\alpha,\delta)+q_1q_2\phi(\beta,\delta)}{q_1q_2[6]_{q_1}[6]_{q_2}} \\ & -\frac{1}{q_1q_2[6]_{q_2}(\beta-\alpha)}\int_\alpha^\beta\left[\phi(\varkappa,\gamma)+q_2^2[4]_{q_2}\phi\left(\varkappa,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+q_2\phi(\varkappa,\delta)\right]{}^{\beta}d_{q_1}\varkappa \\ & -\frac{1}{q_1q_2[6]_{q_1}(\delta-\gamma)}\int_\gamma^\delta\left[q_1\phi(\beta,y)+q_1^2[4]_{q_1}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},y\right)+\phi(\alpha,y)\right]{}^{\delta}d_{q_2}y \\ & +\frac{1}{q_1q_2(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\phi(\varkappa,y)\,{}^{\beta}d_{q_1}\varkappa\,{}^{\delta}d_{q_2}y \end{aligned}$$

and

$$\Lambda_{q_1}(\tau)=\begin{cases}\tau-\frac{1}{[6]_{q_1}}, & \tau\in\left[0,\frac{1}{[2]_{q_1}}\right)\\ \tau-\frac{[5]_{q_1}}{[6]_{q_1}}, & \tau\in\left[\frac{1}{[2]_{q_1}},1\right]\end{cases},$$

$$\Lambda_{q_2}(\sigma)=\begin{cases}\sigma-\frac{1}{[6]_{q_2}}, & \sigma\in\left[0,\frac{1}{[2]_{q_2}}\right)\\ \sigma-\frac{[5]_{q_2}}{[6]_{q_2}}, & \sigma\in\left[\frac{1}{[2]_{q_2}},1\right]\end{cases}.$$

Proof. Because of the Lemma 2.9 and Definitions of $\Lambda_{q_1}(\tau)$ and $\Lambda_{q_2}(\sigma)$, it is easy to see that

$$\begin{aligned} & \int_0^1\int_0^1 \Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma \\ = \;& \frac{\left([5]_{q_2}-1\right)\left([5]_{q_1}-1\right)}{[6]_{q_1}[6]_{q_2}}\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma \\ & +\frac{[5]_{q_1}-1}{[6]_{q_1}}\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1\left(\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right)\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma \end{aligned} \tag{9}$$

$$
\begin{aligned}
&+\frac{[5]_{q_2}-1}{[6]_{q_2}}\int_0^1\int_0^{\frac{1}{[2]_{q_2}}}\left(\tau-\frac{[5]_{q_2}}{[6]_{q_2}}\right)\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\,d_{q_2}\sigma\\
&+\int_0^1\int_0^1\left(\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right)\left(\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right)\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\,d_{q_2}\sigma.\\
=\;&I_1+I_2+I_3+I_4.
\end{aligned}
$$

By considering the Definition 2.13, we have

$$
\begin{aligned}
&\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\\
=\;&\frac{1}{(1-q_1)(1-q_2)(\beta-\alpha)(\delta-\gamma)\tau\sigma}\Big[\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)\\
&-\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma\gamma+(1-\sigma)\delta\right)-\phi\left(\tau\alpha+(1-\tau)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)\\
&+\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)\Big].
\end{aligned}
$$

It is necessary to compute the integrals in the right side of (9) to conclude the proof. By using the definition of q_1q_2-integrals, we obtain that

$$
\begin{aligned}
&\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\,d_{q_2}\sigma \qquad (10)\\
=\;&\frac{1}{(1-q_1)(1-q_2)(\beta-\alpha)(\delta-\gamma)}\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}\frac{1}{\tau\sigma}\Big[\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)\\
&-\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma\gamma+(1-\sigma)\delta\right)-\phi\left(\tau\alpha+(1-\tau)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)\\
&+\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)\Big]d_{q_1}\tau\,d_{q_2}\sigma\\
=\;&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,\frac{q_2^{m+1}}{[2]_{q_2}}\gamma+\left(1-\frac{q_2^{m+1}}{[2]_{q_2}}\right)\delta\right)\right.\\
&-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,\frac{q_2^{m}}{[2]_{q_2}}\gamma+\left(1-\frac{q_2^{m}}{[2]_{q_2}}\right)\delta\right)\\
&-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,\frac{q_2^{m+1}}{[2]_{q_2}}\gamma+\left(1-\frac{q_2^{m+1}}{[2]_{q_2}}\right)\delta\right)\\
&\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,\frac{q_2^{m}}{[2]_{q_2}}\gamma+\left(1-\frac{q_2^{m}}{[2]_{q_2}}\right)\delta\right)\right.\\
=\;&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\left[\phi(\beta,\delta)-\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)-\phi\left(\beta,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right].
\end{aligned}
$$

For that reason, we obtain that

$$
I_1=\frac{\left([5]_{q_1}-1\right)\left([5]_{q_2}-1\right)}{[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left[\phi(\beta,\delta)-\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)-\phi\left(\beta,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right].
$$

