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Optimality and Duality Results for (h, φ)-Nondifferentiable Multiobjective Programming Problems with (h, φ)-(b, F, ρ)-Convex Functions

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Abstract. Generalized algebraic operations introduced by Ben-Tal [5] are used to define new classes of generalized convex functions, namely (h, φ)-(b, F, ρ)-convex functions and generalized (h, φ)-(b, F, ρ)-convex functions in the vectorial case. Further, optimality and duality results are proved for the considered (h, φ)-nondifferentiable multiobjective programming problem under assumptions that the functions involved are (generalized) (h, φ)-(b, F, ρ)-convex.

1. Introduction

Multiobjective programming also known as vector optimization, has grown remarkably in different directional in the setting of optimality conditions and duality results since 1980s. The term multiobjective programming is used to denote a type optimization problems where two or more objectives are to be minimized subject to certain constraints. Multiobjective optimization problems typically have conflicting objectives, and a gain in one objective very often is an expense of another. Recently there have been numerous attempts to generalize the concept of convexity in order to weaken the assumptions of the attained results for nondifferentiable multiobjective programming problems (see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10], [11],[12], [13], [16], [17] [18], and others). One of significant convexity generalizations is the concept of (F, ρ)-convexity introduced by Preda [15] for differentiable multiobjective programming problems. He established optimality conditions and duality results for differentiable vector optimization problems under assumptions that the functions involved are differentiable (F, ρ)-convex. The concept of (F, ρ)-convexity and various its generalizations have been used in proving optimality conditions and duality results for various classes of nonconvex multiobjective programming problems (see, for example, [1], [13], [14], and others). The definition of a (b, F, ρ)-convex function was defined by Pandian [14] for a

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differentiable multiobjective programming problem as a generalization of the definition of a (F,ρ)-convex function introduced by Preda [15].

In [5], Ben-Tal introduced generalized algebraic operations and showed that some classes of generalized convex functions can be defined by using them. By using Ben-Tal generalized algebraic operations and Motzkin's alternative theorem, Xu and Liu [19] developed the generalized Kuhn-Tucker necessary optimality conditions for a class of generalized (h,φ)-differentiable single-objective and multiobjective programming problems. Also based upon Ben-Tal's generalized algebraic operations, Yu and Liu [20] introduced (h,φ)-type-I and generalized (h,φ)-type I functions for the considered (h,φ)-differentiable multi-objective programming problem. Using these concepts of generalized convexity, they proved sufficient optimality conditions for a feasible solution to be a Pareto efficient solution for the considered (h,φ)-differentiable multi-objective programming problem and duality results between it and its (h,φ)-differentiable vector dual problem. In [21], Yuan et al. investigated properties of generalized convexities based on algebraic operations introduced by Ben-Tal [5]. Further, they defined the (h,φ)-generalized directional derivative and the (h,φ)-gradient in the sense of Clarke and, by using them, they introduced the concept of (φ,γ)-convexity.

In this paper, we use the generalized algebraic operations given by Ben-Tal to generalize the concept of differentiable (b,F,ρ)-convexity in a (h,φ)-nondifferentiable vectorial case. Namely, we define the concept of (h,φ)-(F,ρ)-convexity, and also classes of generalized (h,φ)-(F,ρ)-convex functions for a (h,φ)-nondifferentiable multiobjective programming problem. However, the main aim of this paper is to prove optimality conditions and duality results for a new class of nonconvex (h,φ)-nondifferentiable multiobjective programming problems. Used the introduced concepts of (h,φ)-(F,ρ)-convexity and generalized (h,φ)-(F,ρ)-convexity, the sufficient optimality conditions are proved for (h,φ)-nondifferentiable multiobjective programming problems involving such nonconvex functions. Further, for the considered (h,φ)-nondifferentiable multiobjective programming problem, its (h,φ)-nondifferentiable vector dual problem in the sense of Mond-Weir is defined. Then, various duality theorems are proved by the primal (h,φ)-nondifferentiable multiobjective programming problem and its (h,φ)-nondifferentiable vector dual problem in the sense of Mond-Weir under assumptions of (h,φ)-(F,ρ)-convexity and/or generalized (h,φ)-(F,ρ)-convexity.

2. Preliminaries and (generalized) (h,φ)-b-(F,ρ)-convexity

Throughout this paper, the following convention for vectors in the n-dimensional Euclidean space will be followed:

for any $x=(x_1,x_2,...,x_n)^T,\ y=(y_1,y_2,...,y_n)^T$, we define:
(i) $x=y$ if and only if $x_i=y_i$ for all $i=1,2,...,n$;
(ii) $x<y$ if and only if $x_i<y_i$ for all $i=1,2,...,n$;
(iii) $x\leqq y$ if and only if $x_i\leqq y_i$ for all $i=1,2,...,n$;
(iv) $x\leq y$ if and only if $x\leqq y$ and $x\neq y$.
Now, let us recall generalized operations of addition and multiplication introduced by Ben-Tal [5].

1) Let h be an n-dimensional vector-valued continuous function defined on R^n and possessing an inverse function h^{-1}. Then, the h-vector addition of $x,y\in R^n$ is defined as follows:

$$x\bigoplus y=h^{-1}\left(h\left(x\right)+h\left(y\right)\right), \tag{1}$$

and the h-scalar multiplication of $x\in R^n$ and $\alpha\in R$ is defined as follows:

$$\alpha\bigotimes x=h^{-1}\left(\alpha h\left(x\right)\right). \tag{2}$$

2) Let φ be a real-valued continuous function defined on R and possessing the inverse function φ^{-1}. Then the φ-scalar addition of two numbers α and β is defined as follows:

$$\alpha\left[+\right]\beta=\varphi^{-1}\left(\varphi\left(\alpha\right)+\varphi\left(\beta\right)\right), \tag{3}$$

and the φ-scalar multiplication is defined as follows:

$$\beta\,[\cdot]\,\alpha = \varphi^{-1}\left(\beta\varphi\left(\alpha\right)\right). \tag{4}$$

3) The (h, φ)-inner product of $x \in R^n$ and $y \in R^n$ is defined by

$$\left(x^T y\right)_{(h,\varphi)} = \varphi^{-1}\left(h\left(x\right)^T h\left(y\right)\right). \tag{5}$$

Denote

$$\bigoplus_{i=1}^{m} x^i = x^1 \bigoplus x^2 \bigoplus ... \bigoplus x^m,\ x^i \in R^n,\ i = 1, ..., m, \tag{6}$$

$$\left[\sum_{i=1}^{m}\right] \alpha_i = \alpha_1\,[+]\,\alpha_2\,[+] ... [+]\,\alpha_m,\ \ \alpha_i \in R,\ i = 1, ..., m, \tag{7}$$

$$\alpha\,[-]\,\beta = \alpha\,[+]\left((-1)\,[\cdot]\,\beta\right),\ \ \alpha, \beta \in R. \tag{8}$$

Remark 2.1. *Note that, for the generalized algebraic operations given above, the following properties are true:*

a) It is worth noting $\beta\,[\cdot]\,\alpha$ may not be equal to $\alpha\,[\cdot]\,\beta$ for any $\alpha, \beta \in R$.
b) $1 \bigotimes x = x$ for any $x \in R^n$ and $1\,[\cdot]\,\alpha = \alpha$ for any $\alpha \in R$.
c) $\varphi\left(\alpha\,[\cdot]\,\beta\right) = \alpha\varphi\left(\beta\right)$ for any $\alpha, \beta \in R$.
d) $h\left(\alpha \bigotimes x\right) = \alpha h\left(x\right)$ for any $\alpha \in R$ and $x \in R^n$.
e) $\alpha\,[-]\,\beta = \varphi^{-1}\left(\varphi\left(\alpha\right) - \varphi\left(\beta\right)\right)$ for any $\alpha, \beta \in R$.

In [21], Yuan et al. gave the definition of the (h, φ)-generalized directional derivative of a Lipschitz function and the definition of its (h, φ)-generalized gradient of f at x. Now, we re-call these definitions for a common reader.

Let h be an n-dimensional vector-valued continuous function defined on R^n and φ be such a real-valued continuous function defined on R that it has the inverse function φ^{-1}.

Definition 2.2. *[21] The (h, φ)-generalized Clarke directional derivative of a Lipschitz function $f : R^n \to R$ with respect to the direction d is defined as follows:*

$$f^*\left(x; d\right) = \lim_{\substack{y \to x \\ t \downarrow 0}} \frac{1}{t}\,[\cdot]\left(f\left(y \bigoplus t \bigotimes d\right)[-]\,f\left(y\right)\right).$$

Definition 2.3. *[21] The (h, φ)-generalized gradient (in the sense of Clarke) of a Lipschitz function $f : R^n \to R$ at x is defined as follows:*

$$\partial^* f\left(x\right) = \left\{\xi^* \in R^n : f^*\left(x; d\right) \geqq \left(\xi^{*T} d\right)_{(h,\varphi)},\ \forall d \in R^n\right\}.$$

Let f be a vector-valued function defined on R^n, where each its component is a Lipschitz function. Then f will be said (h, φ)-nondifferentiable on R^n if each component has the (h, φ)-generalized gradient at each point of R^n.

