Filomat 36:12 (2022), 4157–4172
https://doi.org/10.2298/FIL2212157B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Wave Equation with Internal Source and Boundary Damping Terms: Global Existence and Stability

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Abstract. In this work, we consider the wave equation with internal source and boundary damping terms. First, we use the stable set method to prove the existence of the global solution. Then, we use some integral inequalities to prove the stability of this solution.

1. Introduction

During the last few decades, many researchers have been interested in the following wave equation with internal damping and source terms:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u & in \quad (0,T)\times\Omega, \\ u = 0 & on \quad (0,T)\times\partial\Omega, \\ u(0) = u_0 \ \ and \ \ u_t(0) = u_1 & in \quad \Omega. \end{cases}$$

Where $T > 0$, $m, p \geq 2$ and Ω is a bounded domain of $\mathrm{I\!R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$.

In 1977, Ball [1] showed that in the absence of the dissipatif term $u_t|u_t|^{m-2}$, the source term $u|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy. Haraux and Zuazua [4] in 1988 proved that in the absence of the source term, the damping term assures the global existence for arbitrary initial data. In the linear damping case $m = 2$, Levine [8] in 1974 established a finite time blow up result for negative initial energy. In 1994, Georgiev and Todorova [6] extended Levine's result to the nonlinear damping case $m \geq 2$. They gave two results :

- if $m \geq p$ then the global solution exists for arbitrary initial data,
- if $p > m$ then solution with sufficiently negative initial energy blows up in finite time.

In 2001, Messaoudi [9] improved the result of Georgiev and Todorova and proved a finite time blow up result for solutions with negative initial energy only. Ikehata [5] in 1995 used the stable set method, introduced by Sattinger [11] in 1968, to show that the global solution exists for small enough initial energy. In addition, authors in [3], [12] and [13] have addressed this issue.

2020 *Mathematics Subject Classification*. [2010] 35L05; 35B40; 35L70; 93D20
Keywords. Wave equation, Internal source term, Boundary damping term, Local solution, Global solution, Stability.
Received: 02 April 2021; Accepted: 20 April 2022
Communicated by Marko Nedeljkov
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Park and Ha [10] in 2008 used the multiplier technique to get the existence and the uniform decay rates concerning the semilinear wave equation with boundary damping and source terms.

In 2015, Fiscella and Vittilaro [2] showed two results. The first result is about the existence of the global solution when the initial data are posed in the energy space and the second result is about the blow up in finite time of the solution for positive initial energy.

We consider the following system

$$\begin{cases} u_{tt} - \Delta u = f(x,u) & in \quad (0,T)\times\Omega, \\ u = 0 & on \quad (0,T)\times\Gamma_0, \\ \partial_\nu u = -(h.\nu)g(x,u_t) & on \quad (0,T)\times\Gamma_1, \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & in \quad \Omega. \end{cases} \tag{1}$$

Where the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. f is a nonlinear internal source function, g represents a nonlinear boundary damping function and $h \in \mathrm{lR}^n$. The objective of this paper is to apply the stable set method to prove the existence of the global solution of (1) then to use some integral inequalities to obtain the stability of this solution. To the best of knowledge, the application of those techniques is new for this kind of problems.

This paper is organized as follows: in addition to the introduction, section 2 contains assumptions on the parameters of (1) needed to obtain our results. In section 3, we present the result of the existence of the maximal solution of our problem. In section 4, we prove that this solution is global. In section 5, we prove that the obtained global solution is stable.

2. Assumptions

The following assumptions are made.

Assumptions on the partition $\{\Gamma_0, \Gamma_1\}$ of Γ:

Let $x_0 \in \mathrm{lR}^n$ and $h_0 > 0$. Put

$$h = h(x) = x - x_0 \qquad for\ all\ x \in \overline{\Omega},$$

$$\Gamma_0 = \{x \in \partial\Omega / h.\nu \le 0\},$$

and

$$\Gamma_1 = \{x \in \partial\Omega / h.\nu \ge h_0\}.$$

Assumptions on the source term f:

f is a Carathéodory real function in $\Omega \times \mathrm{lR}$, $f(x,0) = 0$ and there exist $C_1, C_2, p > 0$, with

$$\begin{cases} 2 \le p & if \quad n = 1,2, \\ 2 \le p \le 2\frac{n-1}{n-2} & if \quad n \ge 3. \end{cases}$$