Now from Definition 2.11, we obtain the following

$$\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1 \sigma \frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma \tag{11}$$
$$= \frac{1}{(1-q_1)(1-q_2)(\beta-\alpha)(\delta-\gamma)}\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1 \frac{1}{\tau}\Big[\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)$$
$$-\phi\left(\tau q_1\alpha+(1-\tau q_1)\beta,\sigma\gamma+(1-\sigma)\delta\right)-\phi\left(\tau\alpha+(1-\tau)\beta,\sigma q_2\gamma+(1-\sigma q_2)\delta\right)$$
$$+\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)\Big]d_{q_1}\tau\, d_{q_2}\sigma$$
$$= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\right)\right.$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\right)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\right)$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\right)\right\}$$
$$= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\sum_{m=0}^{\infty}q_2^m\left[\sum_{n=0}^{\infty}\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\right.\right.$$
$$\left.-\sum_{n=0}^{\infty}\left(\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\right)\right]$$
$$+\sum_{m=0}^{\infty}q_2^m\left[\sum_{n=0}^{\infty}\phi\left(\frac{q_1^{n}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n}}{[2]_{q_1}}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\right.$$
$$\left.\left.-\sum_{n=0}^{\infty}\phi\left(\frac{q_1^{n+1}}{[2]_{q_1}}\alpha+\left(1-\frac{q_1^{n+1}}{[2]_{q_1}}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\right]\right\}$$
$$= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\sum_{m=0}^{\infty}q_2^m\phi\left(\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)-\sum_{m=0}^{\infty}q_2^m\phi\left(\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\right.$$
$$\left.+\sum_{m=0}^{\infty}q_2^m\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)-\sum_{m=0}^{\infty}q_2^m\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\right\}$$
$$= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\frac{1-q_2}{q_2}\sum_{m=0}^{\infty}q_2^m\phi\left(\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)-\frac{1}{q_2}\phi(\beta,\gamma)\right.$$
$$\left.-\frac{1-q_2}{q_2}\sum_{m-0}^{\infty}q_2^m\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)+\frac{1}{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)\right.$$
$$= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left[\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\beta,y)\ {}^{\delta}d_{q_2}y-\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},y\right)\ {}^{\delta}d_{q_2}y\right.$$
$$\left.-\frac{1}{q_2}\phi(\beta,\gamma)+\frac{1}{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)\right].$$

By using the similar operations used in (10), we have

$$\begin{aligned}
&\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1 \frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\ d_{q_2}\sigma \qquad (12)\\
=\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left[\phi(\beta,\delta)-\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)-\phi(\beta,\gamma)+\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)\right].
\end{aligned}$$

From (11) and (12), we obtain that

$$\begin{aligned}
I_2 =\ & \frac{[5]_{q_1}-1}{[6]_{q_1}(\beta-\alpha)(\delta-\gamma)}\left\{\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\beta,y)\ {}^{\delta}d_{q_2}y-\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},y\right)\ {}^{\delta}d_{q_2}y\right\}\\
&+\frac{[5]_{q_2}\left([5]_{q_1}-1\right)}{[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left[\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)-\phi(\beta,\delta)\right]\\
&+\frac{[5]_{q_1}-1}{q_2[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left[\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)-\phi(\beta,\gamma)\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_3 =\ & \frac{[5]_{q_2}-1}{[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left\{\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta \phi(\varkappa,\delta)\ {}^{\beta}d_{q_1}\varkappa-\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta \phi\left(\varkappa,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\ {}^{\beta}d_{q_1}\varkappa\right\}\\
&+\frac{[5]_{q_2}-1}{q_1[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left[\phi\left(\alpha,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)-\phi(\alpha,\delta)\right]\\
&+\frac{[5]_{q_1}\left([5]_{q_2}-1\right)}{[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\left[\phi\left(\beta,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)-\phi(\beta,\delta)\right].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\ d_{q_2}\sigma \qquad (13)\\
=\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left[\phi(\beta,\delta)-\phi(\alpha,\delta)-\phi(\beta,\gamma)+\phi(\alpha,\gamma)\right],
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 \sigma\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\ d_{q_2}\sigma \qquad (14)\\
=\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\beta,y)\ {}^{\delta}d_{q_2}y-\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\alpha,y)\ {}^{\delta}d_{q_2}y\right.\\
&\left.-\frac{1}{q_2}\phi(\beta,\gamma)+\frac{1}{q_2}\phi(\alpha,\gamma)\right\},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 \tau\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\ d_{q_2}\sigma \qquad (15)\\
=\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\left\{\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta \phi(\varkappa,\delta)\ {}^{\beta}d_{q_1}\varkappa-\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta \phi(\varkappa,\gamma)\ {}^{\beta}d_{q_1}\varkappa\right.\\
&\left.-\frac{1}{q_1}\phi(\alpha,\delta)+\frac{1}{q_1}\phi(\alpha,\gamma)\right\}
\end{aligned}$$

and

$$\int_0^1\int_0^1 \tau\sigma \frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}d_{q_1}\tau\, d_{q_2}\sigma \tag{16}$$

$$\begin{aligned}
&= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\Bigg\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n+1}\alpha+\left(1-q_1^{n+1}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\\
&-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n+1}\alpha+\left(1-q_1^{n+1}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\\
&-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m+1}\gamma+\left(1-q_2^{m+1}\right)\delta\right)\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\Bigg\}\\
&= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\Bigg\{\frac{1}{q_1q_2}\Bigg[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\\
&-\sum_{m=0}^{\infty}q_2^m\phi\left(\alpha,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)-\sum_{n=0}^{\infty}q_1^n\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,\gamma\right)+\phi(\alpha,\gamma)\Bigg]\\
&-\frac{1}{q_1}\Bigg[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\\
&-\sum_{m=0}^{\infty}q_2^m\phi\left(\alpha,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\Bigg]-\frac{1}{q_2}\Bigg[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\\
&-\sum_{n=0}^{\infty}q_1^n\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,\gamma\right)\Bigg]+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\Bigg\}\\
&= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\Bigg\{\frac{(1-q_1)(1-q_2)}{q_1q_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)\\
&-\frac{1-q_2}{q_1q_2}\sum_{m=0}^{\infty}q_2^m\phi\left(\alpha,q_2^{m}\gamma+\left(1-q_2^{m}\right)\delta\right)-\frac{1-q_1}{q_1q_2}\sum_{n=0}^{\infty}q_1^n\phi\left(q_1^{n}\alpha+\left(1-q_1^{n}\right)\beta,\gamma\right)+\frac{1}{q_1q_2}\phi(\alpha,\gamma)\Bigg\}\\
&= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\Bigg\{\frac{1}{q_1q_2(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\phi(\varkappa,y)\ {}^{\beta}d_{q_1}\varkappa\ {}^{\delta}d_{q_2}y\\
&-\frac{1}{q_1q_2(\delta-\gamma)}\int_\gamma^\delta\phi(\alpha,y)\ {}^{\delta}d_{q_2}y-\frac{1}{q_1q_2(\beta-\alpha)}\int_\alpha^\beta\phi(\varkappa,\gamma)\ {}^{\beta}d_{q_1}\varkappa+\frac{1}{q_1q_2}\phi(\alpha,\gamma)\Bigg\}.
\end{aligned}$$