The following results will be needed in the sequel (see [5], [20]):

Lemma 2.4. *Assume that f is a real-valued function defined on R^n and (h, φ)-nondifferentiable at $\overline{x} \in R^n$. Then, the following statements hold:*

a) Let $x^i \in R^n$, $\lambda_i \in R$, $i = 1, ..., m$. Then

$$\bigoplus_{i=1}^{m} \left(\lambda_i \bigotimes x^i\right) = h^{-1}\left(\sum_{i=1}^{m} \lambda_i h\left(x^i\right)\right),$$

$$\bigoplus_{i=1}^{m} x^i = h^{-1}\left(\sum_{i=1}^{m} h\left(x^i\right)\right).$$

b) Let $\mu_i, \alpha_i \in R$, $i = 1, ..., m$. Then

$$\left[\sum_{i=1}^{m}\right] (\mu_i [\cdot] \alpha_i) = \varphi^{-1}\left(\sum_{i=1}^{m} \mu_i \varphi(\alpha_i)\right),$$

$$\left[\sum_{i=1}^{m}\right] \alpha_i = \varphi^{-1}\left(\sum_{i=1}^{m} \varphi(\alpha_i)\right).$$

Lemma 2.5. *For $\alpha \in R$, $\alpha [\cdot] f$ is (h, φ)-nondifferentiable at $\overline{x} \in R^n$ and $\partial^* (\alpha [\cdot] f(\overline{x})) = \alpha \bigotimes \partial^* f(\overline{x})$.*

The following properties of generalized algebraic operations were established by Ben-Tal [5].

Lemma 2.6. *[5]The following statements hold:*

a) $\alpha [\cdot] (\beta [\cdot] \gamma) = \beta [\cdot] (\alpha [\cdot] \gamma) = (\alpha\beta) [\cdot] \gamma$ for $\alpha, \beta, \gamma \in R$.
b) $\beta [\cdot] \left[\sum_{i=1}^{m}\right] (\alpha_i) = \left[\sum_{i=1}^{m}\right] (\beta [\cdot] \alpha_i)$, $\beta, \alpha_i \in R$, $i = 1, ..., m$.
c) $\gamma [\cdot] (\alpha [-] \beta) = (\gamma [\cdot] \alpha) [-] (\gamma [\cdot] \beta)$ for $\alpha, \beta, \gamma \in R$.
d) $\left[\sum_{i=1}^{m}\right] (\alpha_i [+] \beta_i) = \left[\sum_{i=1}^{m}\right] (\alpha_i) [+] \left[\sum_{i=1}^{m}\right] (\beta_i)$,
$\left[\sum_{i=1}^{m}\right] (\alpha_i [-] \beta_i) = \left[\sum_{i=1}^{m}\right] (\alpha_i) [-] \left[\sum_{i=1}^{m}\right] (\beta_i)$, $\alpha_i, \beta_i \in R$, $i = 1, ..., m$.

Lemma 2.7. *[5] Assume that the function φ appears in generalized algebraic operations is strictly monotone with $\varphi(0) = 0$. Then, the following statements hold:*

a) Let $\alpha, \beta, \gamma \in R$, $\gamma \geqq 0$. If $\alpha \leqq \beta$, then $\gamma [\cdot] \alpha \leqq \gamma [\cdot] \beta$.
b) Let $\alpha, \beta, \gamma \in R$, $\gamma \geqq 0$. If $\alpha < \beta$, then $\gamma [\cdot] \alpha < \gamma [\cdot] \beta$.
c) Let $\alpha, \beta, \gamma \in R$, $\gamma > 0$. If $\alpha < \beta$, then $\gamma [\cdot] \alpha < \gamma [\cdot] \beta$.
d) Let $\alpha, \beta, \gamma \in R$, $\gamma < 0$. If $\alpha \geqq \beta$, then $\gamma [\cdot] \alpha \leqq \gamma [\cdot] \beta$.
e) Let $\alpha_i, \beta_i \in R$, $i = 1, ..., m$. If $\alpha_i \leqq \beta_i$, $i = 1, ..., m$, then $\left[\sum_{i=1}^{m}\right] \alpha_i \leqq \left[\sum_{i=1}^{m}\right] \beta_i$.
If $\alpha_i \leqq \beta_i$, $i = 1, ..., m$ and there exists at least one $i^* \in \{1, ..., m\}$ such that $\alpha_{i^*} < \beta_{i^*}$, then $\left[\sum_{i=1}^{m}\right] \alpha_i < \left[\sum_{i=1}^{m}\right] \beta_i$.

Lemma 2.8. *[5] Assume that the function φ appears in generalized algebraic operations is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Then, the following statements hold:*

a) $\alpha < \beta \Longleftrightarrow \alpha [-] \beta < 0$, $\alpha, \beta \in R$,
b) $\alpha \leqq \beta \Longleftrightarrow \alpha [-] \beta \leqq 0$, $\alpha, \beta \in R$,
c) $\alpha [+] \beta < 0 \Longrightarrow \alpha < (-1) [\cdot] \beta$, $\alpha, \beta \in R$,
d) $\alpha [+] \beta \leqq 0 \Longrightarrow \alpha \leqq (-1) [\cdot] \beta$, $\alpha, \beta \in R$.

Definition 2.9. *Let X be a nonempty subset of R^n. A functional $F : X \times X \times R^n \to R$ is called (h, φ)-sublinear if, for any $x, z \in X$,*

$$F\left(x, z; a_1 \bigoplus a_2\right) \leqq F(x, z; a_1) [+] F(x, z; a_2), \ \forall a_1, a_2 \in R^n, \tag{9}$$

$$F\left(x, z; \alpha \bigotimes a\right) \leqq \alpha [\cdot] F(x, z; a), \ \forall a \in R^n, \ \alpha \geqq 0. \tag{10}$$

Lemma 2.10. *Let φ be such a continuous function with $\varphi(0) = 0$ that it has the inverse function φ^{-1}. If F is a (h, φ)-sublinear functional, then*

$$F(x, z; 0) = 0, \ \forall x, z \in X. \tag{11}$$

It can be proved, by Definition 2.9, that if F is a (h, φ)-sublinear functional, then

$$F\left(x, z; \bigoplus_{i=1}^{m} a_i\right) \leqq \Big[\sum_{i=1}^{m}\Big] F(x, z; a_i), \ \forall a_i \in R^n, \ i = 1, ..., m. \tag{12}$$

Throughout of the rest of this paper, we further assume that h is a continuous one-to-one and onto function with $h(0) = 0$, and φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Under the above assumption, it is clear that $0\,[\cdot]\,\alpha = \alpha\,[\cdot]\,0 = 0$ for any $\alpha \in R$.

Now, we introduce the definitions of (h, φ)-(b, F, ρ)-convex functions and generalized (h, φ)-(b, F, ρ)-convex functions in a (h, φ)-nondifferentiable vectorial case.

Let $F : X \times X \times R^n \to R$ be a (h, φ)-sublinear functional, $\rho_f = \left(\rho_{f_1}, ..., \rho_{f_m}\right) \in R^m$, $b = (b_1, ..., b_m) : X \times X \to R^m$, where $b_i : X \times X \to R_+ \backslash \{0\}$, $i \in I = \{1, ..., m\}$, and, moreover, $d : X \times X \to R$.

Further, let $f = (f_1, ..., f_m) : X \to R^m$, where f_i, $i = 1, ..., m$, is a Lipschitz function on a nonempty open set $X \subset R^n$ and $\overline{x} \in X$ be given.

Definition 2.11. *f is said to be a (h, φ)-(b, F, ρ)-convex (strictly (h, φ)-(F, ρ)-convex) function at $\overline{x}$ on X if the following inequalities*

$$b_i(x, \overline{x})\,[\cdot]\,(f_i(x)\,[-]\,f_i(\overline{x})) \geqq F\left(x, \overline{x}; \xi_i^*\right)[+]\left(\rho_{f_i}\,[\cdot]\,d^2(x, \overline{x})\right), \ (>) \ i \in I, \tag{13}$$

hold for all $x \in X$, $(x \neq \overline{x})$ and each $\xi_i^* \in \partial^* f_i(\overline{x})$, $i \in I$. If (13) is satisfied for each $\overline{x} \in X$, then f is an (h, φ)-(F, ρ)-convex (strictly (h, φ)-(F, ρ)-convex) function on X.

