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Hermite–Hadamard Type Inequalities for Harmonically-convex Functions Using Fuzzy Integrals

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Abstract. In this study, Hermite–Hadamard type inequalities for harmonically-convex functions using fuzzy integrals are presented. Some examples are also given to illustrate the obtained results.

1. Introduction

As a tool for modeling non-deterministic issues, Sugeno in [41] started the study of theory of fuzzy measures and fuzzy integrals. The hypnotizing properties the fuzzy integrals from a mathematical point of view have been studied by many authors. Ralescu and Adams [33] reviewed several comparable characterizations of fuzzy integrals, whereas Wang and Klir [42, 43] offered a summarized version of fuzzy measure theory and gave a broad view of fuzzy measure theory. Researchers have efficaciously applied the fuzzy measures and Sugeno integrals to various fields, such as applying to decision-making [29] and artificial intelligence [45]. Several theoretical and applied fields use integral inequalities as handy tools, see [15, 16]. The interested readers are also referred to Ref. [20] for more information on classical inequalities. The study of inequalities for the Sugeno integrals was initiated by Román-Flores et al. [17, 34–39] and then further developed by Ouyang et al. in [30–32]. Many researchers studied celebrated inequalities using Sugeno integrals, for example, Hu [19] proved Chebyshev type inequalities for Sugeno-like integral by using binary operation called ϱ-seminorm, Caballero and Sadarangani [11] developed a Cauchy–Schwarz type inequality for the Sugeno integral, Agahi et al. [4] showed a generalization of Stolarsky inequality for Sugeno integral and Ouyang et al. established Minkowski type for the Sugeno integral on abstract spaces. For more results on several other types of inequalities based on Sugeno integrals, see [3], [5]-[9], [13], [14], [23], [24], [44] and the references cited therein.

Recently, the integral inequalities for Sugeno integrals using different kinds of convexities is an thought-provoking topic to many authors in the field of fuzzy integrals, see for instance [1], [2], [10], [18], [22], [25]-[28] and the references cited therein. Caballero and Sadarangani in [12] have shown that the classical Hermite–Hadamard inequalities:

$$\varrho\left(\frac{\zeta+\mu}{2}\right) \leq \frac{1}{\mu-\zeta}\int_{\zeta}^{\mu}\varrho(s)ds \leq \frac{\varrho(\zeta)+\varrho(\mu)}{2}, \tag{1}$$

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where $\varrho : [\zeta, \mu] \to \mathbb{R}$ is a convex function, do not hold true for fuzzy integrals in general. They established some Hermite–Hadamard type inequalities for the Sugeno integral and illustrated their results by providing certain examples.

Motivated by the ongoing research about the integral inequalities for the Sugeno integrals involving the different kinds of convex functions, the main objective of this paper is to find an upper bound of the Sugeno integrals for harmonically-convex functions.

In order to proceed to our results, we first give some basic notations and properties of Sugeno integrals. For more details on Sugeno integrals, we refer the interested readers to [41] and [43].

Suppose that $\mathcal{W}$ is σ-algebra of subsets of $\mathbb{R}$ and that $\theta : \mathcal{W} \to [0, \infty)$ is non-negative extended real valued set function, then θ is said to be fuzzy measure if and only if:

1. $\theta(\emptyset) = 0$,
2. $\mathcal{T}, \mathcal{S} \in \mathcal{W}$ and $\mathcal{T} \subset \mathcal{S}$ imply that $\theta(\mathcal{T}) \leq \theta(\mathcal{S})$ (monotonicity),
3. $\{\mathcal{T}_n\} \subset \mathcal{W}, \mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots$, imply $\lim\limits_{n\to\infty} \theta(\mathcal{T}_n) = \theta\left(\bigcup\limits_{n=1}^{\infty} \mathcal{T}_n\right)$ (continuity from below),
4. $\{\mathcal{T}_n\} \subset \mathcal{W}, \mathcal{T}_1 \supset \mathcal{T}_2 \supset \ldots, \theta(\mathcal{T}_1) < \infty$, imply $\lim\limits_{n\to\infty} \theta(\mathcal{T}_n) = \theta\left(\bigcap\limits_{n=1}^{\infty} \mathcal{T}_n\right)$ (continuity from above).

If ϱ is a non-negative real-valued function defined on $\mathbb{R}$, then we will denote by $L_\alpha \varrho = \{s \in \mathbb{R} : \varrho(s) \geq \alpha\} = \{\varrho \geq \alpha\}$ the α-level of ϱ, for $\alpha > 0$ and $L_0 \varrho = \{s \in \mathbb{R} : \varrho(s) \geq 0\}$ =suppϱ, the support of ϱ. It may be noted that if $\alpha \leq \beta$, then $\{\varrho \leq \alpha\} \subset \{\varrho \leq \beta\}$. If θ is a fuzzy measure on $(\mathbb{R}, \mathcal{W})$ by $\mathcal{F}^\theta(\mathbb{R})$, then we mean all θ-measurable functions from $\mathbb{R}$ to $[0, \infty)$.

