Filomat 36:12 (2022), 4111–4122
https://doi.org/10.2298/FIL2212111Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strong Convergence of an Iterative Procedure for Pseudomonotone Variational Inequalities and Fixed Point Problems

Tzu-Chien Yin[a], Ariana Pitea[b]

[a] *Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[b] *Department of Mathematics and Informatics, University Polithenica of Bucharest, 060042 Bucharest, Romania*

Abstract. Pseudomonotone variational inequalities have been investigated by many authors, a common assumption "weak sequential continuity" being imposed on pseudomonotone operators. In this paper, we propose an iterative procedure for solving pseudomonotone variational inequalities and fixed point problems of asymptotically pseudocontractive operators by using self-adaptive techniques. Under a weaker assumption than weak sequential continuity imposed on pseudomonotone operators, we prove that the suggested procedure has strong convergence.

1. Introduction

Let H be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed, convex subset of H.

In this paper, we focus on the following variational inequality of finding a point $u^\dagger \in C$ such that

$$\langle \varphi(u^\dagger), u - u^\dagger \rangle \geq 0, \ \forall u \in C, \tag{1}$$

where $\varphi : C \to C$ is a nonlinear operator. Denote the solution set to the variational inequality (1) by $Sol(C, \varphi)$.

The variational inequality (1) was proposed by Stampacchia [20] in 1964. It has been shown that this variational inequality provides a natural, convenient and unified framework for the study of many problems in economics, operations research and engineering, see [1, 2, 10–12, 25, 31, 37, 39, 43, 44]. The variational inequality (1) contains, as special cases, well-known problems in mathematical programming such as: systems of nonlinear equations, optimization problems ([4, 9, 13, 35, 42]), complementarity problems and fixed point problems ([21, 28, 36, 38]).

Numerous algorithms for solving (1) have been proposed, including proximal point algorithms ([7, 19, 45]), projection algorithms [17, 18, 32, 40, 48], extragradient algorithms ([6, 16, 34, 41, 46]), subgradient algorithms ([23]) and splitting algorithms ([30]). Ceng, Teboulle and Yao [5] demonstrated the convergence analysis of extragradient algorithms for solving the pseudomonotone variational inequality and fixed point problems. In order to achieve a weak convergence result in [5], an additional condition "sequentially weak-to-strong continuity" was imposed on the pseudomonotone operator φ. However, this additional condition

2020 *Mathematics Subject Classification*. 47H10; 47J25; 47J40.
Keywords. Pseudomonotone variational inequality, Asymptotically pseudocontractive operator, Fixed point, Projection.
Received: 22 March 2021; Accepted: 16 February 2022
Communicated by Naseer Shahzad
Corresponding author: Ariana Pitea
Email addresses: yintzuchien@mail.cmuh.org.tw (Tzu-Chien Yin), arianapitea@yahoo.com (Ariana Pitea)

is not satisfied even for the identity operator. Subsequently, Vuong [22] weakened this assumption imposed on φ to "sequentially weak-to-weak continuity". In this paper, we will relax "sequentially weak-to-weak continuity" to a weaker condition.

At the same time, in order to solve the variational inequality (1), the Lipschitz constant of φ may be difficult to estimate, even if the underlying mapping is linear. In order to overcome this difficulty, some self-adaptive methods for solving variational inequality problems have been developed. The advantage of self-adaptive method lies in the fact that prior information on Lipschitz constant of φ is not required, and convergence is still guaranteed, see [14, 15].

On the other hand, we are interested in an iterative approximation of fixed point problems. It is well known that fixed point theory acts as an important tool for many branches of mathematical analysis and its applications. Especially, iterative algorithms by using fixed point techniques come to be useful in numerous mathematical formulations and theorems ([26, 29]). Often, approximations and solutions to iterative guess strategies utilized in dynamic engineering problems are sought using this method. Recently, fixed point algorithms have attracted much attention, see [3, 27, 33].

Our purpose in this paper is to propose an iterative procedure for solving pseudomonotone variational inequalities and fixed point problems of asymptotically pseudocontractive operators by using self-adaptive techniques. Under a weaker condition than weak sequential continuity imposed on φ, we prove that the suggested procedure converges strongly to a common element of the solution of pseudomonotone variational inequalities and fixed point of asymptotically pseudocontractive operators.

2. Preliminaries

Let C be a nonempty, closed, convex subset of a real Hilbert space H. The symbol " $\rightharpoonup$" stands for the weak convergence and the symbol " $\to$" stands for the strong convergence. Let $\omega_w(x_n)$ be the set of all weak cluster points of the sequence $\{x_n\}$, namely, $\omega_w(x_n) = \{u^\dagger : \exists \{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup u^\dagger$ as $i \to \infty\}$.

A bounded linear operator A is said to be $\hat{\mu}$-strongly positive on H if there exists a constant $\hat{\mu} > 0$ such that

$$\langle A(x), x\rangle \geq \hat{\mu}\|x\|^2, \ \forall x \in H.$$

An operator $\phi : C \to H$ is said to be L_1-Lipschitz if there exists a constant $L_1 \geq 0$ such that

$$\|\phi(x) - \phi(x^\dagger)\| \leq L_1\|u - u^\dagger\|, \ \forall x, x^\dagger \in C.$$

If $L_1 < 1$, then ϕ is said to be L_1-contractive. If $L_1 = 1$, then ϕ is said to be nonexpansive.

Recall that an operator $T : C \to C$ is said to be

(i) τ_n-asymptotically pseudocontractive if for all $x, x^\dagger \in C$, we have

$$\langle T^n(x) - T^n(x^\dagger), x - x^\dagger\rangle \leq \tau_n\|x - x^\dagger\|^2, \forall n \geq 1,$$

where $\{\tau_n\}$ is a real number sequence satisfying $\tau_n \geq 1 (\forall n \geq 1)$ and $\lim_{n\to\infty} \tau_n = 1$; we can rewrite this relation as

$$\|T^n(x) - T^n(x^\dagger)\|^2 \leq (2\tau_n - 1)\|x - x^\dagger\|^2 + \|(I - T^n)x - (I - T^n)x^\dagger\|^2. \tag{2}$$

(ii) uniformly L_2-Lipschitzian if there exists a positive constant L_2 such that

$$\|T^n(x) - T^n(x^\dagger)\| \leq L_2\|x - x^\dagger\|,$$

for all $n \geq 1$ and for all $x, x^\dagger \in C$.