Definition 2.12. *f is said to be a (h, φ)-(b, F, ρ)-quasi-convex function at $\overline{x}$ on X if the following relation*

$$\left[\sum_{i=1}^{m}\right] b_i(x, \overline{x})\,[\cdot]\,f_i(x) \leqq \left[\sum_{i=1}^{m}\right] b_i(x, \overline{x})\,[\cdot]\,f_i(\overline{x}) \implies \left[\sum_{i=1}^{m}\right]\left(F\left(x, \overline{x}; \xi_i^*\right)[+]\left(\rho_{f_i}\,[\cdot]\,d^2(x, \overline{x})\right)\right) \leqq 0 \tag{14}$$

holds for all $x \in X$ and each $\xi_i^* \in \partial^* f_i(\overline{x})$, $i \in I$. If (14) is satisfied for each $\overline{x} \in X$, then f is a (h, φ)-(F, ρ)-quasi-convex function on X.

Definition 2.13. *f is said to be a (h, φ)-(b, F, ρ)-pseudo-convex function at $\overline{x}$ on X if the following relation*

$$\left[\sum_{i=1}^{m}\right](b_i(x, \overline{x})\,[\cdot]\,f_i(x)) < \left[\sum_{i=1}^{m}\right](b_i(x, \overline{x})\,[\cdot]\,f_i(\overline{x})) \implies \left[\sum_{i=1}^{m}\right]\left(F\left(x, \overline{x}; \xi_i^*\right)[+]\left(\rho_{f_i}\,[\cdot]\,d^2(x, \overline{x})\right)\right) < 0 \tag{15}$$

holds for all $x \in X$ and each $\xi_i^* \in \partial^* f_i(\overline{x})$, $i \in I$. If (15) is satisfied for each $\overline{x} \in X$, then f is a (h, φ)-(F, ρ)-pseudo-convex function on X.

Definition 2.14. *f is said to be a strictly (h, φ)-(b, F, ρ)-pseudo-convex function at $\overline{x}$ on X if the following relation*

$$\left[\sum_{i=1}^{m}\right]\left(F\left(x, \overline{x}; \xi_i^*\right)[+]\left(\rho_{f_i}\,[\cdot]\,d^2(x, \overline{x})\right)\right) \geqq 0 \tag{16}$$

$$\implies \left[\sum_{i=1}^{m}\right](b_i(x, \overline{x})\,[\cdot]\,f_i(x)) > \left[\sum_{i=1}^{m}\right](b_i(x, \overline{x})\,[\cdot]\,f_i(\overline{x}))$$

holds for all $x \in X$. If (16) is satisfied for each $\overline{x} \in X$, then f is a strictly (h, φ)-(b, F, ρ)-pseudo-convex function on X.

3. (h,φ)-nondifferentiable multiobjective programming problem and optimality

In the paper, consider the following (h,φ)-nondifferentiable multiobjective programming problem:

$$\begin{array}{rl} V\text{-}\min & f(x) = (f_1(x), ..., f_m(x)) \\ \text{subject to} & g_j(x) \leqq 0, \ j \in J = \{1, ..., p\}, \quad (\mathrm{VP})_{(h,\varphi)} \\ & x \in X \end{array}$$

where $f = (f_1, ..., f_m) : X \to R^m$, $g = \left(g_1, ..., g_p\right) : X \to R^p$, are (h,φ)-nondifferentiable on a nonempty open set $X \subset R^n$, where f_i, $i \in I$, g_j, $j \in J$, are locally Lipschitz functions on X.

The vector optimization problem in which each functions involved is (h,φ)-nondifferentiable is called a (h,φ)-nondifferentiable multiobjective programming problem.

Let D denote the set of all feasible solutions in the vector optimization problem $(\mathrm{VP})_{(h,\varphi)}$, that is,

$$D = \left\{x \in X : g_j(x) \leqq 0, \ j \in J\right\}.$$

Further, for a feasible solution $\overline{x} \in D$, $J(\overline{x}) = \left\{j \in J : g_j(\overline{x}) = 0\right\}$ and $g_{J(\overline{x})}$ is the set of the inequality constraints that are active at a feasible solution $\overline{x} \in D$, that is, $g_{J(\overline{x})} = \left\{g_j : j \in J(\overline{x})\right\}$.

For the considered (h,φ)-nondifferentiable multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$, we give the definition of its optimal solutions in the sense of Pareto.

Definition 3.1. *A point $\overline{x} \in D$ is said to be a weak Pareto solution or a weak minimum in the vector optimization problem $(VP)_{(h,\varphi)}$ if $f(x) \nless f(\overline{x})$ for all $x \in D$.*

Definition 3.2. *A point $\overline{x} \in D$ is said to be a Pareto solution in the vector optimization problem $(VP)_{(h,\varphi)}$ if $f(x) \nleq f(\overline{x})$ for all $x \in D$.*

Now, we establish the sufficient optimality conditions for the considered (h,φ)-nondifferentiable multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$.

Theorem 3.3. *Let $\overline{x} \in D$. Assume that there exist $\overline{\lambda} \in R^m$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^p$, $\overline{\mu} \geqq 0$ such that the following relations*

$$0 \in \left(\bigoplus_{i=1}^{m} \left(\overline{\lambda}_i \bigotimes \partial^* f_i(\overline{x})\right)\right) \bigoplus \left(\bigoplus_{j=1}^{p} \left(\overline{\mu}_j \bigotimes \partial^* g_j(\overline{x})\right)\right), \tag{17}$$

$$\overline{\mu}_j [\cdot] g_j(\overline{x}) = 0, \ j \in J \tag{18}$$

are fulfilled at $\overline{x}$. Further, assume that f is a (h,φ)-$\left(b_f, F, \rho_f\right)$-convex function at $\overline{x}$ on D and $g_{J(\overline{x})}$ is a (h,φ)-$\left(b_g, F, \rho_g\right)$-convex function at $\overline{x}$ on D, where the following inequality

$$\left(\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot]\, d^2(x,\overline{x})\right)\right)[+]\left(\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j \rho_{g_j}\right)[\cdot]\, d^2(x,\overline{x})\right)\right) \geqq 0$$

holds for all $x \in D$. Then $\overline{x}$ is a weak Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(VP)_{(h,\varphi)}$.

Proof. By assumption, f is a (h,φ)-$\left(F, \rho_f\right)$-convex function at $\overline{x}$ on D. Hence, by Definition 2.11, the following inequalities

$$b_i(x,\overline{x})[\cdot](f_i(x)[-] f_i(\overline{x})) \geqq F\left(x,\overline{x};\xi_i^*\right)[+]\left(\rho_{f_i}[\cdot]\, d^2(x,\overline{x})\right), \ i \in I \tag{19}$$

hold for all $x \in D$ and for each $\xi_i^* \in \partial^* f_i(\overline{x})$, $i \in I$. We proceed by contradiction. Suppose, contrary to the result, that is not a weak Pareto solution in the considered (h, φ)-nondifferentiable multiobjective programming problem. Then, by Definition 3.1, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{20}$$

As it follows from Definition 2.11, $b_i(\widetilde{x}, \overline{x}) > 0$, $i \in I$. Hence, by Lemma 2.7 a) and b), (20) gives, respectively,

$$b_i(\widetilde{x}, \overline{x})[\cdot] f_i(\widetilde{x}) \leqq b_i(\widetilde{x}, \overline{x})[\cdot] f_i(\overline{x}), \; i \in I,$$

$$b_{i^*}(\widetilde{x}, \overline{x})[\cdot] f_{i^*}(\widetilde{x}) < b_{i^*}(\widetilde{x}, \overline{x})[\cdot] f_{i^*}(\overline{x}) \text{ for at least one } i^* \in I. \tag{21}$$

By Lemma 2.6 c) and Lemma 2.8 a) and b), the inequalities above yield, respectively,

$$b_i(\widetilde{x}, \overline{x})[\cdot](f_i(\widetilde{x})[-] f_i(\overline{x})) \leqq 0, \; i \in I, \tag{22}$$

$$b_{i^*}(\widetilde{x}, \overline{x})[\cdot](f_{i^*}(\widetilde{x})[-] f_{i^*}(\overline{x})) < 0 \text{ for at least one } i^* \in I. \tag{23}$$