From (13)-(16), we obtain that

$$\begin{aligned}
I_4 &= \frac{1}{(\beta-\alpha)(\delta-\gamma)}\Bigg\{\frac{1}{q_1q_2(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\phi(\varkappa,y)\ {}^{\beta}d_{q_1}\varkappa\ {}^{\delta}d_{q_2}y\\
&-\frac{1}{q_1q_2(\beta-\alpha)}\int_\alpha^\beta\phi(\varkappa,\gamma)\ {}^{\beta}d_{q_1}\varkappa-\frac{[5]_{q_2}}{[6]_{q_2}}\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta\phi(\varkappa,\delta)\ {}^{\beta}d_{q_1}\varkappa\\
&+\frac{[5]_{q_2}}{[6]_{q_2}}\frac{1}{q_1(\beta-\alpha)}\int_\alpha^\beta\phi(\varkappa,\gamma)\ {}^{\beta}d_{q_1}\varkappa-\frac{1}{q_1q_2(\delta-\gamma)}\int_\gamma^\delta\phi(\alpha,y)\ {}^{\delta}d_{q_2}y
\end{aligned}$$

$$
\begin{aligned}
&-\frac{[5]_{q_1}}{[6]_{q_1}}\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\beta,y)\ {}^\delta d_{q_2}y+\frac{[5]_{q_1}}{[6]_{q_1}}\frac{1}{q_2(\delta-\gamma)}\int_\gamma^\delta \phi(\alpha,y)\ {}^\delta d_{q_2}y\\
&+\frac{\phi(\alpha,\gamma)}{q_1q_2[6]_{q_1}[6]_{q_2}}+\frac{[5]_{q_1}[5]_{q_2}}{[6]_{q_1}[6]_{q_2}}\phi(\beta,\delta)+\frac{[5]_{q_2}}{q_1[6]_{q_1}[6]_{q_2}}\phi(\alpha,\delta)+\frac{[5]_{q_1}}{q_2[6]_{q_1}[6]_{q_2}}\phi(\beta,\gamma)\Bigg\}.
\end{aligned}
$$

Now, the calculated integrals from I_1 to I_4 yield

$$
\begin{aligned}
& I_1+I_2+I_3+I_4\\
= & \frac{1}{[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\Bigg\{\frac{q_1[4]_{q_1}}{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\gamma\right)+q_1[4]_{q_1}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\delta\right)\\
& +q_1q_2[4]_{q_1}[4]_{q_2}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+\frac{q_2[4]_{q_2}}{q_1}\phi\left(\alpha,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+q_2[4]_{q_2}\phi\left(\beta,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\Bigg\}\\
& +\frac{\phi(\alpha,\gamma)+q_1\phi(\beta,\gamma)+q_2\phi(\alpha,\delta)+q_1q_2\phi(\beta,\delta)}{q_1q_2[6]_{q_1}[6]_{q_2}(\beta-\alpha)(\delta-\gamma)}\\
& -\frac{1}{q_1q_2[6]_{q_2}(\beta-\alpha)^2(\delta-\gamma)}\int_\alpha^\beta\left[\phi(\varkappa,\gamma)+q_2^2[4]_{q_2}\phi\left(\varkappa,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)+q_2\phi(\varkappa,\delta)\right]{}^\beta d_{q_1}\varkappa\\
& -\frac{1}{q_1q_2[6]_{q_1}(\beta-\alpha)(\delta-\gamma)^2}\int_\gamma^\delta\left[q_1\phi(\beta,y)+q_1^2[4]_{q_1}\phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},y\right)+\phi(\alpha,y)\right]{}^\delta d_{q_2}y\\
& +\frac{1}{q_1q_2(\beta-\alpha)^2(\delta-\gamma)^2}\int_\alpha^\beta\int_\gamma^\delta \phi(\varkappa,y)\ {}^\beta d_{q_1}\varkappa\ {}^\delta d_{q_2}y
\end{aligned}
$$

and multiplying the resultant one with $(\beta-\alpha)(\delta-\gamma)$, we obtain the desirable equality (8) which accomplishes the proof. □

Remark 3.2. *Under the given conditions of Lemma 3.1 with $q_1, q_2 \to 1^-$, we obtain the succeeding identity*

$$
\begin{aligned}
& \frac{\phi\left(\frac{\alpha+\beta}{2},\gamma\right)+\phi\left(\frac{\alpha+\beta}{2},\delta\right)+4\phi\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}\right)+\phi\left(\alpha,\frac{\gamma+\delta}{2}\right)+\phi\left(\beta,\frac{\gamma+\delta}{2}\right)}{9}\\
& +\frac{\phi(\alpha,\gamma)+\phi(\alpha,\delta)+\phi(\beta,\gamma)+\phi(\beta,\delta)}{36}\\
& -\frac{1}{6(\beta-\alpha)}\int_\alpha^\beta\left[\phi(\varkappa,\gamma)+4\phi\left(\varkappa,\frac{\gamma+\delta}{2}\right)+\phi(\varkappa,\delta)\right]d\varkappa\\
& -\frac{1}{6(\delta-\gamma)}\int_\gamma^\delta\left[\phi(\alpha,y)+4\phi\left(\frac{\alpha+\beta}{2},y\right)+\phi(\beta,y)\right]dy\\
& +\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\phi(\varkappa,y)\,d\varkappa dy\\
= & (\beta-\alpha)(\delta-\gamma)\int_0^1\int_0^1\Lambda(\tau)\Lambda(\sigma)\frac{\partial^2\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{\partial\tau\,\partial\sigma}d\tau\,d\sigma
\end{aligned}
\tag{17}
$$

