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Higher Dimensional $[m, C]$-Isometric Commuting d-Tuple of Operators

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Abstract. In this paper we recover an $[m, C]$-isometric operators and (m, C)-isometric commuting tuples of operators on a Hilbert space studied respectively in [11] and [16], we introduce the class of [m,C]-isometries for tuple of commuting operators. This is a generalization of the class of $[m, C]$-isometric commuting operators on a Hilbert spaces. A commuting tuples of operators $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{B}(\mathcal{H})^p$ is said to be $[m, C]$-isometric p-tuple of commuting operators if

$$\Psi_m\Big(\mathbf{S}, C\Big) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\alpha|=j} \frac{j!}{\alpha!} C\mathbf{S}^{\alpha} C\mathbf{S}^{\alpha} \Big) = 0$$

for some positive integer m and some conjugation C. We consider a multi-variable generalization of these single variable $[m, C]$-isometric operators and explore some of their basic properties.

1. Introduction

Let $\mathcal{H}$ be an separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We shall use the notations $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{C}$ be the set of natural numbers, set of positive integers and set of complex numbers respectively .

A conjugation is a conjugate-linear operator $C : \mathcal{H} \longrightarrow \mathcal{H}$, which is both involutive (i.e., $C^2 = I$) and isometric (i.e., $\langle Cx \mid Cy \rangle = \langle y \mid x \rangle \quad (\forall\ x, y \in \mathcal{H})$). In particular, if C is a conjugation on $\mathcal{H}$, then $\|C\| = 1$, $\Big(CTC\Big)^k = CT^kC$ and $\Big(CTC\Big)^* = CT^*C$ for every positive integer k. Isometries, play a critical role in operator theory. This class of operator has been generalized by many authors to non-isometric operators. The invention of m-isometric operators was well-received (as can be checked out in [1–3]) and was naturally followed by a many papers.

In 1995, J. Agler and M. Stankus [1] introduced the class of m-isometric operators and showed some basic results. For an integer $m \in \mathbb{N}$ and an operator $S \in \mathcal{L}(\mathcal{H})$, S is said to be an m-isometric operator if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} S^{*k} S^k = 0. \tag{1.1}$$

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In [9] and [10], the authors introduced the concepts of an (m, C)-isometric operator and an (∞, C)-isometric operator for a single variable operator as follows: an operator $S \in \mathcal{L}(\mathcal{H})$ is said to be an (m, C)-isometric operator if there exists a conjugation C such that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{*k}CS^{k}C = 0 \tag{1.2}$$

and it is called an (∞, C)-isometric operator if

$$\limsup_{m\to\infty}\left(\Big\|\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^{*j}CS^{k}C\Big\|\right)^{\frac{1}{m}} = 0. \tag{1.3}$$

In [11], M. Chō, J. Lee and H. Motoyoshi introduced the concept of $[m, C]$-isometric operators with conjugation C as follows: an operator $S \in \mathcal{L}(\mathcal{H})$ is said to be $[m, C]$-isometric if there exists a conjugation C on $\mathcal{H}$ such that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{k}CS^{k}C = 0. \tag{1.4}$$

Equations (1.1)-(1.2) and (1.4) was extended to so called n-quasi-m-isometric, n-quasi-(m, C)-isometric and n-quasi-$[m, C]$-isometric operators as follows. An operator $S \in \mathcal{L}(\mathcal{H})$ is called

(i) n-quasi-m-isometric operator if

$$S^{*n}\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{*k}S^{k}\Big)S^{n} = 0, \tag{1.5}$$

for some positive integers n and m. (See [18, 21]).

(ii) n-quasi-(m, C)-isometric operator if

$$S^{*n}\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{*k}CS^{k}C\Big)S^{n} = 0, \tag{1.6}$$

for some positive integers n, m and a conjugation C. (See [16]).

(iii) n-quasi-$[m, C]$-isometric operator if

$$S^{*n}\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}CS^{k}CS^{k}\Big)S^{n} = 0, \tag{1.7}$$

for some positive integers $n$and m and a conjugation C. (See [23]).

The investigation of multivariable operators belonging to some specific classes has been quite fashionable since the beginning of the century, and sometimes it is indeed relevant. The originally much of this involved study of tuples of commuting operators was introduced by many authors. Some developments on this subject have been done in [4–8, 12–15, 17, 22, 24] and the references there in. For $p \in \mathbb{N}$, let $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ be a tuple of bounded linear operators. Let $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{N}_0^p$ denote tuple of nonnegative integers, and set $|\alpha| := \sum_{j=1}^{p}|\alpha_j|$, $\alpha! := \alpha_1!\cdots\alpha_p!$. Further, define $\mathbf{S}^\alpha := S_1^{\alpha_1}S_2^{\alpha_2}\cdots S_p^{\alpha_p}$ where $S_j^{\alpha_j} = \underbrace{S_jS_j\cdots S_j}_{\alpha_j-\text{times}}$ $(1 \leq j \leq p)$ and $\mathbf{S}^* = (S_1^*, \cdots, S_p^*)$.