Suppose that θ is a fuzzy measure on $(\mathbb{R}, \Omega)$. If $\varrho \in \mathcal{F}^\theta(\mathbb{R})$ and $\mathcal{T} \subset \Omega$, then the Sugeno integral (or fuzzy integral) of ϱ on $\mathcal{T}$ with respect to the fuzzy measure θ is defined as:

$$\int_{\mathcal{T}} \varrho d\theta = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \theta\left(\mathcal{T} \cap \{\varrho \geq \alpha\}\right)\right],$$

where $\vee$ and $\wedge$ denote the supremum and infimum on $[0, \infty)$, respectively. The following properties of the Sugeno integral are given in [43].

Proposition 1.1. *If θ is a fuzzy measure on $(\mathbb{R}, \mathcal{S})$, $\mathcal{T} \subset \mathcal{S}$ and $\varrho, \psi \in \mathcal{F}^\theta(\mathbb{R})$, then*

1. $\int_{\mathcal{T}} \varrho d\theta \leq \theta(\mathcal{T})$.
2. $\int_{\mathcal{T}} k\, d\theta = k \wedge \theta(\mathcal{T})$, *$k$ for a non-negative constant.*
3. *If $\varrho \leq \psi$ on $\mathcal{T}$ then* $\int_{\mathcal{T}} \varrho d\theta \leq \int_{\mathcal{T}} \psi d\theta$.
4. $\theta(\mathcal{T} \cap \{\varrho \geq \alpha\}) \geq \alpha \Rightarrow \int_{\mathcal{T}} \varrho d\theta \geq \alpha$.
5. $\theta(\mathcal{T} \cap \{\varrho \geq \alpha\}) \leq \alpha \Rightarrow \int_{\mathcal{T}} \varrho d\theta \leq \alpha$.
6. $\int_{\mathcal{T}} \varrho d\theta < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\theta(\mathcal{T} \cap \{\varrho \geq \gamma\}) < \alpha$.
7. $\int_{\mathcal{T}} \varrho d\theta > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\theta(\mathcal{T} \cap \{\varrho \geq \gamma\}) > \alpha$.

Remark 1.2. *Consider the distribution function Y associated to ϱ on $\mathcal{T}$, that is, $Y(\alpha) = \theta(\mathcal{T} \cap \{\varrho \geq \gamma\})$. Then from properties 4 and 5 of Proposition 1.1*

$$Y(\alpha) = \alpha \Rightarrow \int_{\mathcal{T}} \varrho d\theta = \alpha.$$

Therefore, it follows that any fuzzy integral can be calculated by solving the equation $Y(\alpha) = \alpha$.

2. Main Results

We begin this section with the well-known generalization of the convex functions which is familiar as harmonic convexity of functions and its properties.

Definition 2.1. *[40] A set $\mathcal{I}_{\mathbb{R}} \subset \mathbb{R}\setminus\{0\}$ is said to be harmonic convex set. if*

$$\frac{st}{xs+(1-x)t} \in \mathcal{I}_{\mathbb{R}}, \ \ \forall s,t \in \mathcal{I}_{\mathbb{R}}, x \in [0,1].$$

Definition 2.2. *[21] Let $\mathcal{I}_{\mathbb{R}} \subset \mathbb{R}\setminus\{0\}$ be a real harmonic convex set. A function $\varrho : \mathcal{I}_{\mathbb{R}} \to \mathbb{R}$ is said to be harmonically convex, if*

$$\varrho\left(\frac{st}{xs+(1-x)t}\right) \leq x\varrho(t) + (1-x)\varrho(s) \tag{2}$$

for all $s, t \in \mathcal{I}_{\mathbb{R}}$ and $x \in [0,1]$. If the inequality in (2) is reversed, then ϱ is said to be harmonically concave.

Proposition 2.3. *[21] Let $\mathcal{I}_{\mathbb{R}} \subset \mathbb{R}\setminus\{0\}$ be a real interval and $\varrho : \mathcal{I}_{\mathbb{R}} \to \mathbb{R}$ be a function. Then we have:*

1. *if $\mathcal{I}_{\mathbb{R}} \subset (0,+\infty)$ and ϱ is a convex and non-decreasing function then ϱ is harmonically convex,*
2. *if $\mathcal{I}_{\mathbb{R}} \subset (0,+\infty)$ and ϱ is a harmonically convex and non-increasing function then ϱ is convex,*
3. *if $\mathcal{I}_{\mathbb{R}} \subset (-\infty,0)$ and ϱ is a harmonically convex and non-decreasing function then ϱ is convex,*
4. *if $\mathcal{I}_{\mathbb{R}} \subset (-\infty,0)$ and ϱ is a convex and non-increasing function then ϱ is harmonically convex.*

A result which connects the usual convexity to the harmonic convexity is also given in [21]. We state the result as follows.