Denote the set of all fixed points of T by $Fix(T)$.

An operator φ is said to be

- monotone on C if

$$\langle \varphi(x) - \varphi(x^\dagger), x - x^\dagger \rangle \geq 0, \ \forall x, x^\dagger \in C.$$

- pseudomonotone on H if

$$\langle \varphi(\tilde{x}), x - \tilde{x} \rangle \geq 0 \Rightarrow \langle \varphi(x), x - \tilde{x} \rangle \geq 0, \ \forall x, \tilde{x} \in H.$$

- weakly sequentially continuous, if, for given sequence $\{x_n\} \subset C$ satisfying $x_n \rightharpoonup \tilde{x}$, we conclude that $\varphi(x_n) \rightharpoonup \varphi(\tilde{x})$.

For any $x, x^\dagger \in H$ and constant $\eta \in \mathbb{R}$, we have

$$\|\eta x + (1-\eta)x^\dagger\|^2 = \eta\|x\|^2 + (1-\eta)\|x^\dagger\|^2 - \eta(1-\eta)\|x - x^\dagger\|^2. \quad (3)$$

For given $u^\dagger \in H$, there exists a unique point in C, denoted by $proj_C[u^\dagger]$ such that

$$\|u^\dagger - proj_C[u^\dagger]\| \leq \|x - u^\dagger\|, \ \forall x \in C.$$

It is known that $proj_C$ is firmly nonexpansive, that is, $proj_C$ satisfies

$$\|proj_C[q^*] - proj_C[q^\dagger]\|^2 \leq \langle proj_C[q^*] - proj_C[q^\dagger], q^* - q^\dagger \rangle, \ \forall q^*, q^\dagger \in H.$$

Moreover, $proj_C$ satisfies the following inequality

$$\langle q^* - proj_C[q^*], q^\dagger - proj_C[q^*] \rangle \leq 0, \ \forall q^* \in H, q^\dagger \in C. \quad (4)$$

Lemma 2.1 ([47]). *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let $T: C \to C$ be a uniformly L_2-Lipschtzian and asymptotically pseudocontractive operator. Then, $I - T$ is demiclosed at zero.*

Lemma 2.2 ([8]). *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let φ be a continuous and pseudomonotone operator on H. Then $x^\dagger \in Sol(C, \varphi)$ if and only if $x^\dagger$ satisfies*

$$\langle \varphi(p^\dagger), p^\dagger - x^\dagger \rangle \geq 0, \ \forall p^\dagger \in C.$$

Lemma 2.3 ([24]). *Let $\{s_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 1)$ and $\{t_n\}$ be three real number sequences. Suppose that the following conditions are satisfied:*

- $s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n t_n, \forall n \geq 0$;
- $\sum_{n=1}^{\infty} \lambda_n = \infty$;
- $\limsup_{n\to\infty} t_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n t_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

3. Main results

In this section, we give our main results.

Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let $\phi : C \to H$ be a θ-contractive operator. Let A be a $\hat{\mu}$-strongly positive, bounded, linear operator on H. Let the operator φ be pseudomonotone on H and L_1-Lipschitz continuous on C. Let $T : C \to C$ be an L_2-Lipschitz τ_n-asymptotically pseudocontractive operator. Let $\{\lambda_n\}$, $\{\gamma_n\}$ and $\{\eta_n\}$ be three real number sequences in $(0, 1)$. Let ν, κ, ϖ, ς and σ be five constants. Suppose that the following conditions are satisfied:

(C1): $0 < a_1 < \gamma_n < a_2 < \eta_n < \frac{1}{2+\sqrt{4+L_2^2}}$ for all $n \geq 1$;

(C2): $\sigma \in (0,\infty)$, $\nu \in (0,1)$, $\kappa \in (0,1)$, $\varpi \in (0,1)$, $\varsigma \in (0,2)$ and $\sigma\theta < \hat{\mu} \leq 1$;

(C3): $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(C4): $\tau_n \in [1,2](\forall n \geq 1)$, $\sum_{n=1}^{\infty}(\tau_n - 1) < +\infty$ and $\lim_{n\to\infty} \frac{\tau_n - 1}{\lambda_n} = 0$.

In this position, we state our algorithm below.

Algorithm 3.1. *Choose an initial point $x_0 \in C$ and set $n = 0$.*
Step 1. For given x_n, find the smallest nonnegative integer $sni(x_n)$ satisfying

$$u_n = proj_C[x_n - \nu\kappa^{sni(x_n)}\varphi(x_n)], \tag{5}$$

and

$$\nu\kappa^{sni(x_n)}\|\varphi(u_n) - \varphi(x_n)\| \leq \varpi\|u_n - x_n\|. \tag{6}$$

If $u_n = x_n$, then set $y_n = x_n$ and go to Step 2. Otherwise, calculate

$$y_n = proj_C\Big[x_n + \varsigma(1-\varpi)\|u_n - x_n\|^2 \frac{\hat{u}_n}{\|\hat{u}_n\|^2}\Big], \tag{7}$$

where $\hat{u}_n = u_n - x_n - \nu\kappa^{sni(x_n)}\varphi(u_n)$.
Step 2. Compute

$$\begin{cases} \hat{v}_n = (1-\eta_n)y_n + \eta_n T^n(y_n), \\ v_n = (1-\gamma_n)y_n + \gamma_n T^n(\hat{v}_n). \end{cases} \tag{8}$$

Step 3. Calculate

$$x_{n+1} = proj_C[\lambda_n\sigma\phi(x_n) + (I - \lambda_n A)v_n]. \tag{9}$$

Step 4. Set $n := n + 1$ and return to Step 1.