where

$$
\Lambda(\tau)=\begin{cases}\tau-\frac{1}{6}, & \tau\in\left[0,\frac{1}{2}\right)\\[2ex] \tau-\frac{5}{6}, & \tau\in\left[\frac{1}{2},1\right]\end{cases}
$$

and

$$\Lambda(\sigma)=\begin{cases}\sigma-\frac{1}{6}, & \sigma\in\left[0,\frac{1}{2}\right)\\ \sigma-\frac{5}{6}, & \sigma\in\left[\frac{1}{2},1\right]\end{cases}$$

which is proved by Özdemir et al. in [33, Lemma 1].

For reasons of brevity, let us prove another lemma that we will use frequently in our transactions:

Lemma 3.3. *The following q-integrals hold:*

$$A_1(q)=\int_0^{\frac{1}{[2]_q}}\tau\left|\tau-\frac{1}{[6]_q}\right|d_q\tau=q^2\left(\frac{2[2]_q^2+q[3]_q[6]_q^2}{[2]_q^3[3]_q[6]_q^3}\right), \tag{18}$$

$$A_2(q)=\int_0^{\frac{1}{[2]_q}}(1-\tau)\left|\tau-\frac{1}{[6]_q}\right|d_q\tau=q\left(\frac{2[2]_q^2[3]_q[6]_q-2q[2]_q^2+\left(-1+q^3+q^4\right)[3]_q[6]_q^2}{[2]_q^3[3]_q[6]_q^3}\right), \tag{19}$$

$$\begin{aligned}A_3(q)&=\int_{\frac{1}{[2]_q}}^1\tau\left|\tau-\frac{[5]_q}{[6]_q}\right|d_q\tau\\&=\frac{2q^2[2]_q^2[5]_q^3-\left(q+3q^2+5q^3+6q^4+4q^5+2q^6\right)[6]_q^2}{[2]_q^3[3]_q[6]_q^3},\end{aligned} \tag{20}$$

$$\begin{aligned}A_4(q)&=\int_{\frac{1}{[2]_q}}^1\left|\tau-\frac{[5]_q}{[6]_q}\right|(1-\tau)\,d_q\tau\\&=\frac{2\left[\left(q^5+q^6-1\right)[3]_q+[2]_q[5]_q\right][5]_q^2}{[2]_q[3]_q[6]_q^3}+\frac{1+[2]_q^2-q[2]_q[5]_q}{[2]_q^3[6]_q}-\frac{[2]_q^3+1}{[2]_q^3[3]_q},\end{aligned} \tag{21}$$

$$A_5(q)=\int_0^{\frac{1}{[2]_q}}\left|\tau-\frac{1}{[6]_q}\right|d_q\tau=\frac{2q[2]_q^2+[6]_q^2-[2]_q^2[6]_q}{[2]_q^3[6]_q^2}, \tag{22}$$

$$A_6(q)=\int_{\frac{1}{[2]_q}}^1\left|\tau-\frac{[5]_q}{[6]_q}\right|d_q\tau=\frac{\left(1+[2]_q^2\right)[6]_q^2-\left(1+[7]_q\right)[2]_q^2[5]_q}{[2]_q^3[6]_q^2}. \tag{23}$$

Proof. Since $\frac{1}{[2]_q}\geq\frac{1}{[6]_q}$ for $q\in(0,1)$, we have the following equality:

$$\begin{aligned}A_1(q)&=\int_0^{\frac{1}{[2]_q}}\tau\left|\tau-\frac{1}{[6]_q}\right|d_q\tau\\&=\int_0^{\frac{1}{[6]_q}}\tau\left(\frac{1}{[6]_q}-\tau\right)d_q\tau+\int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}}\tau\left(\tau-\frac{1}{[6]_q}\right)d_q\tau\end{aligned}$$

$$= \int_0^{\frac{1}{[6]_q}} \left(\frac{\tau}{[6]_q} - \tau^2\right) d_q\tau + \int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}} \tau\left(\tau - \frac{1}{[6]_q}\right) d_q\tau$$

$$= \left(\frac{\tau^2}{[2]_q [6]_q} - \frac{\tau^3}{[3]_q}\right)_0^{\frac{1}{[6]_q}} + \left(\frac{\tau^3}{[3]_q} - \frac{\tau^2}{[2]_q [6]_q}\right)_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}}$$

$$= q^2 \frac{2[2]_q^2 + q[3]_q [6]_q^2}{[2]_q^3 [3]_q [6]_q^3}.$$

Similarly, the q-integrals (19)-(23) can be obtained and the proof is completed. □

4. Some new q_1q_2-Simpson's type inequalities

In this section, we prove some new quantum boundaries for quantum Simpson's inequalities using the Lemma 3.1.