Theorem 2.4. *[21] Let $\mathcal{I}_{\mathbb{R}} \subset \mathbb{R}\setminus\{0\}$ be a harmonically convex function and $\zeta, \mu \in \mathcal{I}_{\mathbb{R}}$ with $\zeta < \mu$. If $\varrho \in L([\zeta,\mu])$ then the following inequalities hold*

$$\varrho\left(\frac{2\zeta\mu}{\zeta+\mu}\right) \leq \frac{\zeta\mu}{\mu-\zeta}\int_{\zeta}^{\mu}\frac{\varrho(s)}{s^2}ds \leq \frac{\varrho(\zeta)+\varrho(\mu)}{2}. \tag{3}$$

The above inequalities are sharp.

We will see that the inequalities (3) do not hold for fuzzy integrals in general. To prove our assertion we consider the following examples.

Example 2.5. *Take $\mathcal{T} = [1,2]$ and let θ be the usual Lebesgue measure on $\mathcal{T}$. Let $\varrho : [1,2] \to [0,\infty)$ be defined as $\varrho(s) = s^3$, then the function is convex and nondecreasing on $\mathcal{T}$. The function $\varrho(s) = s^3$ is harmonically convex on $\mathcal{T}$. Now to calculate the Sugeno integral $2\int_1^2 s d\theta$, consider the distribution function Y associated to $\frac{\varrho(s)}{s^2} = s$ on $[1,2]$, then*

$$\begin{aligned} Y(\alpha) &= \theta([1,2] \cap \{\varrho \geq \alpha\}) = \theta([1,2] \cap \{s \geq \alpha\}) \\ &= 2-\alpha \end{aligned}$$

and we solve the equation $2-\alpha = \alpha$. It can be easily seen that the solution of this equation is 1. Therefore by Remark 1.2, we have

$$\frac{\zeta\mu}{\mu-\zeta}\int_{\zeta}^{\mu}\frac{\varrho(s)}{s^2}ds = 2\int_1^2 s d\theta = 2.$$

Also

$$\varrho\left(\frac{2\zeta\mu}{\zeta+\mu}\right)=\varrho\left(\frac{4}{3}\right)=\frac{64}{27}.$$

Therefore,

$$\varrho\left(\frac{2\zeta\mu}{\zeta+\mu}\right)=\frac{64}{27}\geq 2=2\int_1^2 sd\theta=\frac{\zeta\mu}{\mu-\zeta}\int_\zeta^\mu\frac{\varrho(s)}{s^2}ds,$$

which shows that left part of (3) is not satisfied in the fuzzy context.

Example 2.6. *Take $\mathcal{T}=[1,2]$ and let θ be the usual Lebesgue measure on $\mathcal{T}$. Let $\varrho:[1,2]\to[0,\infty)$ be defined as $\varrho(s)=\frac{s}{5}$, then the function is convex and non-decreasing on $\mathcal{T}$. The function $\varrho(s)=\frac{s}{5}$ is harmonically convex on $\mathcal{T}$. Now to calculate the Sugeno integral $2\int_1^2\frac{1}{5s}d\theta$, consider the distribution function Υ associated to $\frac{\varrho(s)}{s^2}=\frac{1}{5s}$ on $[1,2]$, then*

$$\begin{aligned}\Upsilon(\alpha)&=\theta\left([1,2]\cap\{\varrho\geq\alpha\}\right)=\theta\left([1,2]\cap\left\{\frac{1}{5s}\geq\alpha\right\}\right)\\&=\theta\left([1,2]\cap\left\{\frac{1}{s}\geq 5\alpha\right\}\right)\\&=\theta\left([1,2]\cap\left\{s\leq\frac{1}{5\alpha}\right\}\right)=\frac{1}{5\alpha}-1\end{aligned}$$

and we solve the equation $\frac{1}{5\alpha}-1=\alpha$. It can be easily seen that the solution of this equation is $\alpha\approx 1.7913$ (the second solution is negative and hence can be neglected), therefore by Remark 1.2, we have

$$2\int_1^2\frac{5}{s}d\theta\approx 2\times(0.17082)=3.34164.$$

Also

$$\frac{\varrho(\zeta)+\varrho(\mu)}{2}=\frac{\frac{1}{5}+\frac{2}{5}}{2}=0.3.$$

Therefore,

$$\frac{\zeta\mu}{\mu-\zeta}\int_\zeta^\mu\frac{\varrho(s)}{s^2}ds=2\int_1^2\frac{5}{s}d\theta\approx 3.34164\geq 0.3=\frac{\varrho(\zeta)+\varrho(\mu)}{2},$$

which shows that right part of (3) is also not satisfied in the fuzzy framework.