Throughout the paper, assume that $\Gamma := Fix(T) \cap Sol(C,\varphi) \neq \emptyset$. In order to prove the convergence of Algorithm 3.1, we need to impose an additional assumption (referred to as ASUMP) on operator φ: If the sequence $\{s_n\} \subset C$ satisfies $s_n \rightharpoonup s^\dagger \in C$ and $\liminf_{n\to\infty} \|\varphi(s_n)\| = 0$, then $\varphi(s^\dagger) = 0$.

Remark 3.2. *It is easy to check that if φ is sequentially weakly continuous, then φ possesses the above assumption (ASUMP).*

Remark 3.3. *We have the following assertions:*

(i) There exists $sni(x_n)$ satisfying (5) *and* (6)*.*

(ii) $0 < \frac{\kappa\varpi}{\nu L_1} < \kappa^{sni(x_n)} \leq 1 (n \geq 0)$.

(iii) If $x_n = proj_C[x_n - \nu\kappa^{sni(x_n)}\varphi(x_n)]$, then $x_n \in Sol(C,\varphi)$.

Next, we show the convergence of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.4. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $q^\dagger = proj_\Gamma(I - A + \sigma\phi)q^\dagger$.*

Proof. Let $\hat{p} \in \Gamma$. From (7), we have

$$\begin{aligned} \|y_n - \hat{p}\|^2 &= \Big\| proj_C\Big[x_n + \varsigma(1-\varpi)\|u_n - x_n\|^2 \frac{\hat{u}_n}{\|\hat{u}_n\|^2}\Big] - proj_C[\hat{p}]\Big\|^2 \\ &\le \Big\| x_n - \hat{p} + \varsigma(1-\varpi)\|u_n - x_n\|^2 \frac{\hat{u}_n}{\|\hat{u}_n\|^2}\Big\|^2 \\ &= \|x_n - \hat{p}\|^2 + 2\varsigma(1-\varpi)\frac{\|u_n - x_n\|^2}{\|\hat{u}_n\|^2}\langle \hat{u}_n, x_n - \hat{p}\rangle + \varsigma^2(1-\varpi)^2 \frac{\|u_n - x_n\|^4}{\|\hat{u}_n\|^2}. \end{aligned} \tag{10}$$

Now, we estimate $\langle \hat{u}_n, x_n - \hat{p}\rangle$. First, note that

$$\begin{aligned} \langle \hat{u}_n, x_n - \hat{p}\rangle &= \langle u_n - x_n - \nu\kappa^{sni(x_n)}\varphi(u_n), x_n - \hat{p}\rangle \\ &= \langle u_n - x_n + \nu\kappa^{sni(x_n)}\varphi(x_n), x_n - \hat{p}\rangle - \nu\kappa^{sni(x_n)}\langle \varphi(x_n), x_n - \hat{p}\rangle \\ &\quad - \nu\kappa^{sni(x_n)}\langle \varphi(u_n), x_n - u_n\rangle - \nu\kappa^{sni(x_n)}\langle \varphi(u_n), u_n - \hat{p}\rangle \\ &= \nu\kappa^{sni(x_n)}\langle \varphi(u_n), \hat{p} - u_n\rangle + \nu\kappa^{sni(x_n)}\langle \varphi(x_n), \hat{p} - x_n\rangle \\ &\quad + \langle u_n - x_n + \nu\kappa^{sni(x_n)}(\varphi(x_n) - \varphi(u_n)), x_n - u_n\rangle \\ &\quad + \langle u_n - x_n + \nu\kappa^{sni(x_n)}\varphi(x_n), u_n - \hat{p}\rangle. \end{aligned} \tag{11}$$

Next, we focus on the four items of the last equality in (11). Taking advantage of the fact that $\hat{p} \in Sol(C, \varphi)$, we have $\langle \varphi(\hat{p}), x_n - \hat{p}\rangle \ge 0$ and $\langle \varphi(\hat{p}), u_n - \hat{p}\rangle \ge 0$. Further, by the pseudomonotonicity of φ, we deduce

$$\langle \varphi(x_n), \hat{p} - x_n\rangle \le 0, \tag{12}$$

and

$$\langle \varphi(u_n), \hat{p} - u_n\rangle \le 0. \tag{13}$$

Applying the characteristic inequality (4) of $proj_C$ to (5), we achieve

$$\langle u_n - x_n + \nu\kappa^{sni(x_n)}\varphi(x_n), u_n - \hat{p}\rangle \le 0. \tag{14}$$

It follows from (11)-(14) that

$$\begin{aligned} \langle \hat{u}_n, x_n - \hat{p}\rangle &\le \langle u_n - x_n + \nu\kappa^{sni(x_n)}(\varphi(x_n) - \varphi(u_n)), x_n - u_n\rangle \\ &\le -\|u_n - x_n\|^2 + \nu\kappa^{sni(x_n)}\|\varphi(x_n) - \varphi(u_n)\|\|x_n - u_n\|. \end{aligned} \tag{15}$$

Combining (6) and (15), we get

$$\langle \hat{u}_n, x_n - \hat{p}\rangle \le -\|u_n - x_n\|^2 + \varpi\|x_n - u_n\|^2 = -(1-\varpi)\|x_n - u_n\|^2.$$

This fact, together with (10) implies that

$$\begin{aligned} \|y_n - \hat{p}\|^2 &\le \|x_n - \hat{p}\|^2 - 2\varsigma(1-\varpi)^2\frac{\|u_n - x_n\|^4}{\|\hat{u}_n\|^2} + \varsigma^2(1-\varpi)^2\frac{\|u_n - x_n\|^4}{\|\hat{u}_n\|^2} \\ &= \|x_n - \hat{p}\|^2 - (2-\varsigma)\varsigma(1-\varpi)^2\frac{\|u_n - x_n\|^4}{\|\hat{u}_n\|^2} \\ &\le \|x_n - \hat{p}\|^2. \end{aligned} \tag{16}$$