Theorem 4.1. *We assume that the conditions of Lemma 3.1 are satisfy. Then, we obtain the succeeding inequality provided that* $\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|$ *is convex on* Δ

$$\begin{aligned} \left|{}^{\beta,\delta}\mathcal{I}_{q_1,q_2}(\phi)\right| \leq\ & (\beta-\alpha)(\delta-\gamma)\Bigg[(A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right| \\ &+(A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right| \\ &+(A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right| \\ &+(A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\Bigg]. \end{aligned} \tag{24}$$

Proof. By taking the modulus of the identity in Lemma 3.1, because of the properties of the modulus, we find that

$$\begin{aligned} &\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi)\right| \\ \leq\ & (\beta-\alpha)(\delta-\gamma)\int_0^1\int_0^1 \left|\Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\right| \left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right| d_{q_1}\tau\, d_{q_2}\sigma. \end{aligned} \tag{25}$$

Using the convexity of $\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|$, the inequality (25) becomes

$$\begin{aligned} &\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi)\right| \\ \leq\ & (\beta-\alpha)(\delta-\gamma)\int_0^1 \Lambda_{q_2}(\sigma)\left[\int_0^1 \Lambda_{q_1}(\tau)\left\{\tau\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\right.\right. \\ &\left.\left.+(1-\tau)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\right\} d_{q_1}\tau\right] d_{q_2}\sigma. \end{aligned} \tag{26}$$

Now, we compute the integrals appeared in the right side of the inequality (26)

$$\begin{aligned}
&\int_0^1 \Lambda_{q_1}(\tau)\left\{\tau\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|+(1-\tau)\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\right\}d_{q_1}\tau\\
=\;&\int_0^{\frac{1}{[2]_q}}\tau\left|\tau-\frac{1}{[6]_q}\right|\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|d_{q_1}\tau\\
&+\int_0^{\frac{1}{[2]_q}}(1-\tau)\left|\tau-\frac{1}{[6]_q}\right|\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|d_{q_1}\tau\\
&+\int_{\frac{1}{[2]_q}}^1\tau\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|d_{q_1}\tau\\
&+\int_0^{\frac{1}{[2]_q}}(1-\tau)\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|d_{q_1}\tau.
\end{aligned}$$

From (18)-(21), we obtain that

$$\begin{aligned}
&\int_0^1 \Lambda_{q_1}(\tau)\left\{\tau\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|+(1-\tau)\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\right\}d_{q_1}\tau\\
=\;&\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|(A_1(q_1)+A_3(q_1))\\
&+\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|(A_2(q_1)+A_4(q_1)).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left|^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi)\right|\\
\leq\;&(\beta-\alpha)(\delta-\gamma)\int_0^1\Lambda_{q_2}(\sigma)\left[\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|(A_1(q_1)+A_3(q_1))\right.\\
&\left.+\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|(A_2(q_1)+A_4(q_1))\right]d_{q_2}\sigma\\
\leq\;&(\beta-\alpha)(\delta-\gamma)\int_0^1\Lambda_{q_2}(\sigma)\left[\left\{\sigma\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|+(1-\sigma)\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\right.\right.\\
&\times(A_1(q_1)+A_3(q_1))\}+\left\{\sigma\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|+(1-\sigma)\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\right.\\
&\times(A_2(q_1)+A_4(q_1))\}]\,d_{q_2}\sigma\\
=\;&(\beta-\alpha)(\delta-\gamma)(A_1(q_1)+A_3(q_1))\left[\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\int_0^{\frac{1}{[2]_q}}\sigma\left|\sigma-\frac{1}{[6]_q}\right|d_{q_2}\sigma\right.\\
&+\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\int_0^{\frac{1}{[2]_q}}(1-\sigma)\left|\sigma-\frac{1}{[6]_q}\right|d_{q_2}\sigma+\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\int_{\frac{1}{[2]_q}}^1\sigma\left|\sigma-\frac{[5]_q}{[6]_q}\right|d_{q_2}\sigma\\
&\left.+\left|\frac{^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{^{\beta}\partial_{q_1}\tau\,^{\delta}\partial_{q_2}\sigma}\right|\int_{\frac{1}{[2]_q}}^1(1-\sigma)\left|\sigma-\frac{[5]_q}{[6]_q}\right|d_{q_2}\sigma\right]
\end{aligned}$$

$$+(\beta-\alpha)(\delta-\gamma)(A_2(q_1)+A_4(q_1))\left[\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\int_0^{\frac{1}{[2]_q}}\sigma\left|\sigma-\frac{1}{[6]_q}\right|d_{q_2}\sigma\right.$$
$$+\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\int_0^{\frac{1}{[2]_q}}(1-\sigma)\left|\sigma-\frac{1}{[6]_q}\right|d_{q_2}\sigma+\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\int_{\frac{1}{[2]_q}}^1\sigma\left|\sigma-\frac{[5]_q}{[6]_q}\right|d_{q_2}\sigma$$
$$\left.+\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\int_{\frac{1}{[2]_q}}^1(1-\sigma)\left|\sigma-\frac{[5]_q}{[6]_q}\right|d_{q_2}\sigma\right].$$

From (18)-(21), we have

$$\begin{aligned}\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,q_2}(\phi)\right| \leq\ & (\beta-\alpha)(\delta-\gamma)\left[(A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\right.\\ &+(A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\\ &+(A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\\ &\left.+(A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{\beta,\delta}\partial_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|\right].\end{aligned}$$