Since $\overline{\lambda} \geq 0$, by Lemma 2.8 a) and b), (22) and (23) imply, respectively,

$$\overline{\lambda}_i[\cdot](b_i(\widetilde{x}, \overline{x})[\cdot](f_i(\widetilde{x})[-] f_i(\overline{x}))) \leqq 0, \; i \in I, \tag{24}$$

$$\overline{\lambda}_{i^*}[\cdot](b_{i^*}(\widetilde{x}, \overline{x})[\cdot](f_{i^*}(\widetilde{x})[-] f_{i^*}(\overline{x}))) < 0 \text{ for at least one } i^* \in I. \tag{25}$$

Since $\widetilde{x} \in D$, (19) gives

$$b_i(\widetilde{x}, \overline{x})[\cdot](f_i(\widetilde{x})[-] f_i(\overline{x})) \geqq F\left(\widetilde{x}, \overline{x}; \xi_i^*\right)[+]\left(\rho_{f_i}[\cdot] d^2(\widetilde{x}, \overline{x})\right), \; i \in I. \tag{26}$$

Using again $\overline{\lambda} \geq 0$, by 2.6 a) and Lemma 2.8 a) and b), it follows that

$$\overline{\lambda}_i[\cdot](b_i(\widetilde{x}, \overline{x})[\cdot](f_i(\widetilde{x})[-] f_i(\overline{x}))) \geqq \overline{\lambda}_i[\cdot] F\left(\widetilde{x}, \overline{x}; \xi_i^*\right)[+]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right), \; i \in I. \tag{27}$$

Combining (20), (21) and (23), we have, respectively,

$$\left(\overline{\lambda}_i[\cdot] F\left(\widetilde{x}, \overline{x}; \xi_i^*\right)\right)[+]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) \leqq 0, \; i \in I, \tag{28}$$

$$\left(\overline{\lambda}_{i^*}[\cdot] F\left(\widetilde{x}, \overline{x}; \xi_i^*\right)\right)[+]\left(\left(\overline{\lambda}_{i^*} \rho_{f_{i^*}}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) < 0 \text{ for at least one } i^* \in I. \tag{29}$$

Since F is a (h, φ)-sublinear functional, by Definition 2.9 and Lemma 2.6 a), (28) and (29) yield, respectively,

$$F\left(\widetilde{x}, \overline{x}; \overline{\lambda}_i \bigotimes \xi_i^*\right)[+]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) \leqq 0, \; i \in I, \tag{30}$$

$$F\left(\widetilde{x}, \overline{x}; \overline{\lambda}_{i^*} \bigotimes \xi_i^*\right)[+]\left(\left(\overline{\lambda}_{i^*} \rho_{f_{i^*}}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) < 0 \text{ for at least one } i^* \in I. \tag{31}$$

Hence, by Lemma 2.3 e), (30) and (31) yield

$$\left[\sum_{i=1}^{m}\right] F\left(\widetilde{x}, \overline{x}; \overline{\lambda}_i \bigotimes \xi_i^*\right)[+]\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) < 0. \tag{32}$$

Thus, by (10) and Definition 2.9, (32) implies

$$F\left(\widetilde{x}, \overline{x}; \bigoplus_{i=1}^{m}\left(\overline{\lambda}_i \bigotimes \xi_i^*\right)\right)[+]\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i \rho_{f_i}\right)[\cdot] d^2(\widetilde{x}, \overline{x})\right) < 0. \tag{33}$$

By assumption, $g_{J(\overline{x})}$ is a (h,φ)-$\left(b_g, F, \rho_g\right)$-convex function at $\overline{x}$ on D. Hence, by Definition 2.11, the following inequalities

$$b_{g_j}(x,\overline{x})\,[\cdot]\left(g_j(x)\,[-]\,g_j(\overline{x})\right) \geqq F\left(x,\overline{x};\zeta_j^*\right)[+]\left(\rho_{g_j}\,[\cdot]\,d^2(x,\overline{x})\right),\ j \in J(\overline{x})$$

hold for all $x \in D$ and for each $\zeta_j^* \in \partial^* g_j(\overline{x})$, $j \in J(\overline{x})$. Therefore, they are also satisfied for $x = \widetilde{x} \in D$. Thus,

$$b_{g_j}(\widetilde{x},\overline{x})\,[\cdot]\left(g_j(\widetilde{x})\,[-]\,g_j(\overline{x})\right) \geqq F\left(\widetilde{x},\overline{x};\zeta_j^*\right)[+]\left(\rho_{g_j}\,[\cdot]\,d^2(\widetilde{x},\overline{x})\right),\ j \in J(\overline{x}). \tag{34}$$

Since $\overline{\mu}_j \geqq 0$, $j \in J$, by Lemma 2.6 a) and c), (34) gives

$$b_{g_j}(\widetilde{x},\overline{x})\,[\cdot]\left(\left(\overline{\mu}_j\,[\cdot]\,g_j(\widetilde{x})\right)[-]\left(\overline{\mu}_j\,[\cdot]\,g_j(\overline{x})\right)\right) \geqq \overline{\mu}_j\,[\cdot]\,F\left(\widetilde{x},\overline{x};\zeta_j^*\right)[+]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right),\ j \in J(\overline{x}). \tag{35}$$

Using again $\overline{\mu}_j \geqq 0$, $j \in J$, together with $\widetilde{x} \in D$, by Lemma 2.7 a), we obtain

$$\overline{\mu}_j\,[\cdot]\,g_j(\widetilde{x}) \leqq 0,\ j \in J. \tag{36}$$

Combining (36) and (18), by Lemma 2.7 a) and Lemma 2.8 b), we obtain

$$b_{g_j}(\widetilde{x},\overline{x})\,[\cdot]\left(\overline{\mu}_j\,[\cdot]\,g_j(\widetilde{x})\right)[-]\left(\overline{\mu}_j\,[\cdot]\,g_j(\overline{x})\right) \leqq 0,\ \ j \in J. \tag{37}$$

By (35) and (37), it follows that

$$\left(\overline{\mu}_j\,[\cdot]\,F\left(\widetilde{x},\overline{x};\zeta_j^*\right)\right)[+]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right) \leqq 0,\ j \in J(\overline{x}).$$

Since F is a (h,φ)-sublinear functional, by Definition 2.9 and Lemma 2.6 a), we have

$$F\left(\widetilde{x},\overline{x};\overline{\mu}_j \bigotimes \zeta_j^*\right)[+]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right) \leqq 0,\ j \in J(\overline{x}). \tag{38}$$

Hence, by Lemma 2.7 e), inequalities (38) imply

$$\left[\sum_{j\in J(\overline{x})}\right] F\left(\widetilde{x},\overline{x};\overline{\mu}_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right) \leqq 0.$$

Taking into account $\overline{\mu}_j = 0$, $j \notin J(\overline{x})$, we have

$$\left[\sum_{j=1}^{m}\right] F\left(\widetilde{x},\overline{x};\overline{\mu}_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j=1}^{m}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right) \leqq 0. \tag{39}$$

Thus, by (10) and Definition 2.9, (39) gives

$$F\left(\widetilde{x},\overline{x};\bigoplus_{j=1}^{m}\overline{\mu}_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j=1}^{m}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right) \leqq 0. \tag{40}$$

Combining (33) and (40), we obtain

$$F\left(\widetilde{x},\overline{x};\bigoplus_{i=1}^{m}\left(\overline{\lambda}_i \bigotimes \xi_i^*\right)\right)[+]\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2(\widetilde{x},\overline{x})\right)[+] \tag{41}$$

$$F\left(\widetilde{x},\overline{x};\bigoplus_{j\in J(\overline{x})}\overline{\mu}_j\bigotimes\zeta_j^*\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)<0.$$

By Definition 2.9 and Lemma 2.7 e), (41) gives that the following inequality

$$F\left(\widetilde{x},\overline{x};\left(\bigoplus_{i=1}^{m}\left(\overline{\lambda}_i\bigotimes\xi_i^*\right)\right)\bigoplus\left(\bigoplus_{j=1}^{p}\left(\overline{\mu}_j\bigotimes\zeta_j^*\left(\overline{x}\right)\right)\right)\right)[+]$$

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)<0$$

holds for each $\xi_i^*\in\partial^*f_i\left(\overline{x}\right)$, $i\in I$, and for each $\zeta_j^*\in\partial^*g_j\left(\overline{x}\right)$, $j\in J\left(\overline{x}\right)$. Hence, (17) implies

$$F\left(\widetilde{x},\overline{x};0\right)[+]\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)<0.$$

By (9), it follows that

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)<0. \tag{42}$$

By assumption, the following inequality

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(\widetilde{x},\overline{x}\right)\right)\geqq 0$$

holds, contradicting (42). Thus, the proof of this theorem is completed. □

In order to prove that a feasible solution $\overline{x}$ is a Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(VP)_{(h,\varphi)}$, the stronger assumption of (h,φ)-$\left(F,\rho_f\right)$-convexity should be imposed on the objective function.