By relation (2) regarding T, we receive

$$\|T^n(\hat{v}_n) - \hat{p}\|^2 \le (2\tau_n - 1)\|\hat{v}_n - \hat{p})\|^2 + \|\hat{v}_n - T^n(\hat{v}_n)\|^2, \tag{17}$$

and

$$\|T^n(y_n) - \hat{p}\|^2 \le (2\tau_n - 1)\|y_n - \hat{p}\|^2 + \|y_n - T^n(y_n)\|^2. \tag{18}$$

Since T is uniformly L_2-Lipschitz continuous, we obtain

$$\|T^n(y_n) - T^n(\hat{v}_n)\| \leq L_2\|y_n - \hat{v}_n\| = L_2\eta_n\|y_n - T^n(y_n)\|. \tag{19}$$

Applying (3) to (8), we attain

$$\begin{aligned}\|\hat{v}_n - \hat{p}\|^2 &= \|(1-\eta_n)(y_n - \hat{p}) + \eta_n(T^n(y_n) - \hat{p})\|^2 \\ &= (1-\eta_n)\|y_n - \hat{p}\|^2 + \eta_n\|T^n(y_n) - \hat{p}\|^2 - \eta_n(1-\eta_n)\|y_n - T^n(y_n)\|^2.\end{aligned}$$

It follows from (18) that

$$\begin{aligned}\|\hat{v}_n - \hat{p}\|^2 &\leq (1-\eta_n)\|y_n - \hat{p}\|^2 + \eta_n[(2\tau_n - 1)\|y_n - \hat{p}\|^2 + \|y_n - T^n(y_n)\|^2] \\ &\quad - \eta_n(1-\eta_n)\|y_n - T^n(y_n)\|^2 \\ &= [1 + 2(\tau_n - 1)\eta_n]\|y_n - \hat{p}\|^2 + \eta_n^2\|y_n - T^n(y_n)\|^2.\end{aligned} \tag{20}$$

Again, by (3) and (19), we get

$$\begin{aligned}\|\hat{v}_n - T^n(\hat{v}_n)\|^2 &= \|(1-\eta_n)(y_n - T^n(\hat{v}_n)) + \eta_n(T^n(y_n) - T^n(\hat{v}_n))\|^2 \\ &= (1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2 + \eta_n\|T^n(y_n) - T^n(\hat{v}_n)\|^2 - \eta_n(1-\eta_n)\|y_n - T^n(y_n)\|^2 \\ &\leq (1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2 - \eta_n(1 - \eta_n - L_2^2\eta_n^2)\|y_n - T^n(y_n)\|^2.\end{aligned} \tag{21}$$

By (17), (20) and (21), we obtain

$$\begin{aligned}\|T^n(\hat{v}_n) - \hat{p}\|^2 &\leq (2\tau_n - 1)[1 + 2(\tau_n - 1)\eta_n]\|y_n - \hat{p}\|^2 + (2\tau_n - 1)\eta_n^2\|y_n - T^n(y_n)\|^2 \\ &\quad + (1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2 - \eta_n(1 - \eta_n - L_2^2\eta_n^2)\|y_n - T^n(y_n)\|^2 \\ &= (2\tau_n - 1)[1 + 2(\tau_n - 1)\eta_n]\|y_n - \hat{p}\|^2 + (1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2 \\ &\quad - \eta_n(1 - 2\tau_n\eta_n - L_2^2\eta_n^2)\|y_n - T^n(y_n)\|^2.\end{aligned} \tag{22}$$

By condition (C1), $\eta_n < \frac{1}{2+\sqrt{4+L_2^2}} \leq \frac{1}{\tau_n+\sqrt{L_2^2+\tau_n^2}}$, it follows that $1 - 2\tau_n\eta_n - L_2^2\eta_n^2 > 0$. In view of (22), we attain

$$\|T^n(\hat{v}_n) - \hat{p}\|^2 \leq (2\tau_n - 1)[1 + 2(\tau_n - 1)\eta_n]\|y_n - \hat{p}\|^2 + (1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2. \tag{23}$$

Based on (3), (8) and (23), it follows

$$\begin{aligned}\|v_n - \hat{p}\|^2 &= \|(1-\gamma_n)y_n + \gamma_n T^n(\hat{v}_n) - \hat{p}\|^2 \\ &= (1-\gamma_n)\|y_n - \hat{p}\|^2 + \gamma_n\|T^n(\hat{v}_n) - \hat{p}\|^2 - \gamma_n(1-\gamma_n)\|T^n(\hat{v}_n) - y_n\|^2 \\ &\leq \gamma_n(2\tau_n - 1)[1 + 2(\tau_n - 1)\eta_n]\|y_n - \hat{p}\|^2 + (1-\gamma_n)\|y_n - \hat{p}\|^2 \\ &\quad + \gamma_n(1-\eta_n)\|y_n - T^n(\hat{v}_n)\|^2 - \gamma_n(1-\gamma_n)\|y_n - T^n(\hat{v}_n)\|^2 \\ &= [1 + 2(\tau_n - 1)\gamma_n + 2(\tau_n - 1)(2\tau_n - 1)\eta_n\gamma_n]\|y_n - \hat{p}\|^2 \\ &\quad + \gamma_n(\gamma_n - \eta_n)\|y_n - T^n(\hat{v}_n)\|^2.\end{aligned}$$

On the basis of condition (C1), we have $2(\tau_n - 1)\gamma_n + 2(\tau_n - 1)(2\tau_n - 1)\eta_n\gamma_n \leq 8(\tau_n - 1)$. It follows that

$$\begin{aligned}\|v_n - \hat{p}\|^2 &\leq [1 + 8(\tau_n - 1)]\|y_n - \hat{p}\|^2 + \gamma_n(\gamma_n - \eta_n)\|y_n - T^n(\hat{v}_n)\|^2 \\ &\leq [1 + 8(\tau_n - 1)]\|y_n - \hat{p}\|^2.\end{aligned} \tag{24}$$