Now we prove Hadamard-type inequalities analogous to (3) for Sugeno integrals.

Theorem 2.7. *Let $\varrho:[\zeta,\mu]\subset\mathbb{R}\backslash\{0\}\to[0,\infty)$ be a harmonically-convex function on $[\zeta,\mu]$ such that $\varrho(\zeta)<\varrho(\mu)$. Let θ be the Lebesgue measure on $[\zeta,\mu]$ with $\mu>\zeta$.*

1. *If $\zeta\mu>0$, then*

$$\int_\zeta^\mu\varrho\, d\theta\leq\min\{\mu-\zeta,\alpha\},\tag{4}$$

where α is a positive root of the equation

$$(\zeta-\mu)\alpha^2+\left(\mu\varrho(\mu)-\zeta\varrho(\zeta)+\mu^2-\zeta\mu\right)\alpha+\left(\zeta\mu-\mu^2\right)\varrho(\mu)=0.\tag{5}$$

2. *If $\zeta\mu < 0$, then*

$$\int_{[\zeta,\mu]\setminus\{0\}} \varrho \, d\theta \leq \min\{\mu - \zeta, \alpha\}, \tag{6}$$

where α is a positive root of the equation

$$(\zeta - \mu)\alpha^2 + \left(\mu\varrho(\mu) - \zeta\varrho(\zeta) - \zeta\mu + \zeta^2\right)\alpha + \left(\zeta\mu - \zeta^2\right)\varrho(\zeta) = 0. \tag{7}$$

Proof. Since ϱ is a harmonically-convex function on $[\zeta, \mu]$, we have

$$\begin{aligned}
\varrho(s) &= \varrho\left(\frac{\zeta\mu}{\frac{\mu(s-\zeta)}{s(\mu-\zeta)}\zeta + \left(1 - \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\right)\mu}\right) \\
&\leq \left(1 - \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\right)\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu) \\
&= \frac{\zeta(\mu - s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu) = h(s).
\end{aligned}$$

By property 3 of Proposition 1.1, we have

$$\int_\zeta^\mu \varrho(s)d\theta \leq \int_\zeta^\mu \left[\frac{\zeta(\mu - s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu)\right]d\theta = \int_\zeta^\mu h(s)d\theta.$$

Let us consider the distribution function Y given by

$$\begin{aligned}
Y(\alpha) &= \theta\left([\zeta,\mu] \cap \{h \geq \alpha\}\right) \\
&= \theta\left([\zeta,\mu] \cap \left\{\frac{\zeta(\mu - s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu) \geq \alpha\right\}\right) \\
&= \theta\left([\zeta,\mu] \cap \left\{\frac{\zeta\mu}{s} \leq \frac{\alpha(\mu-\zeta) - \mu\varrho(\mu) + \zeta\varrho(\zeta)}{\varrho(\zeta) - \varrho(\mu)}\right\}\right).
\end{aligned} \tag{8}$$

Now we consider the following cases.

Case 1: If $\zeta < 0$, $\mu < 0$, then $\zeta\mu > 0$ and $s < 0$. Hence from (8), we have

$$\begin{aligned}
Y(\alpha) &= \theta\left([\zeta,\mu] \cap \{h \geq \alpha\}\right) \\
&= \theta\left([\zeta,\mu] \cap \left\{s \geq \frac{\zeta\mu\left(\varrho(\zeta) - \varrho(\mu)\right)}{\alpha(\mu-\zeta) - \mu\varrho(\mu) + \zeta\varrho(\zeta)}\right\}\right) \\
&= \mu - \frac{\zeta\mu\left(\varrho(\zeta) - \varrho(\mu)\right)}{\alpha(\mu-\zeta) - \mu\varrho(\mu) + \zeta\varrho(\zeta)} = \alpha.
\end{aligned}$$

The last equation can be written as

$$(\zeta - \mu)\alpha^2 + \left(\mu\varrho(\mu) - \zeta\varrho(\zeta) + \mu^2 - \zeta\mu\right)\alpha + \left(\zeta\mu - \mu^2\right)\varrho(\mu) = 0.$$

Hence the inequality (4) follows, where α is a positive root of the equation (5).