The authors J. Gleason et al. in [15] extended Eq.(1.1) to the case of commuting p-tuples of bounded linear operators on a Hilbert space $\mathcal{H}$. The defining equation for an m-isometric p-tuple $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ is:

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^{*\alpha}\mathbf{S}^{\alpha}\Big) = 0 \tag{1.8}$$

The authors Sid Ahmed et al. in [17] extended Eq.(1.2) to the case of (m, C)-isometric p-tuple of commuting operators. A p-tuple $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ of commuting operators is called (m, C)-isometric p-tuple if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^{*\alpha}C\mathbf{S}^{\alpha}C\Big) = 0 \tag{1.9}$$

Eq.(1.5) was extended to $n = (n_1, \cdots, n_p)$-quasi-m-isometric p-tuple of commuting operators by the authors in [8] as follows. A p-tuple of commuting operators $\mathbf{S} = (S_1, \cdots, S_p)$ is called $n = (n_1, \cdots, n_p)$-quasi-m-isometric p-tuple if

$$\mathbf{S}^{*n}\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^{*\alpha}\mathbf{S}^{\alpha}\Big)\Big)\mathbf{S}^{n} = 0, \tag{1.10}$$

for some multiindex $n = (n_1, \cdots, n_p)$ and some positive integer m. This paper investigates tuples of m-isometric operators towards the so-called $[m, C]$-isometric p-tuple of commuting operators

2. $[m, C]$-Isometric d-tuple of commuting operators

In the following, we consider a multivariable generalization of these single variable $[m, C]$-isometric operators and explore some of their basic properties.Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{L}(\mathcal{H})^d$ be a commuting d-tuple of operators. Set for $l \in \mathbb{Z}_+$

$$\Psi_l(\mathbf{S},\ C\Big) := \sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}\Big(\sum_{|\alpha|=j}\frac{j!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big). \tag{2.1}$$

For example

$$\Psi_l(\mathbf{S},\ C\Big) = \sum_{j=1}^{p}CS_jCS_j - I_{\mathcal{H}} \tag{2.2}$$

and

$$\Psi_2(\mathbf{S},\ C\Big) = I_{\mathcal{H}} - 2\sum_{j=1}^{p}CS_jCS_j + \sum_{j=1}^{p}CS_j^2CS_j^2 + 2\sum_{j,\,k=1(j\neq k)}^{p}CS_jS_kCS_jS_k. \tag{2.3}$$

Definition 2.1. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{L}(\mathcal{H})^d$ *be a commuting d-tuple of operators. Then* $\mathbf{S}$ *is said to be an* $[m, C]$*-isometric p-tuple if there exists a conjugation C on* $\mathcal{H}$ *such that*

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\alpha|=j}\frac{j!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big) = 0 \tag{2.4}$$

or equivalently if $\Psi_m\Big(\mathbf{S},\ C\Big) = 0$.

Remark 2.2. (1) *If $p = 1$, Definition 2.1 coincides with the definition of an $[m, C]$-isometric operator in single variable operator introduced by M. Chō et al.* [11].

(2) *It is not difficult to see that if $S_jC = CS_j$ for all $j = 1, 2, \cdots, p$, then $\mathbf{S}$ is an $[m, C]$- isometric p-tuple if and only if $\mathbf{S}$ is an m-isometric p-tuple.*

Remark 2.3. (1) *Since the operators $S_1, S_2, \cdots, S_p$ are commuting, every permutation of $[m, C]$-isometric d-tuple is also an $[m, C]$-isometric d-tuple.*

(2) $\mathbf{S} = (S_1, \cdots, S_d)$ *is an $[m, C]$-isometric d-tuple with a conjugation C if and only if $C\mathbf{S}C := \Big(CS_1C, \cdots, CS_pC\Big)$ is a joint an $[m, C]$-isometric d-tuple with a conjugation C.*

Remark 2.4. (1) *If $p = 2$ and $\mathbf{S} = (S_1, S_2) \in \mathcal{L}(\mathcal{H})^2$ is a commuting 2-tuple of operators, then $\mathbf{S}$ is an $[1, C]$-isometric pair if*

$$\Psi_1\Big(\mathbf{S},\ C\Big) = CS_1CS_1 + CS_2CS_2 - I_{\mathcal{H}} = 0$$

and also $[2, C]$-isometric pair if

$$\Psi_1\Big(\mathbf{S},\ C\Big) = CS_1^2CS_1^2 + CS_2^2CS_2^2 + 2S_1S_2CS_1CS_2 - 2\Big(CS_1CS_1 + CS_2CS_2\Big) + I_{\mathcal{H}} = 0. \tag{2.5}$$

(2) *Let $\mathbf{S} = (S_1, \cdots, S_d)$ be a commuting p-tuple of operators. Then $\mathbf{S}$ is a $[1, C]$-isometric d-tuple if and only if*

$$\Psi_1\Big(\mathbf{S},\ C\Big) = 0. \tag{2.6}$$

and also $[2, C]$-isometric p-tuple if and only if

$$\Psi_2\Big(\mathbf{S},\ C\Big) = 0. \tag{2.7}$$

Example 2.5. *Let C be a conjugation defined by $C[x_1, x_2] = [\overline{x}_2, \overline{x}_1]$. Consider*

$$S_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2) \text{ and } S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -1 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2).$$

I will show that $\mathbf{S} = (S_1, S_2)$ is a $[1, C]$-isometric 2-tuple. Indeed, observe that S_1 and S_2 are commuting and moreover, a simple calculation shows that

$$CS_1CS_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \sqrt{2} & \frac{1}{2} \end{pmatrix} \text{and } CS_2CS_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ -\sqrt{2} & \frac{1}{2} \end{pmatrix}.$$

Now using these equalities and we done $\mathcal{Q}_1(\mathbf{S}) = CS_1CS_1 + CS_2CS_2 - I_{\mathbb{C}^2} = 0$.