Such that

$$|f(x,u) - f(x,v)| \le C_1|u-v|(1 + |u|^{p-2} + |v|^{p-2}) \quad for\ all\ x \in \Omega\ and\ u,v \in \mathrm{lR} \tag{2}$$

and

$$F(x,u) \le \frac{C_2}{p}|u|^p \qquad for\ all\ x \in \Omega\ and\ u \in \mathrm{lR}, \tag{3}$$

where F is the primitive of f defined by

$$F(x,u) = \int_0^u f(x,\tau)d\tau \quad for\ all\ x \in \Omega\ and\ u \in \mathrm{I\!R}.$$

Assumptions on the damping term g:

g is a Carathéodory real function in $\Gamma_1 \times \mathrm{I\!R}$ and there exist $C_3, C_4, C_5, C_6, m > 0$, where

$$\begin{cases} 2 \le m \quad if\ n = 1, 2, \\ 2 \le m \le \frac{2n}{n-2} \quad if\ n \ge 3. \end{cases}$$

Such that, for all $x \in \Gamma_1$, we have

$$\begin{aligned} C_3|u|^{m-1} \le |g(x,u)| \le C_4|u|^{1/(m-1)} \quad & if\ |u| \le 1, \\ C_5|u| \le |g(x,u)| \le C_6|u| \qquad & if\ |u| > 1 \end{aligned} \tag{4}$$

and

$$g(x,u)u \ge 0 \quad for\ all\ x \in \Gamma_1\ and\ u \in \mathrm{I\!R}.$$

3. Preliminaries

This section is concerned with the existence of the maximal solution of (1) and the decreasing of the usual energy associated to this solution.

Definition 3.1. *By a weak solution of (1) in $(0,T)$ we mean a function u*

$$u \in C((0,T); H^1_{\Gamma_0}(\Omega)) \cap C^1((0,T); L^2(\Omega)),$$

such that,

$$for\ all\ \varphi \in C((0,T); H^1_{\Gamma_0}(\Omega)) \cap C^1((0,T); L^2(\Omega)) \cap L^m((0,T) \times \Gamma_1),$$

and for all $t \in (0,T)$, we have

$$\int_\Omega u_t\varphi \mid_0^t dx = \int_0^t \int_\Omega (u_t\varphi - \nabla u \nabla\varphi + f(x,u)\varphi)dxd\tau - \int_0^t \int_{\Gamma_1} (h.\nu)g(x,u_t)\varphi d\Gamma d\tau.$$

Where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0\ on\ \Gamma_0\}.$$

Definition 3.2. *We say that a weak solution u is maximal if u cannot be a restriction of a weak solution in $(0,T')$, where $T < T'$.*

Now, we introduce the existence of the maximal weak solution of the problem (1) as following.

Theorem 3.3. *[2] If $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$ then there exist $T > 0$ and a unique maximal weak solution u of the problem (1) in $(0,T)$.*
Also, the following alternative holds:

$$T = +\infty,$$

or

$$T < +\infty \quad and \quad \lim_{t \to T}(\|u_t\|_2 + \|\nabla u\|_2) = +\infty.$$

Next, we consider the energy functional E associated with our system defined by

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u\|_2^2-\int_{\Omega}F(x,u)dx \quad for\ all\ t\in(0,T).$$

We have the following derivative energy identity, which shows that the above energy is a decreasing function.

Lemma 3.4. *[2] Let $u_0\in H^1_{\Gamma_0}(\Omega)$ and $u_1\in L^2(\Omega)$, we have*

$$E(t)-E(s)=-\int_s^t\int_{\Gamma_1}(h.\nu)g(x,u_t)u_t d\Gamma d\tau \quad for\ all\ 0\le s\le t\le T.$$

4. Global property of the maximal solution

In this section, we prove the global property of the solution of our system. For this end, we introduce the following functionals, associated to the maximal solution given in *Theorem* 3.1, defined by

$$J(t)=J(u(t))=\frac{1}{2}\|\nabla u\|_2^2-\int_{\Omega}F(x,u)dx \quad for\ all\ t\in(0,T)$$

and

$$K(t)=K(u(t))=\|\nabla u\|_2^2-p\int_{\Omega}F(x,u)dx \quad for\ all\ t\in(0,T).$$

We consider the following set

$$\mathbf{H}=\{w\in H^1_{\Gamma_0}(\Omega)/K(w)>0\}. \tag{5}$$

Let C_* be the best constant such that

$$\|u\|_p\le C_*\|\nabla u\|_2 \quad for\ all\ u\in H^1_{\Gamma_0}(\Omega). \tag{6}$$

We have the following property of the set $\mathbf{H}$.