Theorem 3.4. *Let $\overline{x}\in D$. Assume that there exist $\overline{\lambda}\in R^m$, $\overline{\lambda}\geq 0$, $\overline{\mu}\in R^p$, $\overline{\mu}\geqq 0$ such that the relations (17) and (18) are fulfilled at $\overline{x}$. Further, assume that f is a strictly (h,φ)-$\left(b_f,F,\rho_f\right)$-convex function at $\overline{x}$ on D and $g_{J(\overline{x})}$ is a (h,φ)-$\left(b_g,F,\rho_g\right)$-convex function at $\overline{x}$ on D, where the following inequality*

$$\left(\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2\left(x,\overline{x}\right)\right)\right)[+]\left(\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2\left(x,\overline{x}\right)\right)\right)\geqq 0$$

holds for all $x\in D$. Then $\overline{x}$ is a Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(VP)_{(h,\varphi)}$.

Proof. Proof of this theorem is similar to the proof of Theorem 3.3 and, therefore, it has been omitted in the paper. □

Now, we prove the sufficient optimality conditions for $\overline{x} \in D$ to be a Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$ under generalized (h,φ)-(F,ρ)-convexity hypotheses.

Theorem 3.5. *Let $\overline{x} \in D$. Assume that there exist $\overline{\lambda} \in R^m$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^p$, $\overline{\mu} \geqq 0$ such that the relations (17) and (18) are fulfilled at $\overline{x}$. Further, assume that $\left(\overline{\lambda}_1 [\cdot] f_1, ..., \overline{\lambda}_m [\cdot] f_m\right)$ is a strictly (h,φ)-$\left(b_f, F, \rho_f\right)$-pseudo-convex function at $\overline{x}$ on D and $\left(\overline{\mu}_1 [\cdot] g_1, ..., \overline{\mu}_p [\cdot] g_p\right)$ is an (h,φ)-$\left(b_g, F, \rho_g\right)$-quasi-convex function at $\overline{x}$ on D, where the inequality*

$$\left(\left[\sum_{i=1}^{m}\right]\left(\rho_{f_i} [\cdot] d^2 (x,\overline{x})\right)\right) [+] \left(\left[\sum_{j=1}^{p}\right]\left(\rho_{g_j} [\cdot] d^2 (x,\overline{x})\right)\right) \geqq 0 \tag{43}$$

holds for all $x \in D$. Then $\overline{x}$ is a Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(VP)_{(h,\varphi)}$.

Proof. Since F is an (h,φ)-sublinear functional, by Lemma 2.10, (11) gives that the relation

$$F(x,\overline{x};0) = 0. \tag{44}$$

holds for all $x \in D$. By assumption, there exist $\overline{\lambda} \in R^m$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^p$, $\overline{\mu} \geqq 0$ such that (17) is satisfied. This means, by Definition 2.3, that there exist $\xi_i^* \in \partial^* f_i(\overline{x})$, $i \in I$, and $\zeta_j^* \in \partial^* g_j(\overline{x})$, $j \in J(\overline{x})$, such that

$$\left(\bigoplus_{i=1}^{m}\left(\overline{\lambda}_i \bigotimes \xi_i^*\right)\right) \bigoplus \left(\bigoplus_{j=1}^{p}\left(\overline{\mu}_j \bigotimes \zeta_j^*\right)\right) = 0. \tag{45}$$

Combining (44) and (45), for each $x \in D$, we get

$$F\left(x,\overline{x};\left(\bigoplus_{i=1}^{m}\left(\overline{\lambda}_i \bigotimes \xi_i^*\right)\right) \bigoplus \left(\bigoplus_{j=1}^{p}\left(\overline{\mu}_j \bigotimes \zeta_j^*\right)\right)\right) = 0. \tag{46}$$

Since F is a (h,φ)-sublinear functional, by (9), inequality (46) gives

$$F\left(x,\overline{x};\left(\bigoplus_{i=1}^{m}\left(\overline{\lambda}_i \bigotimes \xi_i^*\right)\right)\right) [+] F\left(x,\overline{x};\left(\bigoplus_{j=1}^{p}\left(\overline{\mu}_j \bigotimes \zeta_j^*\right)\right)\right) \geqq 0.$$

By (12), it follows that

$$\left[\sum_{i=1}^{m}\right] F\left(x,\overline{x};\overline{\lambda}_i \bigotimes \xi_i^*\right) [+] \left[\sum_{j=1}^{p}\right] F\left(x,\overline{x};\overline{\mu}_j \bigotimes \zeta_j^*\right) \geqq 0. \tag{47}$$

By assumption, (43) holds for all $x \in D$. Hence, combining (43) and (47), by Lemma 2.8, we obtain

$$\left[\sum_{i=1}^{m}\right] F\left(x,\overline{x};\overline{\lambda}_i \bigotimes \xi_i^*\right) [+] \left[\sum_{j=1}^{p}\right] F\left(x,\overline{x};\overline{\mu}_j \bigotimes \zeta_j^*\right) [+] \tag{48}$$

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i \rho_{f_i}\right) [\cdot] d^2 (x,\overline{x})\right) [+] \left[\sum_{j \in J(\overline{x})}\right]\left(\left(\overline{\mu}_j \rho_{g_j}\right) [\cdot] d^2 (x,\overline{x})\right) \geqq 0$$

By assumption, (18) is fulfilled at $\overline{x}$. Using $\overline{\mu} \geqq 0$, together with Lemma 2.7 a), by Lemma 2.8 b), it follows that

$$\left(\overline{\mu}_j \left[\cdot\right] g_j(x)\right) \left[-\right] \left(\overline{\mu}_j \left[\cdot\right] g_j(\overline{x})\right) \leqq 0,\ j \in J. \tag{49}$$

As it follows from Definition 2.12, $b_{g_j}(x,\overline{x}) > 0$, $j \in J$, for all $x \in D$. Hence, by Lemma 2.6 a) and Lemma 2.7, (49) gives

$$b_{g_j}(x,\overline{x}) \left[\cdot\right] \left[\left(\overline{\mu}_j \left[\cdot\right] g_j(x)\right) \left[-\right] \left(\overline{\mu}_j \left[\cdot\right] g_j(\overline{x})\right)\right] \leqq 0,\ j \in J. \tag{50}$$

Hence, by Lemma 2.7 e), it follows that

$$\left[\sum_{i=1}^{m}\right] b_{g_j}(x,\overline{x}) \left[\cdot\right] \left[\left(\overline{\mu}_j \left[\cdot\right] g_j(x)\right) \left[-\right] \left(\overline{\mu}_j \left[\cdot\right] g_j(\overline{x})\right)\right] \leqq 0. \tag{51}$$

Thus, by Lemma 2.6 d) and Lemma 2.8 b), we get

$$\left[\sum_{i=1}^{m}\right] b_{g_j}(x,\overline{x}) \left[\cdot\right] \left(\overline{\mu}_j \left[\cdot\right] g_j(x)\right) \leqq \left[\sum_{i=1}^{m}\right] b_{g_j}(x,\overline{x}) \left[\cdot\right] \left(\overline{\mu}_j \left[\cdot\right] g_j(\overline{x})\right). \tag{52}$$

By assumption, $\left(\overline{\mu}_1 \left[\cdot\right] g_1, ..., \overline{\mu}_p \left[\cdot\right] g_p\right)$ is a (h,φ)-$\left(F,\rho_g\right)$-quasi-convex function at $\overline{x}$ on D. Hence, by Definition 2.12, (52) implies

$$\left[\sum_{i=1}^{m}\right] F\left(\widetilde{x},\overline{x};\overline{\mu}_j \left[\cdot\right] \zeta_j^*\right) \left[+\right] \left(\left(\overline{\mu}_j \rho_{g_j}\right) \left[\cdot\right] d^2\left(\widetilde{x},\overline{x}\right)\right) \leqq 0. \tag{53}$$

Hence, by Lemma 2.6 d), we have

$$\left[\sum_{i=1}^{m}\right] F\left(\widetilde{x},\overline{x};\overline{\mu}_j \left[\cdot\right] \zeta_j^*\right) \left[+\right] \left[\sum_{i=1}^{m}\right] \left(\left(\overline{\mu}_j \rho_{g_j}\right) \left[\cdot\right] d^2\left(\widetilde{x},\overline{x}\right)\right) \leqq 0. \tag{54}$$