Case 2: If $\zeta > 0$, $\mu > 0$, then $\zeta\mu > 0$ and $s > 0$. Hence from (8), we have

$$\begin{aligned}
Y(\alpha) &= \theta\left([\zeta,\mu] \cap \{h \geq \alpha\}\right) \\
&= \theta\left([\zeta,\mu] \cap \left\{s \geq \frac{\zeta\mu\left(\varrho(\zeta) - \varrho(\mu)\right)}{\alpha(\mu-\zeta) - \mu\varrho(\mu) + \zeta\varrho(\zeta)}\right\}\right) \\
&= \mu - \frac{\zeta\mu\left(\varrho(\zeta) - \varrho(\mu)\right)}{\alpha(\mu-\zeta) - \mu\varrho(\mu) + \zeta\varrho(\zeta)} = \alpha.
\end{aligned}$$

Equivalently, the last equation can be written as

$$(\zeta-\mu)\,\alpha^2+\left(\mu\varrho(\mu)-\zeta\varrho(\zeta)+\mu^2-\zeta\mu\right)\alpha+\left(\zeta\mu-\mu^2\right)\varrho(\mu)=0.$$

Hence we obtain the inequality (4), where α is a positive root of the equation (5).
Case 3: If $\zeta<0$, $\mu>0$, then $[\zeta,\mu]\setminus\{0\}=[\zeta,0)\cup(0,\mu]$, $\zeta\mu<0$ and $s>0$ or $s<0$.
If $s>0$, then from (8) we obtain

$$\begin{aligned}
\Upsilon(\alpha)&=\theta\left(([\zeta,\mu]\setminus\{0\})\cap\{h\geq\alpha\}\right)\\
&=\theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{\frac{\zeta\mu}{s}\leq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\varrho(\zeta)-\varrho(\mu)}\right\}\right)\\
&=\theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{\frac{1}{s}\geq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}\right\}\right)\\
&=\theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{s\leq\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}\right\}\right)\\
&=\theta\left([\zeta,0)\cup\left(0,\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}\right]\right)\\
&=\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}-\zeta=\alpha.
\end{aligned}$$

The last equation can be written as

$$(\zeta-\mu)\,\alpha^2+\left(\mu\varrho(\mu)-\zeta\varrho(\zeta)-\zeta\mu+\zeta^2\right)\alpha+\left(\zeta\mu-\zeta^2\right)\varrho(\zeta)=0.$$

Hence we obtain the inequality (6), where α is a positive root of the equation (7).
If $s<0$, then from (8) we obtain

$$\begin{aligned}
\Upsilon(\alpha)&=\theta\left([\zeta,\mu]\cap\{h\geq\alpha\}\right)\\
&=\theta\left([\zeta,0)\cup(0,\mu]\cap\left\{\frac{\zeta\mu}{s}\leq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\varrho(\zeta)-\varrho(\mu)}\right\}\right)\\
&=\theta\left([\zeta,0)\cup(0,\mu]\cap\left\{\frac{1}{s}\geq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}\right\}\right)\\
&=\theta\left([\zeta,0)\cup(0,\mu]\cap\left\{s\leq\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}\right\}\right)\\
&=\theta\left(\left[\zeta,\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}\right]\right)\\
&=\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}-\zeta=\alpha.
\end{aligned}$$

Thus, we get the inequality (6), where α is a positive root of the equation (7). □

Example 2.8. *Take $\mathcal{T}=[1,2]$ and let θ be the usual Lebesgue measure on $\mathcal{T}$. Let $\varrho:[1,2]\to[0,\infty)$ be defined as $\varrho(s)=s^2\ln s$, then the function is convex and non-decreasing on $\mathcal{T}$. Hence by Proposition 2.3, the function $\varrho(s)=s^2\ln s$ is harmonically convex on $\mathcal{T}$. Clearly, $\varrho(\zeta)=0$ and $\varrho(\mu)=4\ln 2$ and $\zeta\mu>0$.*
Hence by (1) of Theorem 2.7, we have

$$\int_1^2 s^2\ln s\,d\theta\leq\min\{1,\alpha\},$$

where is α is a positive root of the equation

$$\alpha^2 + (4\ln 2 + 2)\,\alpha - 8\ln 2 = 0.$$

The solution of this equation are

$$\alpha_1 = 1 + 4\ln 2 + \sqrt{(1 + 4\ln 2)^2 - 8\ln 2} \approx 6.72$$

and

$$\alpha_2 = 1 + 4\ln 2 - \sqrt{(1 + 4\ln 2)^2 - 8\ln 2} \approx 0.825\,17.$$

Thus

$$\int_1^2 s^2 \ln s \, d\theta \le 1 + 4\ln 2 - \sqrt{(1 + 4\ln 2)^2 - 8\ln 2}.$$