Theorem 4.1. *If $u_0\in\mathbf{H}$ and $u_1\in L^2(\Omega)$ with*

$$\beta=C_2C_*^p\left(\frac{2p}{p-2}E(0)\right)^{(p-2)/2}<1, \tag{7}$$

then the maximal solution u of (1) is global.

Proof.
Firstly, we have

$$u(t)\in\mathbf{H} \quad for\ all\ t\in(0,T).$$

Indeed, since

$$u_0\in\mathbf{H},$$

then

$$K(u_0)>0.$$

This implies that there exists $T'\le T$ such that

$$K(t)\ge 0 \quad for\ all\ t\in[0,T']. \tag{8}$$

We have

$$J(t) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx = \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}(\|\nabla u\|_2^2 - p\int_\Omega F(x,u)dx) = \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t).$$

By (8), we find

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|_2^2 \quad \text{for all } t \in [0,T'].$$

Hence

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t).$$

Moreover

$$J(t) = E(t) - \frac{1}{2}\|u_t\|_2^2 \leq E(t),$$

then

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(t). \tag{9}$$

Since E is a decreasing function then we have

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(0). \tag{10}$$

By (3), we obtain

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}\int_\Omega |u|^p dx = \frac{C_2}{p}\|u\|_p^p.$$

(6) leads to

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}C_*^p\|\nabla u\|_2^p = \frac{C_2}{p}C_*^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2.$$

Also, (10) gives

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}C_*^p(\frac{2p}{p-2}E(0))^{(p-2)/2}\|\nabla u\|_2^2.$$

So

$$p\int_\Omega F(x,u)dx \leq C_2C_*^p(\frac{2p}{p-2}E(0))^{(p-2)/2}\|\nabla u\|_2^2 = \beta\|\nabla u\|_2^2.$$

We then use (7) to find

$$p\int_\Omega F(x,u)dx < \|\nabla u\|_2^2 \quad \text{for all } t \in [0,T'].$$

Hence

$$K(u(t)) = K(t) = \|\nabla u\|_2^2 - p\int_\Omega F(x,u)dx > 0 \quad \text{for all } t \in [0,T'].$$

(5) leads to

$$u(t) \in \mathbf{H} \quad \text{for all } t \in [0,T'].$$

By noting that

$$C_2C_*^p(\frac{2p}{p-2}E(T'))^{(p-2)/2} < 1,$$

we can repeat the proceedings above to extend T' to T.
Secondly, from the definition of E and K, we get for all $t \in (0,T)$

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx = \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t).$$

Since

$$K(t) > 0 \quad for\ all\ t \in (0,T),$$

then

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 \quad for\ all\ t \in (0,T).$$

This implies that there exists $C > 0$ such that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(t) \quad for\ all\ t \in (0,T). \tag{11}$$

Furthermore, E is a decreasing function, then

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(0) \quad for\ all\ t \in (0,T).$$

By the alternative statement, we find the desired result. □

5. Stability of the global solution

We have the following stability result.

Theorem 5.1. *If $u_0 \in \mathbf{H}$ and $u_1 \in L^2(\Omega)$ with $\beta < 1$ then, there exist two positive constants C and w, such that the global solution of (1) satisfies for all $t \geq 0$*

$$E(t) \leq Ce^{-wt} \qquad if \quad m = 2,$$
$$E(t) \leq \frac{C}{t^{2/(m-2)}} \qquad if \quad m > 2.$$

Proof. By the integral inequalities due to Komornik [7], it is sufficient to prove that, for all $0 \leq S \leq T \leq \infty$, there exist $C > 0$ such that

$$\int_S^T E^{m/2}(t)dx \leq CE(S). \tag{12}$$

For this end, we proceed in several steps.
Step 1: Energy identity
We put

$$Mu := 2h.\nabla u + (n-1)u.$$

We multiply the first equation of (1) by $E^{(m-2)/2}(t)Mu$. Then, we integrate the obtained result over $[S,T]\times\Omega$, we find

$$0 = \int_S^T E^{(m-2)/2}(t)\int_\Omega Mu(u_{tt} - \Delta u - f(x,u))dxdt = I_1 + I_2 + I_3, \tag{13}$$