Example 2.6. *Let C be a conjugation on $\mathcal{H}$ and $S \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. Then the operator tuple $\mathbf{S} = \Big(\frac{1}{\sqrt{d}}S, \cdots, \frac{1}{\sqrt{d}}S\Big)$ is an $[m, C]$-isometric d-tuple of operators. Indeed, by the multinomial expansion,*

$$\sum_{|\alpha|=\alpha_1+\cdots+\alpha_p=k} \binom{k}{\alpha} = p^k,$$

we get

$$\begin{aligned}
\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha} &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\Big(\frac{1}{d}\Big)^{|\alpha|}CS^{|\alpha|}CS^{|\alpha|} \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}d^j\Big(\frac{1}{d}\Big)^jCS^{|\alpha|}CS^{|\alpha|} \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}CS^jCS^j \\
&= 0.
\end{aligned}$$

.

Remark 2.7. *It was observed in [11] that If $S \in \mathcal{L}(\mathcal{H})$ is an $[m, C]$-isometric and invertible operator, then S^{-1} is an $[m, C]$-isometric operator. It is natural to ask if similar result holds for an $[m, C]$-isometric n-tuple $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ of operators.*

Definition 2.8. *([14]) Let $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^n$, we will call $\mathbf{S}$ entry-wise invertible if the bounded inverse of each operator exists and in which the inverse of a tuple $\mathbf{S} = (S_1, \cdots, S_p)$ is given by the tuple $\mathbf{S}^{-1} := (S_1^{-1}, \cdots, S_n^{-1})$.*

Example 2.9. *Let $\mathbf{S} = (\frac{1}{\sqrt{p}}I, \frac{1}{\sqrt{p}}I, \cdots, \frac{1}{\sqrt{p}}I) \in \mathcal{B}(\mathcal{H})^p$ and C be a conjugation. It is easy to see that $\mathbf{S}$ satisfying*

$$\sum_{j=1}^{p} C\Big(\frac{1}{\sqrt{p}}\Big)^j C\Big(\frac{1}{\sqrt{p}}\Big)^j C - I_{\mathcal{H}} = I_{\mathcal{H}} - I_{\mathcal{H}} = 0.$$

Therefore, $\mathbf{S}$ is an $[1, C]$-isometric p-tuple of operators by (2.6). However, $\mathbf{S}^{-1} = \Big(\sqrt{p}I, \cdots, \sqrt{p}I\Big)$ satisfying

$$\sum_{j=1}^{p} C\Big(\sqrt{p}\Big)^j C\Big(\sqrt{p}\Big)^j - I_{\mathcal{H}} = \sum_{j=1}^{p} p^j - I_{\mathcal{H}} \neq 0,$$

This means that $\mathbf{S}^{-1}$ does not $[1, C]$-isometric p-tuple. We observe that a direct analogue of [11, Theorem 3.4] *does not hold for general $[m, C]$-isometric p-tuples.*

Theorem 2.10. *Let $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ be a commuting p-tuple of operators and C be a conjugation on $\mathcal{H}$. Then the following statements hold.*

(1)

$$\Psi_{m+1}\Big(\mathbf{S},\ C\Big) + \Psi_m\Big(\mathbf{S},\ C\Big) = \sum_{j=1}^{d} \Big(CS_jC\Big)\Psi_m\Big(\mathbf{S},\ C\Big)(S_j). \tag{2.8}$$

(2)

$$\sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^{\alpha} C\mathbf{S}^{\alpha} = \sum_{j=0}^{n} \binom{n}{j} \Psi_j\Big(\mathbf{S}, C\Big), \tag{2.9}$$

for every integer $n \geq 1$.

(3) *$\mathbf{S}$ is an $[m, C]$-isometric p-tuple of operators if and only if*

$$\sum_{|\alpha|=n} \frac{n!}{\beta!} C\mathbf{S}^{\alpha} C\mathbf{S}^{\alpha} = \sum_{0 \leq j \leq m-1} \binom{n}{j} \Psi_j\Big(\mathbf{S}, C\Big); \quad \forall\ n \in \mathbb{N}. \tag{2.10}$$

(4) *If $\mathbf{S}$ is an $[m, C]$-isometric p-tuple of operators, then*

$$\lim_{n\to\infty} \frac{1}{\binom{n}{m-1}} \sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^{\alpha} C\mathbf{S}^{\alpha} = \Psi_{m-1}\Big(\mathbf{S}, C\Big). \tag{2.11}$$