Example 2.9. *Take $\mathcal{T} = \left[-\frac{1}{2}, \frac{1}{3}\right] \setminus \{0\}$ and let θ be the usual Lebesgue measure on $\mathcal{T}$. Let $\varrho : \left[-\frac{1}{2}, \frac{1}{3}\right] \setminus \{0\} \to [0, \infty)$ be defined as $\varrho(s) = \frac{1}{s^2}$, then the function $\varrho(s) = \frac{1}{s^2}$ is harmonically convex on $\mathcal{T}$. We observe that $\varrho(\zeta) = 4$ and $\varrho(\mu) = 9$ and $\zeta\mu < 0$.*
Hence by (2) of Theorem 2.7, we have

$$\int_{[\zeta,\mu]\setminus\{0\}} \varrho d\theta \le \min\left\{\frac{5}{6}, \alpha\right\},$$

where α is a positive root of the equation

$$2\alpha^2 - 13\alpha + 4 = 0.$$

The roots of this equation are $\frac{13+\sqrt{137}}{4}$ and $\frac{13-\sqrt{137}}{4}$.
Thus

$$\int_{\left[-\frac{1}{2},\frac{1}{3}\right]\setminus\{0\}} \frac{1}{s^2}\, d\theta \le \frac{13 - \sqrt{137}}{4}.$$

Theorem 2.10. *Let $\varrho : [\zeta, \mu] \subset \mathbb{R}\setminus\{0\} \to [0, \infty)$ be a harmonically-convex function on $[\zeta, \mu]$ such that $\varrho(\zeta) > \varrho(\mu)$. Let θ be the Lebesgue measure on $[\zeta, \mu]$ with $\mu > \zeta$.*

1. *If $\zeta\mu < 0$, then*

$$\int_{[\zeta,\mu]\setminus\{0\}} \varrho \, d\theta \le \min\{\mu - \zeta, \alpha\}, \tag{9}$$

   where α is a positive root of the equation (7) given in Theorem 2.7.

2. *If $\zeta\mu > 0$, then*

$$\int_\zeta^\mu \varrho \, d\theta \le \min\{\mu - \zeta, \alpha\}, \tag{10}$$

   where α is a positive root of the equation (5) given in Theorem 2.7.

Proof. Since ϱ is a harmonically-convex function on $[\zeta, \mu]$, we have

$$\begin{aligned}
\varrho(s) &= \varrho\left(\frac{\zeta\mu}{\frac{\mu(s-\zeta)}{s(\mu-\zeta)}\zeta + \left(1-\frac{\mu(s-\zeta)}{s(\mu-\zeta)}\right)\mu}\right)\\
&\leq \left(1-\frac{\mu(s-\zeta)}{s(\mu-\zeta)}\right)\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu)\\
&= \frac{\zeta(\mu-s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu) = h(s).
\end{aligned}$$

By property 3 of Proposition 1.1, we have

$$\int_{\zeta}^{\mu} \varrho d\theta \leq \int_{\zeta}^{\mu}\left[\frac{\zeta(\mu-s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu)\right]d\theta = \int_{\zeta}^{\mu} h(s)d\theta.$$

Let us consider the distribution function Y given by

$$\begin{aligned}
Y(\alpha) &= \theta\left([\zeta,\mu]\cap\{h\geq\alpha\}\right)\\
&= \theta\left([\zeta,\mu]\cap\left\{\frac{\zeta(\mu-s)}{s(\mu-\zeta)}\varrho(\zeta) + \frac{\mu(s-\zeta)}{s(\mu-\zeta)}\varrho(\mu)\geq\alpha\right\}\right)\\
&= \theta\left([\zeta,\mu]\cap\left\{\frac{\zeta\mu}{s}\geq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\varrho(\zeta)-\varrho(\mu)}\right\}\right).
\end{aligned} \tag{11}$$

Now we consider the following cases.
Case 1: If $\zeta < 0$, $\mu > 0$, then $[\zeta,\mu]\setminus\{0\} = [\zeta, 0)\cup(0,\mu]$, $\zeta\mu < 0$ and $s > 0$ or $s < 0$.
If $s > 0$, then from (11) we obtain

$$\begin{aligned}
Y(\alpha) &= \theta\left(([\zeta,\mu]\setminus\{0\})\cap\{h\geq\alpha\}\right)\\
&= \theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{\frac{\zeta\mu}{s}\geq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\varrho(\zeta)-\varrho(\mu)}\right\}\right)\\
&= \theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{\frac{1}{s}\leq\frac{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}\right\}\right)\\
&= \theta\left(([\zeta,0)\cup(0,\mu])\cap\left\{s\geq\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)}\right\}\right)\\
&= \theta\left(\left(\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)},\mu\right]\right)\\
&= \mu-\frac{\zeta\mu(\varrho(\zeta)-\varrho(\mu))}{\alpha(\mu-\zeta)-\mu\varrho(\mu)+\zeta\varrho(\zeta)} = \alpha.
\end{aligned}$$

Thus, we get the inequality (9).