where

$$I_1 = \int_S^T E^{(m-2)/2}(t)\int_\Omega u_{tt}Mudxdt,$$

$$I_2 = -\int_S^T E^{(m-2)/2}(t) \int_\Omega \Delta u M u dxdt$$

and

$$I_3 = -\int_S^T E^{(m-2)/2}(t) \int_\Omega f(x,u) M u dxdt.$$

We have

$$I_1 = \int_S^T E^{(m-2)/2}(t) \int_\Omega u_{tt} M u dxdt = [E^{(m-2)/2}(t) \int_\Omega u_t M u dx]_S^T$$

$$-\frac{m-2}{2} \int_S^T E^{(m-4)/2}(t) E_t(t) \int_\Omega u_t M u dxdt - \int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (Mu)_t dxdt.$$

But

$$-\int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (Mu)_t dxdt = -\int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (2h.\nabla u + (n-1)u)_t dxdt$$

$$= -2\int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (h.\nabla u)_t dxdt - \int_S^T E^{(m-2)/2}(t) \int_\Omega u_t ((n-1)u)_t dxdt,$$

which implies that

$$-\int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (Mu)_t dxdt = -2\int_S^T E^{(m-2)/2}(t) \int_\Omega u_t (h.\nabla u_t) dxdt - (n-1)\int_S^T E^{(m-2)/2}(t) \int_\Omega |u_t|^2 dxdt. \tag{14}$$

If we apply the following identity

$$\int_\Omega j_1 (k.\nabla j_2) dx = \int_\Gamma k.\nu (j_1 j_2) d\Gamma - \int_\Omega j_2 div(j_1 k) dx, \tag{15}$$

for all

$$j_1, j_2 \in C^1(\overline{\Omega}) \quad and \quad k \in (C^1(\overline{\Omega}))^n,$$

with

$$j_1 = j_2 = u_t \quad and \quad k = h,$$

we find

$$-\int_\Omega u_t (h.\nabla u_t) dx = -\int_\Gamma (h.\nu)|u_t|^2 d\Gamma + \int_\Omega u_t div(u_t h) dx.$$

Also, if we apply the following identity

$$div(jk) = j div k + k.\nabla j \quad for\ all\ j \in C^1(\overline{\Omega}) \ \ and \ \ k \in (C^1(\overline{\Omega}))^n,$$

with

$$j = u_t \quad and \quad k = h,$$

we obtain

$$-\int_{\Omega} u_t(h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + \int_{\Omega} u_t(u_t divh + h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + \int_{\Omega} divh|u_t|^2 dx + \int_{\Omega} u_t(h.\nabla u_t)dx,$$

this leads to

$$-2\int_{\Omega} u_t(h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + n\int_{\Omega}|u_t|^2 dx.$$

Now, if we replace the above result in (14), we find

$$-\int_S^T E^{(m-2)/2}(t)\int_{\Omega} u_t(Mu)_t dxdt = \int_S^T E^{(m-2)/2}(t)\int_{\Omega}|u_t|^2 dxdt - \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma dt.$$

So, I_1 takes the forme

$$I_1 = [E^{(m-2)/2}(t)\int_{\Omega} u_t Mudx]_S^T - \frac{m-2}{2}\int_S^T E^{(m-4)/2}(t)E_t(t)\int_{\Omega} u_t Mudxdt$$
$$+\int_S^T E^{(m-2)/2}(t)\int_{\Omega}|u_t|^2 dxdt - \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma dt.$$

For I_2, we use the Green formula, then we have

$$I_2 = -\int_S^T E^{(m-2)/2}(t)\int_{\Omega}\Delta uMudxdt = -\int_S^T E^{(m-2)/2}(t)\int_{\Gamma}\frac{\partial u}{\partial \nu}Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u\nabla(Mu)dxdt.$$

For the second term in the above identity, we have

$$\int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u\nabla(Mu)dxdt = \int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u\nabla(2h.\nabla u + (n-1)u)dxdt,$$

it follows that

$$\int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u\nabla(Mu)dxdt = 2\int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u.\nabla(h.\nabla u)dxdt$$
$$+(n-1)\int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2 dxdt. \tag{16}$$

But

$$\int_{\Omega}\nabla u\nabla(h.\nabla u)dx = \int_{\Omega}|\nabla u|^2 dx + \frac{1}{2}\int_{\Omega} h.\nabla(|\nabla u|^2)dx.$$