Combining (48) and (54), by Lemma 2.8, we have that the inequality

$$\left[\sum_{i=1}^{m}\right] F\left(x,\overline{x};\overline{\lambda}_i \bigotimes \xi_i^*\right) \left[+\right] \left[\sum_{i=1}^{m}\right] \left(\left(\overline{\lambda}_i \rho_{f_i}\right) \left[\cdot\right] d^2(x,\overline{x})\right) \geqq 0 \tag{55}$$

holds. By assumption, f is a strictly (h,φ)-$\left(b_f, F, \rho_f\right)$-pseudo-convex function at $\overline{x}$ on D. Hence, by Definition 2.14, (55) implies that the inequality

$$\left[\sum_{i=1}^{m}\right] \left(b_{f_i}(x,\overline{x}) \left[\cdot\right] \left(\overline{\lambda}_i \left[\cdot\right] f_i(x)\right)\right) > \left[\sum_{i=1}^{m}\right] \left(b_{f_i}(x,\overline{x}) \left[\cdot\right] \left(\overline{\lambda}_i \left[\cdot\right] f_i(\overline{x})\right)\right)$$

holds. Hence, by Lemma 2.6, it follows that, for all $x \in D$,

$$\left[\sum_{i=1}^{m}\right] \left(b_{f_i}(x,\overline{x}) \left[\cdot\right] \left(\left(\overline{\lambda}_i \left[\cdot\right] f_i(x)\right) \left[-\right] \left(\overline{\lambda}_i \left[\cdot\right] f_i(\overline{x})\right)\right)\right) > 0. \tag{56}$$

Since $b_{f_i}(x,\overline{x}) > 0$, $i \in I$, and for all $x \in D$, by Lemma 2.7, we have that, for each $x \in D$, there exists at least one $i \in I$ such that

$$\left(\overline{\lambda}_i \left[\cdot\right] f_i(x)\right) \left[-\right] \left(\overline{\lambda}_i \left[\cdot\right] f_i(\overline{x})\right) > 0.$$

Thus, by Lemma 2.7 c) and Lemma 2.8 a), it follows that, for each $x \in D$, there exists at least one $i \in I$ such that

$$f_i(x) > f_i(\overline{x}).$$

Hence, by Definition 3.1, $\overline{x}$ is a Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$. □

In order to prove that a feasible solution $\overline{x}$ is a weak Pareto solution in the considered nondifferentiable (h,φ)-multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$, the weaker assumption of generalized (h,φ)-$\left(F,\rho_f\right)$-convexity should be imposed on the objective function.

Theorem 3.6. *Let $\overline{x} \in D$. Assume that there exist $\overline{\lambda} \in R^m$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in R^p$, $\overline{\mu} \geqq 0$ such that the following relations (17) and (18) are fulfilled at $\overline{x}$. Further, assume that $\left(\overline{\lambda}_1 [\cdot] f_1, ..., \overline{\lambda}_m [\cdot] f_m\right)$ is a (h,φ)-$\left(b_f, F, \rho_f\right)$-pseudo-convex function at $\overline{x}$ on D and $\left(\overline{\mu}_1 [\cdot] g_1, ..., \overline{\mu}_p [\cdot] g_p\right)$ is a (h,φ)-$\left(b_g, F, \rho_g\right)$-quasi-convex function at $\overline{x}$ on D, where the inequality*

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\, d^2(x,\overline{x})\right)[+]\left[\sum_{j\in J(\overline{x})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\, d^2(x,\overline{x})\right) \geqq 0$$

holds for all $x \in D$. Then $\overline{x}$ is a weak Pareto solution in the considered (h,φ)-nondifferentiable multiobjective programming problem $(VP)_{(h,\varphi)}$.

4. Mond-Weir duality

In this section, for the considered (h,φ)-nondifferentiable multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$, we define the vector dual problem in the sense of Mond-Weir. Further, under (h,φ)-(b,F,ρ)-convexity and/or generalized (h,φ)-(F,ρ)-convexity hypotheses, we prove duality results between the primal multiobjective programming problem $(\mathrm{VP})_{(h,\varphi)}$ and its vector dual problem in the sense of Mond-Weir.

We define for problem $(\mathrm{VP})_{(h,\varphi)}$ its (h,φ)-nondifferentiable vector dual problem in the sense of Mond-Weir as follows:

$$f(y) = (f_1(y), ..., f_m(y)) \to \max$$

$$0 \in \left(\bigoplus_{i=1}^{m}\left(\lambda_i \bigotimes \partial^* f_i(y)\right)\right) \bigoplus \left(\bigoplus_{j=1}^{p}\left(\mu_j \bigotimes \partial^* g_j(y)\right)\right), \tag{57}$$

$$\mu_j [\cdot] g_j(y) \geqq 0, \ j \in J, \qquad (\mathrm{VD})_{(h,\varphi)} \tag{58}$$

$$y \in X, \lambda \geq 0, \mu \geqq 0. \tag{59}$$

We denote by Ω the set of all feasible solutions in problem $(\mathrm{VD})_{(h,\varphi)}$, that is, $\Omega = \{(y,\lambda,\mu) : (y,\lambda,\mu)$ verifying the constraints (57), (58) and (59)$\}$. By Y, we denote the projection of Ω on X, that is, $Y = pr_X\Omega = \{y \in X : (y,\lambda,\mu) \in \Omega\}$. Further, for $y \in Y$, $J(y) = \left\{j \in J : g_j(y) = 0\right\}$.

Theorem 4.1. *(Weak duality). Let x and (y,λ,μ) be arbitrary solutions in problems $(VP)_{(h,\varphi)}$ and $(VD)_{(h,\varphi)}$, respectively. Further, assume that any one of the following hypotheses is fulfilled:*

a) *f is an (h,φ)-$\left(b_f, F, \rho_f\right)$-convex function at y on $D \cup Y$ and $g_{J(y)}$ is an (h,φ)-$\left(b_g, F, \rho_g\right)$-convex function at y on $D \cup Y$, where $\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\, d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\, d^2(x,y)\right) \geqq 0,$*

b) $(\lambda_1 [\cdot] f_1, ..., \lambda_m [\cdot] f_m)$ *is an* (h,φ)*-*$\left(b_f, F, \rho_f\right)$*-pseudo-convex function at* y *on* $D \cup Y$ *and* $\left(\mu_1 [\cdot] g_1, ..., \mu_p [\cdot] g_p\right)$ *is an* (h,φ)*-*$\left(b_g, F, \rho_g\right)$*-quasi-convex function at* y *on* $D \cup Y$*, where* $\left[\sum_{i=1}^{m}\right]\left(\rho_{f_i} [\cdot] d^2(x,y)\right) [+] \left[\sum_{j=1}^{p}\right]\left(\rho_{g_j} [\cdot] d^2(x,y)\right) \geqq 0$.

Then

$$f(x) \not< f(y).$$

Proof. We proceed by contradiction. Suppose, contrary to the result, that

$$f(x) < f(y). \tag{60}$$

Thus, by Lemma 2.8 a), the inequalities above yield

$$f_i(x) [-] f_i(y) < 0, \ i \in I. \tag{61}$$

a) We now prove this theorem under hypothesis a).

Since f is an (h,φ)-$\left(b_f, F, \rho_f\right)$-convex function at y on $D \cup Y$, by Definition 2.11, the following inequalities

$$b_{f_i}(x,y) [\cdot] (f_i(x) [-] f_i(y)) \geqq F\left(x, y; \xi_i^*\right) [+] \left(\rho_{f_i} [\cdot] d^2(x,y)\right), \ i \in I$$

hold for each $\xi_i^* \in \partial^* f_i(y)$, $i \in I$. Using $b_{f_i}(x,y) > 0$, $i \in I$, by Lemma 2.6 a) and Lemma 2.7, the above inequalities imply

$$b_{f_i}(x,y) [\cdot] (f_i(x) [-] f_i(y)) < 0, \ i \in I. \tag{62}$$

Combining (61) and (62), we have

$$F\left(x, y; \xi_i^*\right) [+] \left(\rho_{f_i} [\cdot] d^2(x,y)\right) < 0, \ i \in I. \tag{63}$$

From the feasibility of (y, λ, μ) in problem $(\mathrm{VD})_{(h,\varphi)}$, it follows that $\lambda \geq 0$. Hence, (63) gives

$$\lambda_i F\left(x, y; \xi_i^*\right) [+] \left(\rho_{f_i} [\cdot] d^2(x,y)\right) \leqq 0, \ i \in I, \tag{64}$$

$$\lambda_{i_0} [\cdot] F\left(x, y; \xi_{i_0}^*\right) [+] \left(\rho_{f_{i_0}} [\cdot] d^2(x,y)\right) < 0 \text{ for at least one } i_0 \in I. \tag{65}$$