Hence, the proof is completed. □

Remark 4.2. *Under the given conditions of Theorem 4.1 with $q_1, q_2 \to 1^-$, we obtain the succeeding inequality*

$$\begin{aligned}&\left|\frac{\phi\left(\frac{\alpha+\beta}{2},\gamma\right)+\phi\left(\frac{\alpha+\beta}{2},\delta\right)+4\phi\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}\right)+\phi\left(\alpha,\frac{\gamma+\delta}{2}\right)+\phi\left(\beta,\frac{\gamma+\delta}{2}\right)}{9}\right.\\ &+\frac{\phi(\alpha,\gamma)+\phi(\alpha,\delta)+\phi(\beta,\gamma)+\phi(\beta,\delta)}{36}\\ &-\frac{1}{6(\beta-\alpha)}\int_\alpha^\beta\left[\phi(\varkappa,\gamma)+4\phi\left(\varkappa,\frac{\gamma+\delta}{2}\right)+\phi(\varkappa,\delta)\right]d\varkappa\\ &-\frac{1}{6(\delta-\gamma)}\int_\gamma^\delta\left[\phi(\alpha,y)+4\phi\left(\frac{\alpha+\beta}{2},y\right)+\phi(\beta,y)\right]dy\\ &\left.+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\phi(\varkappa,y)\,d\varkappa dy\right|\\ \leq\ & \frac{25(\beta-\alpha)(\delta-\gamma)}{72}\left[\frac{\left|\frac{\partial^2\phi(\alpha,\gamma)}{\partial\tau\partial\sigma}\right|+\left|\frac{\partial^2\phi(\alpha,\delta)}{\partial\tau\partial\sigma}\right|+\left|\frac{\partial^2\phi(\beta,\gamma)}{\partial\tau\partial\sigma}\right|+\left|\frac{\partial^2\phi(\beta,\delta)}{\partial\tau\partial\sigma}\right|}{72}\right]\end{aligned}\tag{27}$$

which is shown by Özdemir et al. in [33, Theorem 3].

Theorem 4.3. *We assume that the conditions of Lemma 3.1 hold. If $\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p$ is convex on Δ for some $p>1$ and $\frac{1}{r}+\frac{1}{p}=1$, then we obtain the succeeding inequality*

$$\begin{aligned}&\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,\,q_2}(\phi)\right|\\ \leq\ & (\beta-\alpha)(\delta-\gamma)\frac{\left\{q_1^r[5]_{q_1}^r+q_1^{5r}\left([2]_{q_1}^{r+1}-1\right)\right\}^{\frac{1}{r}}\left\{q_2^r[5]_{q_2}^r+q_2^{5r}\left([2]_{q_2}^{r+1}-1\right)\right\}^{\frac{1}{r}}}{[2]_{q_1}^2[2]_{q_2}^2[6]_{q_1}[6]_{q_2}[r+1]_{q_1}^{\frac{1}{r}}[r+1]_{q_2}^{\frac{1}{r}}}\end{aligned}\tag{28}$$

$$\times\left(\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}+q_2\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}\right.$$
$$\left.+q_1\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}+q_1q_2\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}\right)^{\frac{1}{p}}.$$

Proof. Applying the well-known Hölder's inequality for q_1q_2-integrals to the integrals in the right side of (25), it is found that

$$\begin{aligned}
&\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,\,q_2}(\phi)\right| \qquad (29)\\
\leq\;& (\beta-\alpha)(\delta-\gamma)\left[\left(\int_0^1\int_0^1\left|\Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\right|^{r}d_{q_1}\tau d_{q_2}\sigma\right)^{\frac{1}{r}}\right.\\
&\left.\times\left(\int_0^1\int_0^1\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}\right].
\end{aligned}$$

By applying the convexity of $\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p_1}$, the inequality (29) becomes

$$\begin{aligned}
&\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,\,q_2}(\phi)\right| \qquad (30)\\
\leq\;& (\beta-\alpha)(\delta-\gamma)\left[\left(\int_0^1\int_0^1\left|\Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\right|^{r}d_{q_1}\tau d_{q_2}\sigma\right)^{\frac{1}{r}}\right.\\
&\times\left(\int_0^1\int_0^1\left[\tau\sigma\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}+\tau(1-\sigma)\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}\right.\right.\\
&\left.\left.\left.+(1-\tau)\sigma\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}+(1-\tau)(1-\sigma)\left|\frac{{}^{\beta,\,\delta}\partial^{2}_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^{p}\right]d_{q_1}\tau d_{q_2}\sigma\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Now, for $0\leq\tau,\sigma\leq 1$, we know $\tau-\frac{1}{[6]_{q_1}}\leq\tau-\frac{\tau}{[6]_{q_1}}$, $\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\leq\tau-\frac{\tau[5]_{q_1}}{[6]_{q_1}}$, and this yields