Then, by the identity (15), we get

$$\int_{\Omega}\nabla u\nabla(h.\nabla u)dx = \int_{\Omega}|\nabla u|^2 dx + \frac{1}{2}\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma - \frac{1}{2}\int_{\Omega} divh|\nabla u|^2 dx$$

$$= \frac{2-n}{2}\int_{\Omega}|\nabla u|^2dx + \frac{1}{2}\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma.$$

So, by replacing it in (16), we find

$$\int_S^T E^{(m-2)/2}(t)\int_{\Omega}\nabla u\nabla(Mu)dxdt = (2-n)\int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2dxdt$$

$$+(n-1)\int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2dxdt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma dt$$

$$= \int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2dxdt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma dt.$$

Hence

$$I_2 = \int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2dxdt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)|\nabla u|^2 d\Gamma dt - \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}\frac{\partial u}{\partial\nu}Mud\Gamma dt.$$

Inserting I_1, I_2 and I_3 in (13) to find

$$0 = [E^{(m-2)/2}(t)\int_{\Omega}u_tMudx]_S^T - \frac{m-2}{2}\int_S^T E^{(m-4)/2}(t)E_t(t)\int_{\Omega}u_tMudxdt$$

$$+\int_S^T E^{(m-2)/2}(t)\int_{\Omega}|u_t|^2dxdt + \int_S^T E^{(m-2)/2}(t)\int_{\Omega}|\nabla u|^2dxdt - \int_S^T E^{(m-2)/2}(t)\int_{\Omega}f(x,u)Mudxdt$$

$$-\int_S^T E^{(m-2)/2}(t)\int_{\Gamma}\frac{\partial u}{\partial\nu}Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)(|\nabla u|^2 - |u_t|^2)d\Gamma dt.$$

Thus, we can write it as following

$$\int_S^T E^{(m-2)/2}(t)\int_{\Omega}(|u_t|^2 + |\nabla u|^2)dxdt = I_{\Omega} + I_{[S,T]\times\Omega} + I_{[S,T]\times\Gamma}, \tag{17}$$

where

$$I_{\Omega} = -[E^{(m-2)/2}(t)\int_{\Omega}u_tMudx]_S^T,$$

$$I_{[S,T]\times\Omega} = \frac{m-2}{2}\int_S^T E^{(m-4)/2}(t)E_t(t)\int_{\Omega}u_tMudxdt + \int_S^T E^{(m-2)/2}(t)\int_{\Omega}f(x,u)Mudxdt,$$

and

$$I_{[S,T]\times\Gamma} = \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}\frac{\partial u}{\partial\nu}Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma}(h.\nu)(|u_t|^2 - |\nabla u|^2)d\Gamma dt.$$

Step 2: Energy inequality

For the first term I_Ω, we can see that

$$\int_\Omega |Mu|^2 dx = \int_\Omega |2h.\nabla u + (n-1)u|^2 dx \leq \int_\Omega (|2h.\nabla u| + |(n-1)u|)^2 dx.$$

By the following inequality

$$(a+b)^2 \leq 2(a^2+b^2) \quad for\ all\ a, b \geq 0,$$

we have

$$\int_\Omega |Mu|^2 dx \leq 2\int_\Omega |2h.\nabla u|^2 dx + 2\int_\Omega |(n-1)u|^2 dx.$$

In the rest of the proof, C represents a positive generic constant.

By the Poincare's inequality, we get

$$\int_\Omega |Mu|^2 dx \leq C\|\nabla u\|_2^2. \tag{18}$$

Hence, we have

$$|\int_\Omega u_t Mu dx| \leq \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |Mu|^2 dx \leq C(\|u_t\|_2^2 + \|\nabla u\|_2^2),$$

but, by (11), we obtain

$$|\int_\Omega u_t Mu dx| \leq CE(t). \tag{19}$$

Then, the first term I_Ω became

$$I_\Omega = E^{(m-2)/2}(S)\int_\Omega u_t(S)Mu(S)dx - E^{(m-2)/2}(T)\int_\Omega u_t(T)Mu(S)dx$$
$$\leq CE^{(m-2)/2}E(S) + CE^{(m-2)/2}(T)E(T).$$

Since the energy E is a positive decreasing function, then

$$I_\Omega \leq CE^{(m-2)/2}(S)E(S) \leq CE(S). \tag{20}$$

For the second term $I_{[S,T]\times\Omega}$, we have

$$I_{[S,T]\times\Omega} = \frac{m-2}{2}\int_S^T E^{(m-4)/2}(t)E_t(t)\int_\Omega u_t Mu dx dt + \int_S^T E^{(m-2)/2}(t)\int_\Omega f(x,u)Mu dx dt.$$