Since F is an (h,φ)-sublinear functional, by Definition 2.9 and Lemma 2.6 a), (64) and (65) yield, respectively,

$$F\left(x, y; \lambda_i \bigotimes \xi_i^*\right) [+] \left(\left(\lambda_i \rho_{f_i}\right) [\cdot] d^2(x,y)\right) \leqq 0, \ i \in I, \tag{66}$$

$$F\left(x, y; \lambda_{i_0} \bigotimes \xi_{i_0}^*\right) [+] \left(\left(\lambda_{i^*} \rho_{f_{i^*}}\right) [\cdot] d^2(x,y)\right) < 0 \text{ for at least one } i^* \in I. \tag{67}$$

Thus, by Lemma 2.7 e), (66) and (67) imply

$$\left[\sum_{i=1}^{m}\right] F\left(x, y; \lambda_i \bigotimes \xi_i^*\right) [+] \left[\sum_{i=1}^{m}\right] \left(\left(\lambda_i \rho_{f_i}\right) [\cdot] d^2(x,y)\right) < 0. \tag{68}$$

Thus, by (10) and Definition 2.9, (28) implies

$$F\left(x, y; \bigoplus_{i=1}^{m} \left(\lambda_i \bigotimes \xi_i^*\right)\right) [+] \left[\sum_{i=1}^{m}\right] \left(\left(\lambda_i \rho_{f_i}\right) [\cdot] d^2(x,y)\right) < 0. \tag{69}$$

Since $g_{J(y)}$ is an (h,φ)-$\left(b_g, F, \rho_g\right)$-convex function at y on $D \cup Y$, by Definition 2.11, the following inequalities

$$b_{g_j}(x,y)\,[\cdot]\left(g_j(x)\,[-]\,g_j(y)\right) \geqq F\left(x,y;\nabla^* g_j(y)\right)[+]\left(\rho_{g_j}\,[\cdot]\,d^2(x,y)\right),\ j \in J(y) \tag{70}$$

hold for each $\zeta_j^* \in \partial^* g_j(y)$, $j \in J(y)$. From the feasibility of (y,λ,μ) in problem $(\mathrm{VD})_{(h,\varphi)}$, it follows that $\mu_j \geqq 0$, $j \in J$. Hence, by Lemma 2.6 a), (70) yields

$$b_{g_j}(x,y)\,[\cdot]\left(\mu_j\,[\cdot]\,g_j(x)\right)[-]\left(\mu_j\,[\cdot]\,g_j(y)\right) \geqq \tag{71}$$

$$\left(\mu_j\,[\cdot]\,F\left(x,y;\zeta_j^*\right)\right)[+]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right),\ j \in J(y).$$

Using $x \in D$ and $(y,\lambda,\mu) \in \Omega$ together with $b_{g_j}(x,y) \geqq 0$, $j \in J$, by Lemma 2.8 b), it follows that

$$b_{g_j}(x,y)\,[\cdot]\left(\mu_j\,[\cdot]\,g_j(x)\right)[-]\left(\mu_j\,[\cdot]\,g_j(y)\right) \leqq 0,\ j \in J. \tag{72}$$

Combining (71) and (72), we have

$$\left(\mu_j\,[\cdot]\,F\left(x,y;\zeta_j^*\right)\right)[+]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) \leqq 0,\ j \in J(y). \tag{73}$$

Since F is an (h,φ)-sublinear functional, by Definition 2.9 and Lemma 2.6 a), (73) yields

$$F\left(x,y;\mu_j \bigotimes \zeta_j^*\right)[+]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) \leqq 0,\ j \in J(y). \tag{74}$$

Hence, by Lemma 2.7 e), (74) gives

$$\left[\sum_{j\in J(y)}\right] F\left(x,y;\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) \leqq 0.$$

Taking into account $\mu_j = 0$, $j \notin J(y)$, we get

$$\left[\sum_{j=1}^{p}\right] F\left(x,y;\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) \leqq 0.$$

Thus, by (10) and Definition 2.9, the above inequality implies

$$F\left(x,y;\bigoplus_{j=1}^{p}\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) \leqq 0. \tag{75}$$

By (74) and (75), it follows that

$$F\left(x,y;\bigoplus_{i=1}^{m}\left(\lambda_i \bigotimes \xi_i^*\right)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+] \tag{76}$$

$$F\left(x,y;\bigoplus_{j\in J(\overline{x})}\mu_j \bigotimes \nabla^* g_j(y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right) < 0.$$

By Definition 2.9 and Lemma 2.7 e), (76) implies that the following inequality

$$F\left(x,y;\left(\bigoplus_{i=1}^{m}\left(\lambda_i\bigotimes\xi_i^*\right)\right)\bigoplus\left(\bigoplus_{j=1}^{p}\left(\mu_j\bigotimes\zeta_j^*\right)\right)\right)[+]$$

$$\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right)<0$$

holds for each $\xi_i^*\in\partial^* f_i(y)$, $i\in I$, and for each $\zeta_j^*\in\partial^* g_j(y)$, $j\in J(y)$. Using the feasibility of (y,λ,μ) in problem $(\mathrm{VD})_{(h,\varphi)}$ again, by the first constraint of $(\mathrm{VD})_{(h,\varphi)}$, we conclude that

$$F(x,y;0)\,[+]\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right)<0.$$

By (9), it follows that

$$\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right)<0. \tag{77}$$

By assumption, we have that the following inequality

$$\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right)\geqq 0$$

holds, contradicting (77). Thus, the proof of this theorem under hypothesis a) is completed.

Now, we proof of this theorem under hypothesis b).

By hypothesis b), $(\lambda_1[\cdot]f_1,....,\lambda_m[\cdot]f_m)$ is a (h,φ)-$\left(b_f,F,\rho_f\right)$-pseudo-convex function at y on $D\cup Y$. As it follows from Definition 2.13, $b_{f_i}(x,y)>0$, $i\in I$. Hence, by Lemma 2.6 a) and Lemma 2.7, (61) gives

$$b_{f_i}(x,y)\,[\cdot]\,(f_i(x)\,[-]\,f_i(y))<0,\ i\in I. \tag{78}$$

Since $\lambda\geq 0$, by Lemma 2.6 c) and Lemma 2.7, the inequalities above imply, respectively,

$$b_{f_i}(x,y)\,[\cdot]\,(\lambda_i[\cdot]f_i(x)\,[-]\,\lambda_i[\cdot]f_i(y))<0,\ i\in I. \tag{79}$$

$$b_{f_i}(x,y)\,[\cdot]\,(\lambda_i[\cdot]f_i(x)\,[-]\,\lambda_i[\cdot]f_i(y))\leqq 0 \text{ for at least one } i\in I. \tag{80}$$

Hence, by Lemma 2.7 e), (79) and (80) yield

$$\left[\sum_{i=1}^{m}\right]\left(b_{f_i}(x,y)\,[\cdot]\,(\lambda_i[\cdot]f_i(x)\,[-]\,\lambda_i[\cdot]f_i(y))\right)<0.$$

Using Lemma 2.6 c), Lemma 2.7 e) and Lemma 2.8 a), we get

$$\left[\sum_{i=1}^{m}\right]\left(b_{f_i}(x,y)\,[\cdot]\,(\lambda_i[\cdot]f_i(x))\right)<\left[\sum_{i=1}^{p}\right]\left(b_{f_i}(x,y)\,[\cdot]\,(\lambda_i[\cdot]f_i(y))\right). \tag{81}$$

Since $(\lambda_1 [\cdot] f_1, ..., \lambda_m [\cdot] f_m)$ is an (h,φ)-$\left(b_f, F, \rho_f\right)$-pseudo-convex function at y on $D \cup Y$, by Definition 2.13, (81) implies that the inequality

$$\left[\sum_{i=1}^{m}\right]\left(F\left(x,y;\lambda_i \bigotimes \xi_i^*\right)[+]\left(\rho_{f_i}[\cdot]\, d^2(x,y)\right)\right) < 0$$

holds for each $\xi_i^* \in \partial^* f_i(y)$, $i \in I$. Then, by Lemma 2.6 d), we obtain

$$\left[\sum_{i=1}^{m}\right] F\left(x,y;\lambda_i \bigotimes \xi_i^*\right)[+]\left[\sum_{i=1}^{m}\right]\left(\rho_{f_i}[\cdot]\, d^2(x,y)\right) < 0.$$

Since F is an (h,φ)-sublinear functional, by Definition 2.9, it follows that

$$F\left(x,y;\bigoplus_{i=1}^{m}\left(\lambda_i \bigotimes \xi_i^*\right)\right)[+]\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\, d^2(x,y)\right) < 0. \tag{82}$$