If $s<0$, then from (11) we obtain

$$\begin{aligned}
\Upsilon(\alpha) &= \theta\left(\left([\zeta,\mu]\setminus\{0\}\right)\cap\{h\geq\alpha\}\right)\\
&= \theta\left(\left([\zeta,0)\cup(0,\mu]\right)\cap\left\{\frac{\zeta\mu}{s}\geq\frac{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}{\varrho\left(\zeta\right)-\varrho(\mu)}\right\}\right)\\
&= \theta\left(\left([\zeta,0)\cup(0,\mu]\right)\cap\left\{\frac{1}{s}\leq\frac{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}\right\}\right)\\
&= \theta\left(\left([\zeta,0)\cup(0,\mu]\right)\cap\left\{s\geq\frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}\right\}\right)\\
&= \theta\left(\left[\frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)},0\right)\cup(0,\mu]\right)\\
&= \mu-\frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}=\alpha.
\end{aligned}$$

Hence we get inequality (9).

Case 2: If $\zeta>0$, $\mu>0$, then $\zeta\mu>0$ and $s>0$. Hence from (11), we have

$$\begin{aligned}
\Upsilon(\alpha) &= \theta\left([\zeta,\mu]\cap\{h\geq\alpha\}\right)\\
&= \theta\left([\zeta,\mu]\cap\left\{s\leq\frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}\right\}\right)\\
&= \frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}-\zeta=\alpha.
\end{aligned}$$

Hence we obtain the inequality (10).

Case 3: If $\zeta<0$, $\mu<0$, then $\zeta\mu>0$ and $s<0$. Hence from (11), we have

$$\begin{aligned}
\Upsilon(\alpha) &= \theta\left([\zeta,\mu]\cap\{h\geq\alpha\}\right)\\
&= \theta\left([\zeta,\mu]\cap\left\{s\leq\frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}\right\}\right)\\
&= \frac{\zeta\mu\left(\varrho\left(\zeta\right)-\varrho(\mu)\right)}{\alpha\left(\mu-\zeta\right)-\mu\varrho\left(\mu\right)+\zeta\varrho\left(\zeta\right)}-\zeta=\alpha.
\end{aligned}$$

From the last equation, we obtain the inequality (10). □

Example 2.11. *Take $\mathcal{T}=[-2,-1]$ and let θ be the usual Lebesgue measure on $\mathcal{T}$. Let $\varrho:[-2,-1]\to[0,\infty)$ be defined as $\varrho(s)=\ln(-s)$, then the function is convex and nonincreasing on $\mathcal{T}$. Hence by Proposition 2.3, the function $\varrho(s)=\ln(-s)$ is harmonically convex on $\mathcal{T}$. Clearly, $\varrho(\zeta)=\ln 2$ and $\varrho(\mu)=0$ and $\zeta\mu>0$.*
Hence by (2) of Theorem 2.10, we have

$$\int_{\zeta}^{\mu}\varrho\,d\theta\leq\min\{1,\alpha\},$$

where α is a positive root of the equation

$$\alpha^2-(2\ln 2-1)\,\alpha=0.$$

The roots of this equation are 0 *and* $\ln 2-1$.
Thus

$$\int_{-2}^{-1}\ln(-s)\;d\theta\leq 2\ln 2-1\approx 0.386294.$$

Example 2.12. *Take* $\mathcal{T} = \left[-\frac{1}{3}, \frac{1}{2}\right] \setminus \{0\}$ *and let* θ *be the usual Lebesgue measure on* $\mathcal{T}$. *Let* $\varrho : \left[-\frac{1}{3}, \frac{1}{2}\right] \setminus \{0\} \to [0, \infty)$ *be defined as* $\varrho(s) = \frac{1}{s^2}$. *Then the function* $\varrho(s) = \frac{1}{s^2}$ *is harmonically convex on* $\mathcal{T}$. *Clearly,* $\varrho(\zeta) = 9$ *and* $\varrho(\mu) = 4$ *and* $\zeta\mu < 0$.
Hence by (1) of Theorem 2.10, we have

$$\int_{\mathcal{T}} \varrho \, d\theta \leq \min\{\mu - \zeta, \alpha\},$$

where α *is a root of the equation*

$$3\alpha^2 - 19\alpha + 9 = 0.$$

The roots of this equation are $\frac{19+\sqrt{253}}{6}$ *and* $\frac{19-\sqrt{253}}{6}$.
Thus

$$\int_{\left[-\frac{1}{3},\frac{1}{2}\right]\setminus\{0\}} \frac{1}{s^2} d\theta \leq \frac{19 - \sqrt{253}}{6} \approx 0.515671.$$