By (19) and the Young inequality, we get for all $\epsilon_1 > 0$

$$I_{[S,T]\times\Omega} \leq C\int_S^T E^{(m-4)/2}(t)(-E_t(t))E(t)dt + \frac{\epsilon_1}{2}\int_S^T E^{(m-2)/2}(t)\int_\Omega |Mu|^2 dx dt + \frac{1}{2\epsilon_1}\int_S^T E^{(m-2)/2}(t)\int_\Omega |f(x,u)|^2 dx dt.$$

Using the assumptions on f, (9) and (18), we find

$$I_{[S,T]\times\Omega} \leq C\int_S^T E^{(m-2)/2}(t)(-E_t(t))dt + \epsilon_1 C\int_S^T E^{m/2}(t)dt + C(\epsilon_1)\int_S^T E^{(m-2)/2}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)} dx)dt$$

$$\leq C[E^{m/2}(S) - E^{m/2}(T)] + \epsilon_1 C \int_S^T E^{m/2}(t)dt + C(\epsilon_1) \int_S^T E^{(m-2)/2}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)} dx)dt,$$

Since the energy is a positive decreasing function, then we obtain

$$I_{[S,T]\times\Omega} \leq CE(S) + \epsilon_1 C \int_S^T E^{m/2}(t)dt + C(\epsilon_1) \int_S^T E^{(m-2)/2}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)} dx)dt, \tag{21}$$

We apply the interpolation inequality

$$\|u\|_r \leq \|u\|_2^\alpha \|u\|_\beta^{1-\alpha} \quad with \ \frac{1}{r} = \frac{\alpha}{2} + \frac{1-\alpha}{\beta} \ and \ \alpha \in [0,1].$$

For

$$r = 2(p-1), \ \alpha = 1/2(p-1) \ and \ \beta = 2(2p-3),$$

we obtain

$$\|u\|_{2(p-1)} \leq \|u\|_2^{1/2(p-1)} \|u\|_{2(2p-3)}^{(2p-3)/2(p-1)},$$

then

$$\|u\|_{2(p-1)}^{2(p-1)} \leq \|u\|_2 \|u\|_{2(2p-3)}^{2p-3}.$$

We use the Young inequality to find for all $\epsilon_2 > 0$

$$\|u\|_p^p \leq \frac{\epsilon_2}{2} \|u\|_{2(2p-3)}^{2(2p-3)} + \frac{1}{2\epsilon_2} \|u\|_2^2.$$

Using the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(2p-3)}(\Omega)$ to have

$$\|u\|_p^p \leq \epsilon_2 C \|\nabla u\|_2^{2(2p-3)} + C(\epsilon_2)\|u\|_2^2.$$

By (9), we get

$$\|u\|_p^p \leq \epsilon_2 CE(t) + C(\epsilon_2)\|u\|_2^2,$$

Then, we replace it in (21), to find

$$I_{[S,T]\times\Omega} \leq CE(S) + \epsilon_1 C \int_S^T E^{m/2}(t)dt + \epsilon_2 C(\epsilon_1) \int_S^T E^{m/2}(t)dt + C(\epsilon_1, \epsilon_2) \int_S^T E^{(m-2)/2}(t) \int_\Omega |u|^2 dxdt, \tag{22}$$

For the third term $I_{[S,T]\times\Gamma}$, we have

$$I_{[S,T]\times\Gamma} = \int_S^T E^{(m-2)/2}(t) \int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t) \int_\Gamma (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt$$

$$= \int_S^T E^{(m-2)/2}(t) \int_{\Gamma_0} \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t) \int_{\Gamma_1} \frac{\partial u}{\partial \nu} Mud\Gamma dt$$

$$+ \int_S^T E^{(m-2)/2}(t) \int_{\Gamma_0} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt + \int_S^T E^{(m-2)/2}(t) \int_{\Gamma_1} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt$$

$$= I_{[S,T]\times\Gamma_0} + I_{[S,T]\times\Gamma_1}, \tag{23}$$

where

$$I_{[S,T]\times\Gamma_0} = \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}\frac{\partial u}{\partial \nu}Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt$$

and

$$I_{[S,T]\times\Gamma_1} = \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}\frac{\partial u}{\partial \nu}Mud\Gamma dt + \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