Using $x \in D$ and $(y,\lambda,\mu) \in \Omega$, by Lemma 2.8, we obtain

$$\mu_j [\cdot]\, g_j(x) \leqq \mu_j [\cdot]\, g_j(y),\ j \in J. \tag{83}$$

By assumption, g is an (h,φ)-$\left(b_g, F, \rho_g\right)$-quasi-convex function at y on $D \cup Y$. As it follows from Definition 2.12, $b_{g_j}(x,y) > 0$, $j \in J$, for all $x \in D$ and $y \in Y$. Hence, by Lemma 2.6 a) and Lemma 2.7, (83) gives

$$b_{g_j}(x,y)[\cdot]\left(\mu_j[\cdot]\, g_j(x)\right) \leqq b_{g_j}(x,y)[\cdot]\left(\mu_j[\cdot]\, g_j(y)\right) \leqq 0,\ j \in J.$$

Thus, by Lemma 2.7 e), it follows that

$$\left[\sum_{j=1}^{p}\right]\left(b_{g_j}(x,y)[\cdot]\left(\mu_j[\cdot]\, g_j(x)\right)\right) \leqq \left[\sum_{j=1}^{p}\right]\left(b_{g_j}(x,y)[\cdot]\left(\mu_j[\cdot]\, g_j(y)\right)\right). \tag{84}$$

By assumption, $g_{J(y)}$ is an (h,φ)-$\left(b_g, F, \rho_g\right)$-quasi-convex function at y on $D \cup Y$. Therefore, by Definition 2.12, (84) implies that the inequality

$$\left[\sum_{j=1}^{p}\right]\left(F\left(x,y;\mu_j \bigotimes \zeta_j^*\right)[+]\left(\rho_{g_j}[\cdot]\, d^2(x,y)\right)\right) \leqq 0.$$

holds for each $\zeta_j^* \in \partial^* g_j(y)$, $j \in J$. Then, using Lemma 2.6 d), we obtain

$$\left[\sum_{j=1}^{p}\right] F\left(x,y;\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j=1}^{p}\right]\left(\rho_{g_j}[\cdot]\, d^2(x,y)\right) \leqq 0. \tag{85}$$

Since F is an (h,φ)-sublinear functional, by Definition 2.9 and Lemma 2.6 a), (85) implies

$$F\left(x,y;\bigoplus_{j=1}^{m}\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j=1}^{p}\right]\left(\rho_{g_j}[\cdot]\, d^2(x,y)\right) \leqq 0. \tag{86}$$

Combining (82) and (86), we get

$$F\left(x,y;\bigoplus_{i=1}^{m}\left(\lambda_i \bigotimes \xi_i^*\right)\right)[+]\left[\sum_{i=1}^{m}\right]\left(\rho_{f_i}[\cdot]\, d^2(x,y)\right)[+]F\left(x,y;\bigoplus_{j\in J(y)}\mu_j \bigotimes \zeta_j^*\right)[+]\left[\sum_{j=1}^{p}\right]\left(\rho_{g_j}[\cdot]\, d^2(x,y)\right) < 0.$$

The rest of proof is the same as in the proof of this theorem under hypothesis a). □

If stronger (h,φ)-(F,ρ)-convexity and/ or generalized (h,φ)-(F,ρ)-convexity hypotheses are imposed on the objective function, then a stronger result is true.

Theorem 4.2. *(Weak duality). Let x and (y,λ,μ) be arbitrary solutions in problems (VP)$_{(h,\varphi)}$ and (VD)$_{(h,\varphi)}$, respectively. Further, assume that any one of the following hypotheses is fulfilled:*

a) *f is a strictly (h,φ)-$\left(b_f,F,\rho_f\right)$-convex function at y on $D\cup Y$ and $g_{J(y)}$ is an (h,φ)-$\left(b_g,F,\rho_g\right)$-convex function at y on $D\cup Y$,*

b) *f is a strictly (h,φ)-$\left(b_f,F,\rho_f\right)$-pseudo-convex function at y on $D\cup Y$ and $g_{J(y)}$ is an (h,φ)-$\left(b_g,F,\rho_g\right)$-quasi-convex function at y on $D\cup Y$.*

If the inequality

$$\left[\sum_{i=1}^{m}\right]\left(\left(\lambda_i\rho_{f_i}\right)[\cdot]\,d^2(x,y)\right)[+]\left[\sum_{j\in J(y)}\right]\left(\left(\mu_j\rho_{g_j}\right)[\cdot]\,d^2(x,y)\right)\geqq 0$$

holds, then

$$f(x)\nleq f(y).$$

Theorem 4.3. *(Direct duality). Assume that $\overline{x}$ is a weak Pareto solution (Pareto solution) in problem (VP)$_{(h,\varphi)}$ and there exist $\overline{\lambda}\in R^m$, $\overline{\lambda}\geq 0$, $\overline{\mu}\in R^p$, $\overline{\mu}\geqq 0$ such that (17) and (18) are fulfilled at $\overline{x}$ with these Lagrange multipliers. Then $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ is feasible in problem (VD)$_{(h,\varphi)}$. If all hypotheses of Theorem 4.1 (Theorem 4.2) are satisfied, then $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ is a weak efficient solution (efficient solution) of a maximum type in problem (VD)$_{(h,\varphi)}$ and optimal values in both (h,φ)-nondifferentiable vector optimization problems are the same.*

Proof. The feasibility of $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ in problem (VD)$_{(h,\varphi)}$ follows directly from conditions (17) and (18). The efficiency (weak efficiency) of a maximum type of $\left(\overline{x},\overline{\lambda},\overline{\mu}\right)$ in problem (VD)$_{(h,\varphi)}$ follows from weak duality (Theorem 4.2 or 4.1, respectively). □

Theorem 4.4. *(Converse duality). Let $\left(\overline{y},\overline{\lambda},\overline{\mu}\right)$ be efficient (weakly efficient) of a maximum type in Mond-Weir dual problem (VD)$_{(h,\varphi)}$ with $\overline{y}\in D$. Assume, furthermore, that any one of the following hypotheses is fulfilled:*

a) *f is a strictly (h,φ)-$\left(b_f,F,\rho_f\right)$-convex ($(h,\varphi)$-$\left(b_f,F,\rho_f\right)$-convex) function at $\overline{y}$ on $D\cup Y$ and $g_{J(\overline{y})}$ is an (h,φ)-$\left(b_g,F,\rho_g\right)$-convex function at $\overline{y}$ on $D\cup Y$,*

b) *f is a strictly (h,φ)-$\left(b_f,F,\rho_f\right)$-pseudo-convex ($(h,\varphi)$-$\left(b_f,F,\rho_f\right)$-pseudo-convex) function at $\overline{y}$ on $D\cup Y$ and $g_{J(\overline{y})}$ is a (h,φ)-$\left(b_g,F,\rho_g\right)$-quasi-convex function at $\overline{y}$ on $D\cup Y$.*

If the inequality

$$\left[\sum_{i=1}^{m}\right]\left(\left(\overline{\lambda}_i\rho_{f_i}\right)[\cdot]\,d^2(x,\overline{y})\right)[+]\left[\sum_{j\in J(\overline{y})}\right]\left(\left(\overline{\mu}_j\rho_{g_j}\right)[\cdot]\,d^2(x,\overline{y})\right)\geqq 0$$

holds for all $x\in D$, then $\overline{y}$ is a Pareto (weak Pareto) solution in the considered (h,φ)-nondifferentiable multiobjective programming problem (VP)$_{(h,\varphi)}$.

Proof. Follows directly from weak duality (Theorem 4.2 or 4.1, respectively). □

5. Conclusion

In the paper, a class of nonconvex (h, φ)-nondifferentiable multiobjective problems has been considered in which every component of involved functions is a Lipschitz function. Generalized algebraic operations introduced by Ben-Tal [5] have been used to define new classes of (h, φ)-nondifferentiable generalized convex functions, namely (h, φ)-(b, F, ρ)-convex functions and generalized (h, φ)-(b, F, ρ)-functions. The introduced concepts of (h, φ)-nondifferentiable generalized convexity turned out to be useful to development optimality conditions for a feasible solution to be a (weak Pareto) Pareto solution and several duality results for the considered nonconvex (h, φ)-nondifferentiable multiobjective problem. Namely, the sufficient optimality conditions and various duality results have been established for the considered nonconvex (h, φ)-nondifferentiable multiobjective problem under assumptions that the functions constituting it are (h, φ)-(F, ρ)-convex and/ or generalized (h, φ)-(b, F, ρ)-convex functions (not necessarily, with respect to the same b and the same ρ). Thus, the sufficiency of Karush-Kuhn-Tucker necessary optimality conditions and duality results in the sense of Mond-Weir have been proved for the larger class of (h, φ)-nondifferentiable multiobjective programming problems than φ-nondifferentiable convex vector optimization problems and, moreover, even than (φ, γ)-nondifferentiable convex ones.

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