Proof. (1) According to equation (2.1), we get

$$\begin{aligned}
&\Psi_{m+1}\Big(\mathbf{S},\ C\Big)\\
=\ &\sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big)\\
=\ &(-1)^{m+1}I_{\mathcal{H}}-\sum_{k=1}^{m}(-1)^{m-k}\Big[\binom{m}{k}+\binom{m}{k-1}\Big]\sum_{|\alpha=k}\frac{k!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}+\sum_{|\beta|=m+1}\frac{(m+1)!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\\
=\ &-\Psi_m\Big(\mathbf{S},\ C\Big)+\sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big)+\sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\\
=\ &-\Psi_m(\mathbf{S},\mathbf{C})+\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!(\alpha_1+\cdots+\alpha_p)}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}+\sum_{|\alpha|=m+1}\frac{m!(\alpha_1+\cdots+\alpha_p)}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\\
=\ &-\Psi_m\Big(\mathbf{S},C\Big)+\sum_{j=1}^{p}\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}(-1)^{m-k}\binom{m}{k}\frac{k!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!}\\
&\times CS_jC\Big[C\Big(S_1^{\alpha_1}\cdots S_j^{\alpha_{j-1}}S_{j+1}^{\alpha_{j+1}}\cdots S_p^{\alpha_p}\Big)C\Big(S_1^{\alpha_1}\cdots S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}\cdots S_p^{\alpha_p}\Big)\Big]S_j\\
&+\sum_{1\le j\le p}\sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!}CS_jC\Big[C\Big(S_1^{\alpha_1}\cdots S_j^{\alpha_{j-1}}S_{j+1}^{\alpha_{j+1}}\cdots S_p^{\alpha_p}\Big)C\Big(S_1^{\alpha_1}\cdots S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}\cdots S_p^{\alpha_p}\Big)\Big]S_j\\
=\ &-\Psi_m(\mathbf{S})+\sum_{j=1}^{p}\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}(-1)^{k}\binom{m}{k}\frac{k!}{\alpha!}(CS_jC)C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}(S_j)+\sum_{j=1}^{p}\sum_{|\alpha|=m}\frac{m!}{\alpha!}(CS_jC)C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}(S_j)\\
=\ &-\Psi_m\Big(\mathbf{S},\mathbf{C}\Big)+\sum_{j=1}^{p}(CS_jC)\Big(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big)(S_j)\\
=\ &-\Psi_m\Big(\mathbf{S},C\Big)+\sum_{1\le j\le p}(CS_jC)\Psi_m(\mathbf{S})(S_j).
\end{aligned}$$

(2) We prove (2.9) by induction on n. Obviously for $n=0$ and $n=1$ it holds. Assume that Eq. (2.9) is hold for n.. By taking into account Eq.(2.1) and Eq. (2.9), we have

$$\begin{aligned}
\sum_{|\alpha|=n+1}\frac{n!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\ &=\ \Psi_{n+1}\Big(\mathbf{S},C\Big)-\sum_{j=0}^{n}(-1)^{n+1-j}\binom{n+1}{j}\sum_{|\alpha|=j}\frac{n!}{\alpha!}C\mathbf{S}^{\beta}C\mathbf{S}^{\alpha}\\
&=\ \Psi_{n+1}\Big(\mathbf{S},C\Big)-\sum_{j=0}^{n}(-1)^{n+1-j}\binom{n+1}{j}\sum_{k=0}^{j}\binom{j}{k}\Psi_k\Big(\mathbf{S},C\Big)\\
&=\ \Psi_{n+1}\Big(\mathbf{S},C\Big)-\sum_{k=0}^{n}\Psi_k\Big(\mathbf{S},C\Big))\sum_{j=k}^{|}n(-1)^{n+1-j}\binom{n+1}{j}\binom{j}{k}\\
&=\ \Psi_{n+1}\Big(\mathbf{S},C\Big)-\sum_{k=0}^{n}\Psi_k(\mathbf{S})\Big(\sum_{k\le j\le n}(-1)^{n+1-j}\binom{n+1}{k}\binom{n+1-k}{j-k}\Big)\\
&=\ \Psi_{n+1}\Big(\mathbf{S},C\Big)-\sum_{k=0}^{n}\binom{n+1}{k}\Psi_k\Big(\mathbf{S},C\Big)\Big(\sum_{j=k}^{n}(-1)^{n+1-j}\binom{n+1-k}{j-k}\Big)
\end{aligned}$$

$$= \Psi_{n+1}\Big(\mathbf{S},C\Big) - \sum_{k=0}^{n}\binom{n+1}{k}\Psi_k\Big(\mathbf{S},C\Big)\Big(-1+\underbrace{\sum_{r=0}^{n+1-k}(-1)^{n+1-k-r}\binom{n+1-k}{r}}_{=0}\Big)$$

$$= \sum_{0\le k\le n+1}\binom{n+1}{k}\Phi_k\Big(\mathbf{S},C\Big).$$

(3) If $\mathbf{S}$ is an $[m,C]$-isometric tuple, then $\Psi_q\Big(\mathbf{S},C\Big) = 0$ for all $q \geq m$ by Corollary2.11. Then Eq.(2.10) follows from Eq.(2.9). However , if Eq.(2.10) holds for all $n \geq 1$. Then $\Psi_k\Big(\mathbf{S},C\Big) = 0$ for $q \geq m$ by Eq.(2.10), so $\mathbf{S}$ is an $[m,C]$-isometric tuple. Therefore the necessity of the existence of the polynomial mapping $n \mapsto \sum_{|\beta|=n}\frac{n!}{\beta!}C\mathbf{S}^\beta C\mathbf{S}^\beta$ follows from Eq.(2.10).