This together with (16) implies that

$$\|v_n - \hat{p}\| \leq [1 + 4(\tau_n - 1)]\|y_n - \hat{p}\| \leq [1 + 4(\tau_n - 1)]\|x_n - \hat{p}\|. \tag{25}$$

Since A is $\hat{\mu}$-strongly positive, $\|1-\lambda_n A\| \leq 1-\hat{\mu}\lambda_n$. From (9) and (25), we get

$$\begin{aligned}\|x_{n+1}-\hat{p}\| &= \|proj_C[\lambda_n\sigma\phi(x_n)+(I-\lambda_n A)v_n]-\hat{p}\|\\ &\leq (I-\lambda_n A)\|v_n-\hat{p}\|+\lambda_n\sigma\|\phi(x_n)-\phi(\hat{p})\|+\lambda_n\|\sigma\phi(\hat{p})-A(\hat{p})\|\\ &\leq (1-\lambda_n\hat{\mu})[1+4(\tau_n-1)]\|x_n-\hat{p}\|+\lambda_n\sigma\theta\|x_n-\hat{p}\|+\lambda_n\|\sigma\phi(\hat{p})-A(\hat{p})\|\\ &\leq [1+4(\tau_n-1)]\max\left\{\|x_n-\hat{p}\|,\frac{\|\sigma\phi(\hat{p})-A(\hat{p})\|}{\hat{\mu}-\sigma\theta}\right\}\\ &\leq \prod_{i=1}^{n}[1+4(\tau_i-1)]\max\left\{\|x_0-\hat{p}\|,\frac{\|\sigma\phi(\hat{p})-A(\hat{p})\|}{\hat{\mu}-\sigma\theta}\right\}.\end{aligned}$$

Using condition (C4), the sequence $\{x_n\}$ is bounded. Thus, the sequences $\{\varphi(x_n)\}$, $\{\phi(x_n)\}$, $\{y_n\}$, $\{v_n\}$, $\{A(v_n)\}$ and $\{\hat{v}_n\}$ are all bounded.

Since $proj_C$ is firmly nonexpansive, from (9), we have

$$\begin{aligned}\|x_{n+1}-\hat{p}\|^2 &= \|proj_C[\lambda_n\sigma\phi(x_n)+(I-\lambda_n A)v_n]-proj_C[\hat{p}]\|^2\\ &\leq \langle\lambda_n\sigma\phi(x_n)+(I-\lambda_n A)v_n-\hat{p},x_{n+1}-\hat{p}\rangle\\ &= \sigma\lambda_n\langle\phi(x_n)-\phi(\hat{p}),x_{n+1}-\hat{p}\rangle+\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle+(I-\lambda_n A)\langle v_n-\hat{p},x_{n+1}-\hat{p}\rangle\\ &\leq \sigma\theta\lambda_n\|x_n-\hat{p}\|\|x_{n+1}-\hat{p}\|+\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle+\|I-\lambda_n A\|\|v_n-\hat{p}\|\|x_{n+1}-\hat{p}\|\\ &\leq [\sigma\theta\lambda_n\|x_n-\hat{p}\|+(1-\hat{\mu}\lambda_n)\|v_n-\hat{p}\|]\|x_{n+1}-\hat{p}\|+\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle\\ &\leq \frac{1}{2}[\sigma\theta\lambda_n\|x_n-\hat{p}\|+(1-\hat{\mu}\lambda_n)\|v_n-\hat{p}\|]^2+\frac{\|x_{n+1}-\hat{p}\|^2}{2}+\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle.\end{aligned} \tag{26}$$

Note that $\sigma\theta<\hat{\mu}$. It follows from (26) that

$$\|x_{n+1}-\hat{p}\|^2 \leq \sigma\theta\lambda_n\|x_n-\hat{p}\|^2+(1-\hat{\mu}\lambda_n)\|v_n-\hat{p}\|^2+2\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle. \tag{27}$$

Since $\{x_n\}$ is bounded, there exists a positive constant M such that $M\geq\sup_n\{8(\hat{\mu}-\sigma\theta)\|x_n-\hat{p}\|^2\}$. On account of (16), (24) and (27), we achieve

$$\begin{aligned}\|x_{n+1}-\hat{p}\|^2 &\leq [1-(\hat{\mu}-\sigma\theta)\lambda_n]\|x_n-\hat{p}\|^2+2\lambda_n\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle\\ &\quad+(1-\hat{\mu}\lambda_n)\gamma_n(\gamma_n-\eta_n)\|y_n-T^n(\hat{v}_n)\|^2+8(\tau_n-1)\|x_n-\hat{p}\|^2\\ &\quad-(1-\hat{\mu}\lambda_n)[1+8(\tau_n-1)](2-\varsigma)\varsigma(1-\varpi)^2\frac{\|u_n-x_n\|^4}{\|\hat{u}_n\|^2}\\ &\leq [1-(\hat{\mu}-\sigma\theta)\lambda_n]\|x_n-\hat{p}\|^2+(\hat{\mu}-\sigma\theta)\lambda_n\left\{\frac{(1-\hat{\mu}\lambda_n)\gamma_n(\gamma_n-\eta_n)}{\hat{\mu}-\sigma\theta}\right.\\ &\quad\times\frac{\|y_n-T^n(\hat{v}_n)\|^2}{\lambda_n}+\frac{2}{\hat{\mu}-\sigma\theta}\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle+\frac{\tau_n-1}{\lambda_n}M\\ &\quad\left.-\frac{(1-\hat{\mu}\lambda_n)[1+8(\tau_n-1)](2-\varsigma)\varsigma(1-\varpi)^2}{\hat{\mu}-\sigma\theta}\frac{\|u_n-x_n\|^4}{\|\hat{u}_n\|^2\lambda_n}\right\}.\end{aligned} \tag{28}$$