For $I_{[S,T]\times\Gamma_0}$, we use the definition of Mu, then we get

$$I_{[S,T]\times\Gamma_0} = 2\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nabla u)\frac{\partial u}{\partial \nu}d\Gamma dt+(n-1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}u\frac{\partial u}{\partial \nu}d\Gamma dt+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

On Γ_0, we have

$$\begin{cases} u = 0, \\ \nabla u = \nu.\frac{\partial u}{\partial \nu}, \end{cases}$$

then

$$\begin{cases} u_t = 0, \\ |\nabla u|^2 = |\frac{\partial u}{\partial \nu}|^2. \end{cases}$$

So, we can write the term on $I_{[S,T]\times\Gamma_0}$ as following

$$I_{[S,T]\times\Gamma_0} = 2\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt - \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Then

$$I_{[S,T]\times\Gamma_0} = \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Since

$$h.\nu \le 0 \ \ on \ \ \Gamma_0,$$

so we arrive at

$$I_{[S,T]\times\Gamma_0} \le 0. \tag{24}$$

For $I_{[S,T]\times\Gamma_1}$, we use the assumptions on the boundary term (4) and the definition of Mu to find

$$I_{[S,T]\times\Gamma_1} = \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(2h.\nabla u)\frac{\partial u}{\partial \nu}d\Gamma dt+(n-1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}u\frac{\partial u}{\partial \nu}d\Gamma dt+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

Then

$$I_{[S,T]\times\Gamma_1} = -\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(2h.\nabla u)(h.\nu)g(x,u_t)d\Gamma dt - (n-1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}u(h.\nu)g(x,u_t)d\Gamma dt$$

$$+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

By the Young inequality, we find for all $\epsilon_1 > 0$

$$I_{[S,T]\times\Gamma_1} \le \int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}[2(\frac{h^2}{2}|g(x,u_t)|^2+\frac{1}{2}|\nabla u|^2)+(n-1)(\frac{\epsilon_1}{2}|u|^2+\frac{1}{2\epsilon_1}|g(x,u_t)|^2)](h.\nu)d\Gamma dt$$

$$-\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)|\nabla u|^2 d\Gamma dt+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)|u_t|^2 d\Gamma dt$$

$$=\epsilon_1 C\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|u|^2 d\Gamma dt+C(\epsilon_1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|g(x,u_t)|^2(h.\nu)d\Gamma dt$$

$$+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(h.\nu)|u_t|^2 d\Gamma dt.$$

We put $d\Gamma_h = h.\nu$ to obtain

$$I_{[S,T]\times\Gamma_1} \le \epsilon_1 C\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|u|^2 d\Gamma_h dt$$

$$+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|u_t|^2 d\Gamma_h dt+C(\epsilon_1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|g(x,u_t)|^2 d\Gamma_h dt.$$

We have

$$\int_{\Gamma_1}|u|^2 d\Gamma_h \le \frac{1}{2}\int_\Omega |\nabla u|^2 dx \le CE(t).$$

So, the term on Γ_1 became

$$I_{[S,T]\times\Gamma_1} \le \epsilon_1 C\int_S^T E^{m/2}(t)dt+C(\epsilon_1)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt. \tag{25}$$

We have

$$\Gamma_1 = \Gamma_2 \cup \Gamma_3,$$

with

$$\Gamma_2 = \{x \in \Gamma_1; |u_t| \le 1\}$$

and

$$\Gamma_3 = \{x \in \Gamma_1; |u_t| > 1\}.$$

Then, we obtain

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|(g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt=\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt$$

$$+\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_3}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt. \quad (26)$$

We have

$$|g(x,u_t)|^2 = |g(x,u_t)|^{2/m}(|g(x,u_t)|^{-2/m}|g(x,u_t)|^2) = |g(x,u_t)|^{2/m}|g(x,u_t)|^{2(m-1)/m}$$

and

$$|u_t|^2 = |u_t|^{2/m}(|u_t|^{-2/m}|u_t|^2) = |u_t|^{2/m}|u_t|^{2(m-1)/m}.$$

For $|u_t| \le 1$, we use the assumptions on g to find

$$|g(x,u_t)|^2 \le C_4^{2(m-1)/m}|g(x,u_t)u_t|^{2/m}$$

and

$$|u_t|^2 \le \frac{1}{C_3^{2/m}}|g(x,u_t)u_t|^{2/m},$$

then

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le C\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_2\subset\Gamma_1}|g(x,u_t)u_t|^{2/m}d\Gamma_h dt.$$