(4) By Eq.(2.10) if $\mathbf{S}$ is an $[m,C[$-isometric p-tuple of operators, we have

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}C\mathbf{S}^\alpha C\mathbf{S}^\alpha = \sum_{j=0}^{m-2}\binom{n}{j}\Psi_j\Big(\mathbf{S},C\Big) + \binom{n}{m-1}\Psi_{m-1}\Big(\mathbf{S},C\Big).$$

Dividing both sides by $\binom{n}{m-1}$, we get that

$$\frac{1}{\binom{n}{m-1}}\sum_{|\alpha|=n}\frac{n!}{\alpha!}C\mathbf{S}^\alpha C\mathbf{S}^\alpha = \sum_{j=0}^{m-2}\frac{1}{\binom{n}{m-1}}\binom{n}{j}\Psi_j\Big(\mathbf{S},C\Big) + \Psi_{m-1}\Big(\mathbf{S},C\Big).$$

By the fact that $\frac{1}{\binom{n}{m-1}}\binom{n}{j} \longrightarrow 0$ for $j = 0,\cdots,m-2$, it follows by taking $n \to \infty$ that

$$\lim_{n\to\infty}\frac{1}{\binom{n}{m-1}}\sum_{|\alpha|=n}\frac{n!}{\alpha!}C\mathbf{S}^\alpha C\mathbf{S}^\alpha = \Psi_{m-1}\Big(\mathbf{S},C\Big).$$

□

Corollary 2.11. *Let $\mathbf{S} = (S_1,\cdots,S_d) \in \mathcal{L}(\mathcal{H})^d$ be a commuting d-tuple of operators and C be a conjugation on $\mathcal{H}$. If $\mathbf{S}$ is an $[m,C]$-isometric d-tuple, then $\mathbf{S}$ is an $[n,C]$-isometric p-tuple for all $n \geq m$.*

Proof. This statement is an immediate consequence of the statement (1) of Theorem 2.10. □

The following Corollary is a particular case of the statements (3) and (4) of Theorem 2.10.

Corollary 2.12. *Let $\mathbf{S} = (S_1,\cdots,S_p) \in \mathcal{L}(\mathcal{H})^p$ be a commuting p-tuple such that $\mathbf{S}$ is an $[2,C]$-isometric p-tuple where C is a conjugation on $\mathcal{H}$. Then the following identities hold.*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}C\mathbf{S}^\alpha C\mathbf{S}^\alpha = n\Big(\sum_{j=1}^{p}CS_jCS_j\Big) - (n-1)I., \quad \forall\ n \in \mathbb{N}. \tag{2.12}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{|\alpha|=n}\frac{n!}{\alpha!}C\mathbf{S}^\alpha C\mathbf{S}^\beta = \sum_{j=1}^{p}CS_jCS_j - I. \tag{2.13}$$

Recall that for an operator $S \in \mathcal{L}(\mathcal{H})$, the numerical range $W(S)$ of S is defined by

$$W(S) = \Big\{\Big\langle Sx \mid x\Big\rangle\ ;\ x \in \mathcal{H},\ \ \|x\| = 1\ \Big\}.$$

An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be convexoid if $\overline{W(S)} = co\sigma(S)$, that is, the closure of the numerical range of S is equal to the convex hull of the spectrum of S.

Definition 2.13. *Let* $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ *be commuting p-tuple of operators. We said that* $\mathbf{S}$ *is* $[C]$*-power bounded if*

$$\sup_n \Big\| \sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^\alpha C\mathbf{S}^\alpha \Big\| < \infty.$$

Theorem 2.14. *Let* $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ *be commuting p-tuple of operators such is* $[2, C]$*-isometric p-tuple. If* $\mathbf{S}$ *is* $[C]$*-power bounded and* $\Psi_1\Big(\mathbf{S}, C\Big)$ *is convexoid, where* $\Psi_1\Big(\mathbf{S}.C\Big)$ *is given in Eq.(2.4). Then* $\mathbf{S}$ *is a* $[1, C]$*-isometric p-tuple.*

Proof. We need to prove that $W\Big(\Psi_1\Big(\mathbf{S}, C\Big)\Big) = W\Big(\sum_{j=1}^{p} CS_jCS_j - I\Big) = \{0\}$. Assume that the claim is not true i.e; $W\Big(\Psi_1\Big(\mathbf{S}, C\Big)\Big) = W\Big(\sum_{j=1}^{p} CS_jCS_j - I\Big) \neq \{0\}$. By the assumption that $\Psi_1(\mathbf{S}, C\Big)$ is convexoid, it follows

$$\overline{W\Big(\Psi_1\Big(\mathbf{S}, C\Big)\Big)} = co\sigma\Big(\Psi_1\Big(\mathbf{S}, C\Big).$$

From which there exist a number $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $(u_n)_n \subset \mathcal{H}$ with $\|u_n\| = 1$ such hat

$$\lim_{n\to\infty} \Big((\Psi_1\Big(\mathbf{S}, C\Big)\Big) - \lambda\Big)u_n = 0.$$

Since $\mathbf{S}$ is an $[2, C]$-isometric p-tuple, it follows in view of Eq. (2.13) that

$$\frac{1}{n}\Big(\sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^\alpha C\mathbf{S}^\alpha - I\Big) = \sum_{j=1}^{p} CS_jCS_j - I = \Psi_1\Big(\mathbf{S}, C\Big).$$