For any $n\geq 0$, set $s_n=\|x_n-\hat{p}\|^2$ and

$$\begin{aligned}t_n &= \frac{(1-\hat{\mu}\lambda_n)\gamma_n(\gamma_n-\eta_n)}{\hat{\mu}-\sigma\theta}\frac{\|y_n-T^n(\hat{v}_n)\|^2}{\lambda_n}+\frac{2}{\hat{\mu}-\sigma\theta}\langle\sigma\phi(\hat{p})-A(\hat{p}),x_{n+1}-\hat{p}\rangle\\ &\quad+\frac{\tau_n-1}{\lambda_n}M-\frac{(1-\hat{\mu}\lambda_n)[1+8(\tau_n-1)](2-\varsigma)\varsigma(1-\varpi)^2}{\hat{\mu}-\sigma\theta}\frac{\|u_n-x_n\|^4}{\|\hat{u}_n\|^2\lambda_n}.\end{aligned} \tag{29}$$

From (28), we have

$$s_{n+1}\leq[1-(\hat{\mu}-\sigma\theta)\lambda_n]s_n+(\hat{\mu}-\sigma\theta)\lambda_n t_n,\ \forall n\geq 0. \tag{30}$$

By condition (C4), we assume that $0 < \frac{\tau_n - 1}{\lambda_n} \leq 1$ for all $n \geq 0$. By virtue of (29), we obtain

$$t_n \leq \frac{2}{\hat{\mu} - \sigma\theta}\langle \sigma\phi(\hat{p}) - A(\hat{p}), x_{n+1} - \hat{p}\rangle + M \leq \frac{2}{\hat{\mu} - \sigma\theta}\|\sigma\phi(\hat{p}) - A(\hat{p})\|\|x_{n+1} - \hat{p}\| + M.$$

It follows that $\limsup_{n\to\infty} t_n < +\infty$. Now, we prove that $-1 \leq \limsup_{n\to\infty} t_n$. If not, there exists a positive integer N_0 such that $t_n < -1$ for all $n \geq N_0$. Taking advantage of (30), we obtain $s_{n+1} \leq s_n - (\hat{\mu} - \sigma\theta)\lambda_n$ for all $n \geq N_0$. Therefore,

$$s_{n+1} \leq s_{N_0} - (\hat{\mu} - \sigma\theta)\sum_{i=N_0}^{n} \lambda_i,$$

which implies that

$$\limsup_{n\to\infty} s_n \leq s_{N_0} - (\hat{\mu} - \sigma\theta)\limsup_{n\to\infty} \sum_{i=N_0}^{n} \lambda_i = -\infty,$$

which is impossible. So, $-1 \leq \limsup_{n\to\infty} t_n < +\infty$.

Let $x^\dagger \in \omega_w(x_n)$. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $x_{n_i} \rightharpoonup x^\dagger \in C$ and

$$\begin{aligned}\limsup_{n\to\infty} t_n = \lim_{i\to\infty} t_{n_i} = \lim_{i\to\infty}\Big[&\frac{(1-\hat{\mu}\lambda_{n_i})\gamma_{n_i}(\gamma_{n_i} - \eta_{n_i})}{\hat{\mu} - \sigma\theta}\frac{\|y_{n_i} - T^{n_i}(\hat{v}_{n_i})\|^2}{\lambda_{n_i}} \\ &+ \frac{2}{\hat{\mu} - \sigma\theta}\langle \sigma\phi(\hat{p}) - A(\hat{p}), x_{n_i+1} - \hat{p}\rangle + \frac{\tau_{n_i} - 1}{\lambda_{n_i}}M \\ &- \frac{(1-\hat{\mu}\lambda_{n_i})[1 + 8(\tau_{n_i} - 1)](2-\varsigma)\varsigma(1-\varpi)^2}{\hat{\mu} - \sigma\theta}\frac{\|u_{n_i} - x_{n_i}\|^4}{\|\hat{u}_{n_i}\|^2\lambda_{n_i}}\Big].\end{aligned} \tag{31}$$

Since $\{x_{n_i+1}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}+1}\}$ of $\{x_{n_i+1}\}$ such that $x_{n_{i_j}+1} \rightharpoonup \hat{z}(j \to \infty)$. Thus, $\lim_{j\to\infty}\langle \sigma\phi(\hat{p}) - A(\hat{p}), x_{n_{i_j}+1} - \hat{p}\rangle = \langle \sigma\phi(\hat{p}) - A(\hat{p}), \hat{z} - \hat{p}\rangle$. Based on (31), we have $\limsup_{n\to\infty} t_n = \lim_{j\to\infty} t_{n_{i_j}}$. For convenience, write $n_{i_j} = n_k$. Thus, from (31), we deduce

$$\begin{aligned}\limsup_{n\to\infty} t_n = \lim_{k\to\infty} t_{n_k} = \lim_{k\to\infty}\Big[&\frac{(1-\hat{\mu}\lambda_{n_k})\gamma_{n_k}(\gamma_{n_k} - \eta_{n_k})}{\hat{\mu} - \sigma\theta}\frac{\|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\|^2}{\lambda_{n_k}} \\ &+ \frac{2}{\hat{\mu} - \sigma\theta}\langle \sigma\phi(\hat{p}) - A(\hat{p}), \hat{z} - \hat{p}\rangle + \frac{\tau_{n_k} - 1}{\lambda_{n_k}}M \\ &- \frac{(1-\hat{\mu}\lambda_{n_k})[1 + 8(\tau_{n_k} - 1)](2-\varsigma)\varsigma(1-\varpi)^2}{\hat{\mu} - \sigma\theta}\frac{\|u_{n_k} - x_{n_k}\|^4}{\|\hat{u}_{n_k}\|^2\lambda_{n_k}}\Big].\end{aligned} \tag{32}$$