$$\begin{aligned}
&\int_0^1\int_0^1\left|\Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\right|^{r}d_{q_1}\tau d_{q_2}\sigma\\
=\;&\int_0^1\int_0^{\frac{1}{[2]_{q_2}}}\left|\Lambda_{q_1}(\tau)\right|^{r}\left|\sigma-\frac{1}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma d_{q_1}\tau\\
&+\int_0^1\int_{\frac{1}{[2]_{q_2}}}^1\left|\Lambda_{q_1}(\tau)\right|^{r}\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma d_{q_1}\tau\\
=\;&\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
&+\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
&+\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{\frac{1}{[2]_{q_2}}}^1\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma d_{q_1}\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{1}{[2]_{q_1}}}^{1}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{\frac{1}{[2]_{q_2}}}^{1}\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma d_{q_1}\tau\\
\leq\;&\int_{0}^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{\tau}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{0}^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{\sigma}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
&+\int_{\frac{1}{[2]_{q_1}}}^{1}\left|\tau-\frac{\tau\,[5]_{q_1}}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{0}^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{\sigma}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
&+\int_{0}^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{\tau}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{\frac{1}{[2]_{q_2}}}^{1}\left|\sigma-\frac{\sigma\,[5]_{q_2}}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
&+\int_{\frac{1}{[2]_{q_1}}}^{1}\left|\tau-\frac{\tau\,[5]_{q_1}}{[6]_{q_1}}\right|^{r}d_{q_1}\tau\int_{\frac{1}{[2]_{q_2}}}^{1}\left|\sigma-\frac{\sigma\,[5]_{q_2}}{[6]_{q_2}}\right|^{r}d_{q_2}\sigma\\
=\;&\left|1-\frac{1}{[6]_{q_1}}\right|^{r}\left|1-\frac{1}{[6]_{q_2}}\right|^{r}\int_{0}^{\frac{1}{[2]_{q_1}}}\int_{0}^{\frac{1}{[2]_{q_2}}}\tau^{r}\sigma^{r}d_{q_2}\sigma d_{q_1}\tau\\
&+\left|1-\frac{[5]_{q_1}}{[6]_{q_1}}\right|^{r}\left|1-\frac{1}{[6]_{q_2}}\right|^{r}\int_{\frac{1}{[2]_{q_1}}}^{1}\int_{0}^{\frac{1}{[2]_{q_2}}}\tau^{r}\sigma^{r}d_{q_2}\sigma d_{q_1}\tau\\
&+\left|1-\frac{1}{[6]_{q_1}}\right|^{r}\left|1-\frac{[5]_{q_2}}{[6]_{q_2}}\right|^{r}\int_{0}^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^{1}\tau^{r}\sigma^{r}d_{q_2}\sigma d_{q_1}\tau\\
&+\left|1-\frac{[5]_{q_1}}{[6]_{q_1}}\right|^{r}\left|1-\frac{[5]_{q_2}}{[6]_{q_2}}\right|^{r}\int_{\frac{1}{[2]_{q_1}}}^{1}\int_{\frac{1}{[2]_{q_2}}}^{1}\tau^{r}\sigma^{r}d_{q_2}\sigma d_{q_1}\tau\\
=\;&\frac{1}{[2]_{q_1}^{r+1}[2]_{q_2}^{r+1}[6]_{q_1}^{r}[6]_{q_2}^{r}[r+1]_{q_1}[r+1]_{q_2}}\\
&\times\Big\{q_1^{r}q_2^{r}[5]_{q_1}^{r}[5]_{q_2}^{r}+q_1^{5r}q_2^{r}[5]_{q_2}^{r}\left([2]_{q_1}^{r+1}-1\right)+q_1^{r}q_2^{5r}[5]_{q_1}^{r}\left([2]_{q_2}^{r+1}-1\right)\\
&+q_1^{5r}q_2^{5r}\left([2]_{q_1}^{r+1}-1\right)\left([2]_{q_2}^{r+1}-1\right)\Big\}\\
=\;&\frac{\left\{q_1^{r}[5]_{q_1}^{r}+q_1^{5r}\left([2]_{q_1}^{r+1}-1\right)\right\}\left\{q_2^{r}[5]_{q_2}^{r}+q_2^{5r}\left([2]_{q_2}^{r+1}-1\right)\right\}}{[2]_{q_1}^{r+1}[2]_{q_2}^{r+1}[6]_{q_1}^{r}[6]_{q_2}^{r}[r+1]_{q_1}[r+1]_{q_2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{0}^{1}\int_{0}^{1}\left|\Lambda_{q_1}(\tau)\Lambda_{q_2}(\sigma)\right|^{r}d_{q_1}\tau d_{q_2}\sigma\right)^{\frac{1}{r}}\\
=\;&\frac{\left\{q_1^{r}[5]_{q_1}^{r}+q_1^{5r}\left([2]_{q_1}^{r+1}-1\right)\right\}^{\frac{1}{r}}\left\{q_2^{r}[5]_{q_2}^{r}+q_2^{5r}\left([2]_{q_2}^{r+1}-1\right)\right\}^{\frac{1}{r}}}{[2]_{q_1}^{\frac{r+1}{r}}[2]_{q_2}^{\frac{r+1}{r}}[6]_{q_1}[6]_{q_2}[r+1]_{q_1}^{\frac{1}{r}}[r+1]_{q_2}^{\frac{1}{r}}}.
\end{aligned}
\tag{31}
$$

Additionally, if we apply the concept of Lemma 2.3 for $\alpha = 0$ to the above quantum integrals, we attain

$$
\int_{0}^{1}\int_{0}^{1}\tau\sigma d_{q_1}\tau d_{q_2}\sigma=\left(\int_{0}^{1}\tau d_{q_1}\tau\right)\left(\int_{0}^{1}\sigma d_{q_2}\sigma\right)=\frac{1}{[2]_{q_1}[2]_{q_2}}, \tag{32}
$$

$$
\int_{0}^{1}\int_{0}^{1}\tau(1-\sigma)d_{q_1}\tau d_{q_2}\sigma=\frac{q_2}{[2]_{q_1}[2]_{q_2}}, \tag{33}
$$

$$\int_0^1\int_0^1 (1-\tau)\,\sigma d_{q_1}\tau d_{q_2}\sigma = \frac{q_1}{[2]_{q_1}[2]_{q_2}}, \tag{34}$$

$$\int_0^1\int_0^1 (1-\tau)(1-\sigma)\,d_{q_1}\tau d_{q_2}\sigma = \frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}. \tag{35}$$

By substituting the calculated integrals (31)-(35) in (30), we obtain the desired inequality (28) which finishes the proof. □

Theorem 4.4. *We suppose that the assumptions of Lemma 3.1 hold. If* $\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p$ *is convex on* Δ *for some* $p\geq 1$*, then we obtain the succeeding inequality*

$$\begin{aligned}
&\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,\,q_2}(\phi)\right| \\
\leq\ & (\beta-\alpha)(\delta-\gamma)\Bigg[A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right) \\
&+A_2(q_1)\left(A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_5^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right) \\
&+A_2(q_1)\left(A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right) \\
&+A_4(q_1)\left(A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right) \\
&+A_4(q_1)\left(A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)\Bigg\}^{\frac{1}{p}}\Bigg].
\end{aligned} \tag{36}$$