By the embedding $L^1(\Gamma_1)$ in $L^{2/m}(\Gamma_1)$, we get

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le C\int_S^T E^{(m-2)/2}(t)[\int_{\Gamma_1} g(x,u_t)u_t d\Gamma_h]^{2/m}dt$$

$$\le C\int_S^T E^{(m-2)/2}(t)(-E_t(t))^{2/m}dt.$$

We apply the following Young inequality

$$ab \le \epsilon_2 a^\alpha + \frac{1}{\epsilon_2^{\beta/\alpha}}b^\beta \quad for\ \ all\ \ a,b\ge 0\ \ and\ \ \frac{1}{\alpha}+\frac{1}{\beta}=1.$$

For

$$a = E^{(m-2)/2}(t), \quad b=(-E_t(t))^{2/m}, \quad \alpha = m/(m-2) \quad and \quad \beta = m/2,$$

to find

$$\int_S^T E^{(m-2)/2}(t)(-E_t(t))^{2/m}dt \le \epsilon_2\int_S^T E^{m/2}(t)dt + C(\epsilon_2)\int_S^T(-E_t(t))dt.$$

This implies that

$$\int_S^T E^{(m-2)/2}(t)(-E_t(t))^{2/m}dt \le \epsilon_2\int_S^T E^{m/2}(t)dt + C(\epsilon_2)E(S).$$

Hence

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le \epsilon_2 C\int_S^T E^{m/2}(t)dt + C(\epsilon_2)E(S). \quad (27)$$

Now, for $|u_t| > 1$, we use the assumption on g to obtain

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_3}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le (\frac{1}{C_5}+C_6)\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_3\subset\Gamma_1} g(x,u_t)u_t d\Gamma_h dt$$

$$\le C\int_S^T E^{(m-2)/2}(-E_t(t))dt,$$

then

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_3}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le CE(S). \tag{28}$$

We insert (27), (28) in (26) to find

$$\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}|(g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le \epsilon_2 C\int_S^T E^{m/2}(t)dt + C(\epsilon_2)E(S). \tag{29}$$

After that, we put the result (29) in (25) to obtain

$$I_{[S,T]\times\Gamma_1} \le \epsilon_1 C\int_S^T E^{m/2}(t)dt + \epsilon_2 C(\epsilon_1)\int_S^T E^{m/2}(t)dt + C(\epsilon_1,\epsilon_2)E(S). \tag{30}$$

Puting (24) and (30) in (23), to find

$$I_{[S,T]\times\Gamma} \le \epsilon_1 C\int_S^T E^{m/2}(t)dt + \epsilon_2 C(\epsilon_1)\int_S^T E^{m/2}(t)dt + C(\epsilon_1,\epsilon_2)E(S). \tag{31}$$

Combining (20), (22) and (31) in (17), we get

$$\int_S^T E^{(m-2)/2}(t)\int_\Omega(|u_t|^2+|\nabla u|^2)dxdt \le C(\epsilon_1,\epsilon_2)\int_S^T E^{(m-2)/2}(t)\int_\Omega |u|^2dxdt+(\epsilon_1 C+\epsilon_2 C(\epsilon_1))\int_S^T E^{m/2}(t)dt+C(\epsilon_1,\epsilon_2)E(S).$$

Taking ϵ_1 sufficiently small, then ϵ_2 sufficiently small and using the definition of the energy, to obtain

$$\int_S^T E^{m/2}(t)dt \le C\int_S^T E^{(m-2)/2}(t)\int_\Omega |u|^2dxdt + CE(S). \tag{32}$$

Step 3: End of the proof

By the uniqueness compacteness argument, we can prove that

$$\int_S^T E^{(m-2)/2}(t)\int_\Omega |u|^2dxdt \le C\int_S^T E^{(m-2)/2}(t)\int_{\Gamma_1}(|g(x,u_t)|^2+|u_t|^2)d\Gamma dt.$$

Then, for all $\epsilon_3 > 0$, we have

$$\int_S^T E^{(m-2)/2}(t)\int_\Omega |u|^2dxdt \le \epsilon_3 C\int_S^T E^{m/2}(t)dt + C(\epsilon_3)E(S).$$

Replacing it in (32) and taking ϵ_3 sufficiently small, then the result (12) is finally obtained. □

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