Note that $\lim_{k\to\infty} \lambda_{n_k} = 0$ and $\lim_{k\to\infty} \frac{\tau_{n_k} - 1}{\lambda_{n_k}} = 0$. By (32), we deduce

$$\lim_{k\to\infty}\Big[\frac{\gamma_{n_k}(\gamma_{n_k} - \eta_{n_k})}{\hat{\mu} - \sigma\theta}\frac{\|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\|^2}{\lambda_{n_k}} - \frac{(2-\varsigma)\varsigma(1-\varpi)^2}{\hat{\mu} - \sigma\theta}\frac{\|u_{n_k} - x_{n_k}\|^4}{\|\hat{u}_{n_k}\|^2\lambda_{n_k}}\Big] \quad \text{exists.}$$

This indicates that

$$\lim_{k\to\infty} \|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\| = 0, \tag{33}$$

and

$$\lim_{k\to\infty} \frac{\|u_{n_k} - x_{n_k}\|^4}{\|\hat{u}_{n_k}\|^2} = 0. \tag{34}$$

Using (19), we derive

$$\begin{aligned}\|y_{n_k} - T^{n_k}(y_{n_k})\| &\le \|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\| + \|T^{n_k}(\hat{v}_{n_k}) - T^{n_k}(y_{n_k})\| \\ &\le \|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\| + L_2\eta_{n_k}\|y_{n_k} - T^{n_k}(y_{n_k})\|.\end{aligned}$$

It follows that

$$\|y_{n_k} - T^{n_k}(y_{n_k})\| \le \frac{1}{1 - L_2\eta_{n_k}}\|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\|.$$

This together with (33) implies that

$$\lim_{k\to\infty} \|y_{n_k} - T^{n_k}(y_{n_k})\| = 0. \tag{35}$$

Taking into account (5), we get that $\|u_n - \hat{p}\| \le \|x_n - \hat{p}\| + \nu\kappa^{sni(x_n)}\|\varphi(x_n)\|$. Hence, $\{u_n\}$ and $\{\hat{u}_n\}$ are bounded. Consequently, from (34), we conclude

$$\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{36}$$

Combining (6) and (36), we obtain

$$\lim_{k\to\infty} \|\varphi(u_{n_k}) - \varphi(x_{n_k})\| = 0.$$

As a result of (7), we have the following estimate

$$\begin{aligned}\|y_{n_k} - x_{n_k}\| &= \left\|proj_C\Big[x_{n_k} + \varsigma(1-\varpi)\|u_{n_k} - x_{n_k}\|^2\frac{\hat{u}_{n_k}}{\|\hat{u}_{n_k}\|^2}\Big] - proj_C[x_{n_k}]\right\| \\ &\le \frac{\varsigma(1-\varpi)\|u_{n_k} - x_{n_k}\|^2}{\|\hat{u}_{n_k}\|}.\end{aligned}$$

This together with (36) implies that

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0. \tag{37}$$

Note that

$$\begin{aligned}\|x_{n_k} - T^{n_k}(x_{n_k})\| &\le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - T^{n_k}(y_{n_k})\| + \|T^{n_k}(y_{n_k}) - T^{n_k}(x_{n_k})\| \\ &\le (1 + L_2)\|x_{n_k} - y_{n_k}\| + \|y_{n_k} - T^{n_k}(y_{n_k})\|.\end{aligned}$$

Combining (35) and (37), we deduce

$$\lim_{k\to\infty} \|x_{n_k} - T^{n_k}(x_{n_k})\| = 0. \tag{38}$$

Taking account of (9), we have

$$\begin{aligned}\|x_{n_k+1} - v_{n_k}\| &= \|proj_C[\lambda_{n_k}\sigma\phi(x_{n_k}) + (I - \lambda_{n_k}A)v_{n_k}] - proj_C[v_{n_k}]\| \\ &\le \lambda_{n_k}\|\sigma\phi(x_{n_k}) - A(v_{n_k})\| \to 0\ (k \to \infty).\end{aligned} \tag{39}$$

Observe that $\|v_{n_k} - y_{n_k}\| = \gamma_{n_k}\|y_{n_k} - T^{n_k}(\hat{v}_{n_k})\| \to 0$ due to (33). This together with (37) and (39) implies that

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{40}$$

Since T is uniformly L_2-Lipschitz, we have

$$\begin{aligned}\|x_{n_k+1} - Tx_{n_k+1}\| &\le \|x_{n_k+1} - T^{n_k+1}x_{n_k+1}\| + \|T^{n_k+1}x_{n_k+1} - T^{n_k+1}x_{n_k}\| + \|T^{n_k+1}x_{n_k} - Tx_{n_k+1}\| \\ &\le \|x_{n_k+1} - T^{n_k+1}x_{n_k+1}\| + L_2\|x_{n_k+1} - x_{n_k}\| + L_2\|T^{n_k}x_{n_k} - x_{n_k+1}\| \\ &\le \|x_{n_k+1} - T^{n_k+1}x_{n_k+1}\| + 2L_2\|x_{n_k+1} - x_{n_k}\| + L_2\|T^{n_k}x_{n_k} - x_{n_k}\|.\end{aligned} \tag{41}$$

By (38), (40) and (41), we have immediately that

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{42}$$

By Lemma 2.1, (42) and noticing that $x_{n_k} \rightharpoonup x^\dagger$ $(k \to \infty)$, we conclude that $x^\dagger \in Fix(T)$. Next, we show that $x^\dagger \in Sol(C, \varphi)$.