This means that

$$\frac{1}{n}\Big(\sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^\alpha C\mathbf{S}^\alpha - I - n\lambda\Big)u_n = \Big(\Psi_1\Big(\mathbf{S}, C\Big) - \lambda\Big)u_n,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n}\Big(\sum_{|\alpha|=n} \frac{n!}{\alpha!} C\mathbf{S}^\alpha C\mathbf{S}^\alpha - I - n\lambda\Big)u_n = 0,$$

□

Proposition 2.15. *Let* $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ *be an* $[m, C]$*-isometric p-tuple where C is a conjugation on* $\mathcal{H}$*. If* $CS_j = S_jC$ *for* $j = 1, \cdots, p$*, then* $\mathbf{S}$ *satisfying the following identity*

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{S}^{2\alpha}\Big) = 0.$$

Proof. Under the assumption that $\mathbf{S}$ is an $[m, C]$-isometric p-tuple, we get that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}C\mathbf{S}^{\alpha}C\mathbf{S}^{\alpha}\Big)=0.$$

By using the conditions $CS_j = S_jC$ for all $j = 1, 2, \cdots, p$, we have

$$C\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{S}^{\alpha}C\Big)=0.$$

From which we deduce that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^{2\alpha}\Big)=0.$$

□

Lemma 2.16. *([8]) Let $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{N}_0^p$, $\mu = (\mu_1, \cdots, \mu_p) \in \mathbb{N}_0^p$, $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ be such that $|\alpha| + |\mu| + k = n + 1$. For $1 \le l \le p$, let 1_l be the element of $\mathbb{N}^p$ such that all its components are zero except the l component which is equal to 1. Then,*

$$\binom{n+1}{\alpha,\mu,k}=\sum_{1\le l\le p}\left(\binom{n}{\alpha-1_l,\mu,k}+\binom{n}{\alpha,\mu-1_l,k}\right)+\binom{n}{\alpha,\mu,k-1}. \tag{2.14}$$

In order to prove that the perturbation of a $[m, C]$-isometric p-tuple by a nilpotent p-tuple is a $[n, C]$-isometric tuple under suitable conditions, we need to introduce the following lemma. It should be noted that some technics of our proof are inspected from [17, Lemma 2.2].

Lemma 2.17. *Let $S = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ and $N = (N_1, \cdots, N_p) \in \mathcal{L}(\mathcal{H})^p$ be two commuting operators such that $S_jN_k = N_kS_j$ for all j, k and let C be a conjugation on $\mathcal{H}$. Then the following identity holds:*

$$\Psi_n\Big(\mathbf{S}+\mathbf{N},\ C\Big)=\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot\Psi_k\Big(\mathbf{S},C\Big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}. \tag{2.15}$$

Proof. By mathematical induction on n., we prove that the identity (2.15) is holds. In fact assume that $n = 1$ we have

$$\begin{aligned}
&\sum_{|\alpha|+|\mu|+k=1}\binom{1}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot\Psi_k\Big(\mathbf{S},C\Big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
=\;&\sum_{j=1}^{p}C\Big(S_j+N_j\Big)CN_j+\sum_{1\le j\le p}CN_jCS_j+\sum_{1\le j\le p}CS_jCS_j-I\\
=\;&\sum_{j=1}^{p}\Big(CS_jCN_j+CN_jCN_j+CN_jCS_j+CS_jCS_j\Big)-I\\
=\;&\sum_{j=1}^{p}\Big(C(S_j+N_j)\Big)\Psi_0\Big(\mathbf{S}+\mathbf{N},C\Big)\Big(C(S_j+N_j)\Big)-\Psi_0\Big(\mathbf{S}+\mathbf{N},\ C\Big)\\
=\;&\Psi_1\Big(\mathbf{S}+\mathbf{N},C\Big)\ \text{(from Eq.2.8)}.
\end{aligned}$$

Now assume that (2.15) holds for n. In view of Eq.2.8, we have

$$\begin{aligned}
&\Psi_{n+1}\Big(\mathbf{S}+\mathbf{N},C\Big)\\
=\;&\sum_{l=1}^{p}\Big(C(S_l+N_l)C\Big)\Psi_n\Big(\mathbf{S}+\mathbf{N},C\Big)\Big(S_l+N_l\Big)-\Psi_n\Big(\mathbf{S}+\mathbf{N},\Big))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{p}\Big(C(S_l+N_l)C\Big)\Big(\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot\Psi_k\big(\mathbf{S},\ C\big)\cdot C\mathbf{S}^{\mu}\cdot C\mathbf{N}^{\alpha}\Big)\big(S_l+N_l\big)\\
&\quad -\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot\Psi_k\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&= \sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}C\cdot\mathbf{N}^{\mu}\Big(\sum_{l=1}^{p}C\big(S_l+N_l\big)C\Psi_k\big(\mathbf{S},C\big)\big(S_l+N_l\big)-\Psi_k\big(\mathbf{S},C\big)\Big)\\
&\qquad\qquad \cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&= \sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}C\cdot\mathbf{N}^{\mu}\Big(\Psi_{k+1}\big(\mathbf{S},C\big)+\sum_{l=1}^{p}\Big\{CS_lC\Psi_k\big(\mathbf{S},C\big)N_l\\
&\quad +N_l\cdot\Psi_k(\mathbf{S},C\big)\cdot C(S_l+N_l)C\cdot\Psi_k\big(\mathbf{S},C\big)\cdot N_l\Big\}\Big)\cdot C\mathbf{S}^{\mu}\cdot C\mathbf{N}^{\alpha}\\
&= \sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot\Psi_{k+1}\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&\quad +\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}\Big\{\sum_{1\le l\le p}C(\mathbf{S}+\mathbf{N})^{\alpha}C(S_l+N_l)C\cdot\mathbf{N}^{\mu}\cdot\Psi_k\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\cdot N_l\Big\}\\
&\quad +\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}\Big\{\sum_{1\le l\le p}C(\mathbf{S}+\mathbf{N})^{\alpha}\cdot C\mathbf{N}^{\mu}\cdot N_l\cdot\Psi_k\big(\mathbf{S}\big)\cdot CS_l\cdot\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\Big\}
\end{aligned}
$$