Theorem 2.13. *Let* $\varrho : [\zeta, \mu] \subset \mathbb{R}\setminus\{0\} \to [0, \infty)$ *be a harmonically-convex function on* $[\zeta, \mu]$ *such that* $\varrho(\zeta) = \varrho(\mu)$. *Let* θ *be the Lebesgue measure on* $[\zeta, \mu]$ *with* $\mu > \zeta$.

1. *If* $\zeta, \mu > 0$, *then*

$$\int_{\zeta}^{\mu} \varrho \, d\theta \leq \min\{\mu - \zeta, \mu, \varrho(\zeta)\}. \tag{12}$$

2. *If* $\zeta, \mu < 0$, *then*

$$\int_{\zeta}^{\mu} \varrho \, d\theta \leq \min\{\mu - \zeta, \varrho(\zeta)\}. \tag{13}$$

3. *If* $\zeta < 0$, $\mu > 0$, *then*

$$\int_{[\zeta,\mu]\setminus\{0\}} \varrho \, d\theta \leq \min\{\mu - \zeta, \varrho(\zeta)\}. \tag{14}$$

Proof. (1) and (2) If $\varrho(\zeta) = \varrho(\mu)$, then from the first part of Theorem 2.7 or the second part of Theorem 2.10, we have

$$\int_{\zeta}^{\mu} \varrho \, d\theta \leq \min\{\mu - \zeta, \alpha\},$$

where α is a positive root of the equation

$$\alpha^2 - (\varrho(\zeta) + \mu)\alpha + \mu\varrho(\zeta) = 0. \tag{15}$$

The solutions of the equation (15) are μ and $\varrho(\zeta)$. Hence the inequalities (12) and (13) are proved.
(3) If $\varrho(\zeta) = \varrho(\mu)$, then from the second part of Theorem 2.7 or the first part of Theorem 2.10, we have

$$\int_{[\zeta,\mu]\setminus\{0\}} \varrho \, d\theta \leq \min\{\mu - \zeta, \alpha\},$$

where α is a positive root of the equation

$$\alpha^2 - (\varrho(\zeta) + \zeta)\alpha - \zeta\varrho(\zeta) = 0.$$

The solutions of this equation are ζ and $\varrho(\zeta)$. Thus, we get the inequality (14). □

Example 2.14. *Take* $\mathcal{T} = \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}$ *and let* θ *be the usual Lebesgue measure on* $\mathcal{T}$. *Let* $\varrho : \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\} \to [0, \infty)$ *be defined as* $\varrho(s) = \frac{1}{s^2}$, *then the function* $\varrho(s) = \frac{1}{s^2}$ *is harmonically convex on* $\mathcal{T}$. *We observe that* $\varrho(\zeta) = 4$ *and* $\varrho(\mu) = 4$ *and* $\zeta < 0$ *and* $\mu > 0$.
Hence by (3) of Theorem 2.13, we have

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]\setminus\{0\}} \frac{1}{s^2} d\theta \leq \min\{\mu - \zeta, \varrho(\zeta)\} = \min\{1, 4\} = 1.$$

Example 2.15. *Take* $\mathcal{T} = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ *and let* θ *be the usual Lebesgue measure on* $\mathcal{T}$. *Let* $\varrho : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to [0, \infty)$ *be defined as* $\varrho(s) = \frac{\pi}{2} + \cos s$, *then the function* $\varrho(s) = 1 + \cos s$ *is harmonically convex on* $\mathcal{T}$. *We observe that* $\varrho(\zeta) = \frac{\pi}{2}$ *and* $\varrho(\mu) = \frac{\pi}{2}$ *and* $\zeta > 0$ *and* $\mu > 0$.
Hence by (1) of Theorem 2.13, we have

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 + \cos s)\ d\theta \leq \min\left\{\pi, \frac{3\pi}{2}, \frac{\pi}{2}\right\} = \frac{\pi}{2}.$$

3. Conclusion

We have proved a series of Hermite–Hadamard type inequalities for fuzzy integrals on a fuzzy measure space based on harmonically-convexity. It generalizes the results of [21]. We believe that the obtained results in this study will contribute to estimation and approximation theory in information sciences systems when considering the importance of fuzzy integrals.

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