In view of (14), we have

$$\langle u_{n_k} + \nu\kappa^{sni(x_{n_k})}\varphi(x_{n_k}) - x_{n_k}, p^\dagger - u_{n_k}\rangle \geq 0, \ \forall p^\dagger \in C.$$

It implies that

$$\langle \varphi(x_{n_k}), p^\dagger - x_{n_k}\rangle \geq \langle \varphi(x_{n_k}), u_{n_k} - x_{n_k}\rangle + \frac{1}{\nu\kappa^{sni(x_{n_k})}}\langle u_{n_k} - x_{n_k}, u_{n_k} - p^\dagger\rangle, \ \forall p^\dagger \in C. \tag{43}$$

According to (36) and (43), we receive

$$\liminf_{k\to\infty} \langle \varphi(x_{n_k}), p^\dagger - x_{n_k}\rangle \geq 0, \ \forall p^\dagger \in C. \tag{44}$$

Now, we consider two possibilities: $\liminf_{k\to\infty} \|\varphi(x_{n_k})\| = 0$ and $\liminf_{k\to\infty} \|\varphi(x_{n_k})\| > 0$.

If $\liminf_{k\to\infty} \|\varphi(x_{n_k})\| = 0$, by the assumption (ASUMP) of φ, we deduce that $\varphi(x^\dagger) = 0$. Therefore, $x^\dagger \in Sol(C, \varphi)$.

Suppose that $\liminf_{k\to\infty} \|\varphi(x_{n_k})\| > 0$. Without loss of generality, we assume that $\|\varphi(x_{n_k})\| \geq \hat{\upsilon}(\forall k \geq 0)$ for some $\hat{\upsilon} > 0$. Set $\hat{x}_{n_k} = \frac{\varphi(x_{n_k})}{\|\varphi(x_{n_k})\|^2}(\forall k \geq 0)$. Then, $\langle \varphi(x_{n_k}), \hat{x}_{n_k}\rangle = 1(\forall k \geq 0)$. From (44), we have

$$\liminf_{k\to\infty} \left\langle \frac{\varphi(x_{n_k})}{\|\varphi(x_{n_k})\|}, p^\dagger - x_{n_k}\right\rangle \geq 0. \tag{45}$$

Let $\{\epsilon_k\}$ be a real number sequence satisfying $\epsilon_k > 0(\forall k \geq 0)$ and $\epsilon_k \to 0$ as $k \to \infty$. By (45), for each ϵ_k, there exists the smallest positive integer m_k such that

$$\left\langle \frac{\varphi(x_{n_k})}{\|\varphi(x_{n_k})\|}, p^\dagger - x_{n_k}\right\rangle + \epsilon_k \geq 0, \ \forall k \geq m_k,$$

which implies that

$$\langle \varphi(x_{n_k}), p^\dagger - x_{n_k}\rangle + \epsilon_k\|\varphi(x_{n_k})\| \geq 0, \ \forall k \geq m_k.$$

Namely,

$$\langle \varphi(x_{n_k}), p^\dagger + \epsilon_k\|\varphi(x_{n_k})\|\hat{x}_{n_k} - x_{n_k}\rangle \geq 0, \ \forall k \geq m_k.$$

It follows from the pseudomonotonicity of φ that

$$\langle \varphi(p^\dagger + \epsilon_k\|\varphi(x_{n_k})\|\hat{x}_{n_k}), p^\dagger + \epsilon_k\|\varphi(x_{n_k})\|\hat{x}_{n_k} - x_{n_k}\rangle \geq 0, \ \forall k \geq m_k. \tag{46}$$

Since $\lim_{k\to\infty} \epsilon_k\|\varphi(x_{n_k})\|\|\hat{x}_{n_k}\| = \lim_{k\to\infty} \epsilon_k = 0$, letting $k \to \infty$ in (46), we deduce

$$\langle \varphi(p^\dagger), p^\dagger - x^\dagger\rangle \geq 0, \forall p^\dagger \in C. \tag{47}$$

By Lemma 2.2 and (47), we conclude that $x^\dagger \in Sol(C, \varphi)$. Therefore, $x^\dagger \in \Gamma$.

Finally, we prove that $x_n \to proj_\Gamma(I - A + \sigma\phi)q^\dagger = q^\dagger$. Thanks to (28), we get

$$\|x_{n+1} - q^\dagger\|^2 \leq [1 - (\hat{\mu} - \sigma\theta)\lambda_n]\|x_n - q^\dagger\|^2 + (\hat{\mu} - \sigma\theta)\lambda_n\left\{\frac{\tau_n - 1}{\lambda_n}M + \frac{2}{(\hat{\mu} - \sigma\theta)}\langle \sigma\phi(q^\dagger) - A(q^\dagger), x_{n+1} - q^\dagger\rangle\right\}. \tag{48}$$

It is obviously that

$$\limsup_{n\to\infty} \langle \sigma\phi(q^\dagger) - A(q^\dagger), x_{n+1} - q^\dagger\rangle \leq 0.$$

Therefore, applying Lemma 2.3 to (48), we conclude that $x_n \to q^\dagger$. This completes the proof. □

Considering T as the identity operator, we propose the next algorithm to determine the solution to the variational inequality (1).

Algorithm 3.5. *Choose an initial point $x_0 \in C$ and set $n = 0$.*

Step 1. For given x_n, find the smallest nonnegative integer $sni(x_n)$ satisfying

$$u_n = proj_C[x_n - \nu\kappa^{sni(x_n)}\varphi(x_n)],$$

and

$$\nu\kappa^{sni(x_n)}\|\varphi(u_n) - \varphi(x_n)\| \leq \varpi\|u_n - x_n\|.$$

If $u_n = x_n$, then set $y_n = x_n$ and go to Step 2. Otherwise, calculate

$$y_n = proj_C\Big[x_n + \varsigma(1-\varpi)\|u_n - x_n\|^2 \frac{\hat{u}_n}{\|\hat{u}_n\|^2}\Big].$$

where $\hat{u}_n = u_n - x_n - \nu\kappa^{sni(x_n)}\varphi(u_n)$.

Step 2. Calculate

$$x_{n+1} = proj_C[\lambda_n\sigma\phi(x_n) + (I - \lambda_n A)y_n].$$

Step 3. Set $n := n + 1$ and return to Step 1.

Corollary 3.6. *The sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to $q_1^\dagger = proj_{Sol(C,\varphi)}(I - A + \gamma\phi)q_1^\dagger$.*

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