By taking into account the identity (2.14), it follows that

$$
\begin{aligned}
&\sum_{|\alpha|+|\mu|+k=n+1}\binom{n+1}{\alpha,\mu,k}C(\mathbf{S}+\mathbf{N})^{\alpha}C\mathbf{N}^{\mu}\cdot\Psi_k\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&= \Psi_{n+1}\big(\mathbf{S},C\big)+\sum_{|\alpha|=n+1}\binom{n+1}{\alpha}C\big(\mathbf{S}+\mathbf{N}\big)^{\alpha}C\cdot\mathbf{N}^{\alpha}+\sum_{|\mu|=n+1}\binom{n+1}{\mu}\mathbf{N}^{\mu}C\mathbf{S}^{\mu}C\\
&\quad +\sum_{|\alpha|+|\mu|+k=n+1}\underbrace{\binom{n+1}{\alpha,\mu,k}}C(\mathbf{S}+\mathbf{N})^{\alpha}C\mathbf{N}^{\mu}\cdot\Psi_k\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&= \Psi_{n+1}\big(\mathbf{S},C\big)+\sum_{|\alpha|=n+1}\binom{n+1}{\alpha}C\big(\mathbf{S}+\mathbf{N}\big)^{\alpha}C\cdot\mathbf{N}^{\alpha}+\sum_{|\mu|=n+1}\binom{n+1}{\mu}\mathbf{N}^{\mu}C\mathbf{S}^{\mu}C\\
&\quad +\sum_{|\alpha|+|\mu|+k=n+1}\underbrace{\Big(\sum_{l=1}^{p}\Big(\binom{n}{\alpha-1_l,\mu,k}+\binom{n}{\alpha,\mu-1_l,k}\Big)+\binom{n}{\alpha,\mu,k-1}.\Big)}C(\mathbf{S}+\mathbf{N})^{\alpha}C\mathbf{N}^{\mu}\cdot\\
&\qquad \Psi_k\big(\mathbf{S},C\big)\cdot C\mathbf{S}^{\mu}C\cdot\mathbf{N}^{\alpha}\\
&= \Psi_{n+1}\big(\mathbf{S},C\big)+\sum_{|\alpha|=n+1}\binom{n+1}{\alpha}C\big(\mathbf{S}+\mathbf{N}\big)^{\alpha}C\cdot\mathbf{N}^{\alpha}+\sum_{|\mu|=n+1}\binom{n+1}{\mu}\mathbf{N}^{\mu}C\mathbf{S}^{\mu}C\\
&\quad +\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}\sum_{l=1}^{p}C\big(S_l+N_l\big)\big(\mathbf{S}+\mathbf{N}\big)^{\alpha}C\mathbf{N}^{\mu}\Psi_k\big(\mathbf{S},C\big)C\mathbf{S}^{\mu}CN_l\mathbf{N}^{\alpha}\\
&\quad +\sum_{|\alpha|+|\mu|+k=n}\binom{n}{\alpha,\mu,k}\sum_{l=1}^{p}C\big(\mathbf{S}+\mathbf{N}\big)^{\alpha}CN_l\mathbf{N}^{\mu}\Psi_k\big(\mathbf{S},C\big)CS_l\mathbf{S}^{\mu}C\mathbf{N}^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{|\alpha|+|\mu|+k=n} \binom{n}{\alpha,\mu,k} C(\mathbf{S}+\mathbf{N})^{\alpha} C\mathbf{N}^{\mu} \cdot \Psi_{k+1}\Big(\mathbf{S},C\Big) \cdot C\mathbf{S}^{\mu} C \cdot \mathbf{N}^{\alpha} \\
= &\sum_{|\alpha|+|\mu|+k=n} \binom{n}{\alpha,\mu,k} \sum_{l=1}^{p} C\Big(\mathbf{S}+\mathbf{N}\Big)^{\alpha} CN_l \mathbf{N}^{\mu} \Psi_k\Big(\mathbf{S},C\Big) CS_l \mathbf{S}^{\mu} C\mathbf{N}^{\alpha} \\
&+ \sum_{|\alpha|+|\mu|+k=n} \binom{n}{\alpha,\mu,k} C(\mathbf{S}+\mathbf{N})^{\alpha} C\mathbf{N}^{\mu} \cdot \Psi_{k+1}\Big(\mathbf{S},C\Big) \cdot C\mathbf{S}^{\mu} C \cdot \mathbf{N}^{\alpha} \\
&+ \sum_{|\alpha|+|\mu|+k=n} \binom{n}{\alpha,\mu,k} C(\mathbf{S}+\mathbf{N})^{\alpha} C\mathbf{N}^{\mu} \cdot \Psi_{k+1}\Big(\mathbf{S},C\Big) \cdot C\mathbf{S}^{\mu} C \cdot \mathbf{N}^{\alpha} \\
= &\ \Psi_{n+1}\Big(\mathbf{S}+\mathbf{N},C\Big).
\end{aligned}
$$