Proof. Applying the well-known power mean inequality for q_1q_2-integrals to the integrals in the right side of (25), it is found that

$$\left|{}^{\beta,\,\delta}\mathcal{I}_{q_1,\,q_2}(\phi)\right| \tag{37}$$

$$
\begin{aligned}
\leq\quad & (\beta-\alpha)(\delta-\gamma)\Bigg[\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|d_{q_1}\tau d_{q_2}\sigma\right)^{1-\frac{1}{p}}\\
&\times\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|\right.\\
&\times\left.\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}\\
&+\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_1}\tau\, d_{q_2}\sigma\right)^{1-\frac{1}{p}}\\
&\times\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\right.\\
&\times\left.\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}\\
&+\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|d_{q_1}\tau d_{q_2}\sigma\right)^{1-\frac{1}{p}}\\
&\times\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|\right.\\
&\times\left.\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}\\
&+\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_1}\tau\, d_{q_2}\sigma\right)\\
&\times\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\right.\\
&\times\left.\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}\Bigg].
\end{aligned}
$$

By applying the convexity of $\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p$, we have

$$
\begin{aligned}
&\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|d_{q_1}\tau d_{q_2}\sigma\right)^{1-\frac{1}{p}}\\
&\times\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|\right.\\
&\times\left.\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}
\end{aligned}
\qquad (38)
$$

$$\leq \left(\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|d_{q_1}\tau\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|d_{q_2}\sigma\right)^{1-\frac{1}{p}}$$

$$\left[\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|\left\{\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{1}{[6]_{q_1}}\right|\left(\tau\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right.\right.\right.$$

$$\left.\left.\left.+(1-\tau)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)d_{q_1}\tau\right\}d_{q_2}\sigma\right]^{\frac{1}{p}}$$

$$= A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_2}\sigma\right.$$

$$\left.+A_2(q_1)\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_2}\sigma\right]^{\frac{1}{p}}$$

$$\leq A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|\left(\sigma\left|\frac{{}^{\beta,\,d}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{d}\partial_{q_2}\sigma}\right|^p\right.\right.$$

$$\left.+(1-\sigma)\left|\frac{{}^{\beta,\,d}\partial^2_{q_1,q_2}\phi(\alpha,d)}{{}^{\beta}\partial_{q_1}\tau\,{}^{d}\partial_{q_2}\sigma}\right|^p\right)d_{q_2}\sigma+A_2(q_1)\int_0^{\frac{1}{[2]_{q_2}}}\left|\sigma-\frac{1}{[6]_{q_2}}\right|$$

$$\left.\times\left(\sigma\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+(1-\sigma)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right)d_{q_2}\sigma\right]^{\frac{1}{p}}$$

$$= A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\left\{A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\right.$$

$$\left.+A_2(q_1)\left\{A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\right]^{\frac{1}{p}}.$$

By applying the similar operations, we obtain that

$$\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_1}\tau\, d_{q_2}\sigma\right)^{1-\frac{1}{p}}\left(\int_0^{\frac{1}{[2]_{q_2}}}\int_{\frac{1}{[2]_{q_1}}}^1\left|\tau-\frac{1}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\right. \tag{39}$$

$$\left.\times\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\right)^{\frac{1}{p}}$$

$$\leq A_5^{1-\frac{1}{p}}A_6^{1-\frac{1}{p}}\left[A_1(q_1)\left\{A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\right.$$

$$\left.+A_2(q_1)\left\{A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\right]^{\frac{1}{p}},$$

$$\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|d_{q_1}\tau d_{q_2}\sigma\right)^{1-\frac{1}{p}}\left(\int_{\frac{1}{[2]_{q_2}}}^1\int_0^{\frac{1}{[2]_{q_1}}}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{1}{[6]_{q_2}}\right|\right. \tag{40}$$

$$
\begin{aligned}
&\times\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\Bigg)^{\frac{1}{p}}\\
\leq\;& A_6^{1-\frac{1}{p}}A_5^{1-\frac{1}{p}}\Bigg[A_3(q_1)\left\{A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\\
&+A_4(q_1)\left\{A_1(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_2(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\Bigg]^{\frac{1}{p}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{\frac{1}{[2]_{q_2}}}^{1}\int_{\frac{1}{[2]_{q_1}}}^{1}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_1}\tau\, d_{q_2}\sigma\right)\Bigg[\int_{\frac{1}{[2]_{q_2}}}^{1}\int_{\frac{1}{[2]_{q_1}}}^{1}\left|\tau-\frac{[5]_{q_1}}{[6]_{q_1}}\right|\left|\sigma-\frac{[5]_{q_2}}{[6]_{q_2}}\right| \qquad (41)\\
&\times\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi\left(\tau\alpha+(1-\tau)\beta,\sigma\gamma+(1-\sigma)\delta\right)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p d_{q_1}\tau\, d_{q_2}\sigma\Bigg]^{\frac{1}{p}}\\
\leq\;& A_6^{1-\frac{1}{p}}A_6^{1-\frac{1}{p}}\Bigg[A_3(q_1)\left\{A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\\
&+A_4(q_1)\left\{A_3(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p+A_4(q_2)\left|\frac{{}^{\beta,\,\delta}\partial^2_{q_1,q_2}\phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\,{}^{\delta}\partial_{q_2}\sigma}\right|^p\right\}\Bigg]^{\frac{1}{p}}.
\end{aligned}
$$

From (37)-(41), we obtain the desired inequality and the proof is ended. □

5. Conclusion

In this research, we have proved a new integral identity involving quantum integrals and quantum numbers. We have proved some new Simpson's inequalities for q_1q_2-differentiable co-ordinated convex functions using the newly derived equality. It is also shown that the results presented in this research transformed into some classical results by taking the limits $q_1, q_2 \to 1^-$ in the main results. It is an interesting and new problem that the upcoming researchers may use the techniques of this research and prove Newton's inequalities and similar inequalities for different kinds of convexities in their future work.

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