Therefore, the proof is completed. □

Theorem 2.18. *Let $S = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H})^p$ be an $[m, C]$-isometric p-tuple of commuting operators where C is a conjugation and let $N = (N_1, \cdots, N_p) \in \mathcal{L}(\mathcal{H})^p$ be a r-nilpotent p-tuple of commuting operators such that $S_jN_k = N_kS_j$ for all $j, k = 1, \cdots, p$. Then $\mathbf{S} + \mathbf{N} := (S_1 + N_1, \cdots, S_p + N_p)$ is an $[m + 2r - 2, C]$-isometric p-tuple.*

Proof. We will prove that $\Psi_{m+2q-2}\Big(\mathbf{S}+\mathbf{N},C\Big) = 0$. From Lemma 2.17, we have

$$\Psi_n\Big(\mathbf{S}+\mathbf{N},C\Big) = \sum_{|\alpha|+|\mu|+k=m+2r-2} \binom{m+2r-2}{\alpha,\mu,k} C(\mathbf{S}+\mathbf{N})^{\alpha} C \cdot \mathbf{N}^{\mu} \cdot \Psi_k\Big(\mathbf{S},C\Big) \cdot C\mathbf{S}^{\mu} C \cdot \mathbf{N}^{\alpha}.$$

(1) If $\max\{|\alpha|, |\mu|\} \geq r$, then $\mathbf{N}^{\mu} = 0$ or $\mathbf{N}^{\alpha} = 0$.
(2) If $\max\{|\alpha|, |\mu|\} \leq r - 1$, then $k \geq m$ and hence $\Psi_k\Big(\mathbf{S},C\Big) = 0$ $\Big($by Corollary 2.11$\Big)$.

By tanking into account (1) and (2), we get that $\Psi_{m+2q-2}\Big(\mathbf{S}+\mathbf{N},C\Big) = 0$.
Therefore, $\mathbf{S} + \mathbf{N}$ is an $[m + 2q - 2, C]$-isometric tuple as required. □

Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. Given nonzero $T, S \in \mathcal{L}(\mathcal{H})$, let $T \otimes S \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denote the tensor product on the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$. A particularly interesting consequences of Theorems 2.18 are the following.

Corollary 2.19. *Let $\mathbf{S} = (S_1, \ldots, S_p) \in \mathcal{L}(\mathcal{H})^p$ be an $[m, C]$-isometric p-tuple of commuting operators with a conjugation C and let $\mathbf{N} = (N_1, \cdots, N_p) \in \mathcal{L}(\mathcal{H})^p$ be a r-nilpotent p-tuple of commuting operators.Then*
$\mathbf{S}\otimes\mathbf{I} + \mathbf{I} \otimes \mathbf{N} := (S_1 \otimes I + I \otimes N_1, \cdots, S_p \otimes I + I \otimes N_p) \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is an $(m + 2q - 2, C \otimes C)$-isometric p-tuple.

Proof. It is well known that $(S_k \otimes I)(I \otimes N_j) = (I \otimes N_j)(S_k \otimes I)$ for all $j, k = 1; \cdots, p$ However, it is easy to check that $\mathbf{S} \otimes \mathbf{I} = (S_1 \otimes I,\ \cdots, S_p \otimes i) \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})^p$ is an $[m, C \otimes C]$-isometric p-tuple and $\mathbf{I} \otimes \mathbf{N} \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})^p$ is a nilpotent p-tuple of order r. So $\mathbf{S} \otimes \mathbf{I}$ and $\mathbf{I} \otimes \mathbf{N}$ satisfy the conditions of Theorem 2.18. From which, $\mathbf{S}\otimes\mathbf{I} + \mathbf{I} \otimes \mathbf{N}$ is an $[m + 2q - 2, C \otimes C]$-isometric p-tuple of operators. □

Corollary 2.20. *Let $\mathbf{Q} = (Q_1, \cdots, Q_p) \in \mathcal{L}(\mathcal{H})^p$ be an $[m, C]$-isometric p-tuple where C is a conjugation on $\mathcal{H}$. If $\mathbf{S} = (S_1, \cdots, S_p) \in \mathcal{L}(\mathcal{H}^{(r)})$ is defined by*

$$S_j = \begin{pmatrix} Q_j & \lambda_j I & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \lambda_j I \\ 0 & \ddots & 0 & Q_j \end{pmatrix} \ \text{on}\ \ \mathcal{H}^{(r)} := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$$

where $\lambda_j \in \mathbb{C}$ *for* $j = 1, \cdots, p$, *then* $\mathbf{S}$ *is an* $[m+2r-2, C^{(r)}]$*-isometric tuple where* $C^{(r)} := C \oplus C \cdots \oplus C$ *is a conjugation on* $\mathcal{H}^{(n)}$.

Proof. Consider the p-tuples of operators $\mathbf{R} = (R_1, \cdots, R_p)$ and $\mathbf{J} = (J_1, \cdots, J_p)$ where

$$R_j = \begin{pmatrix} Q_j & 0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & Q_j \end{pmatrix} \text{ and } J_j = \begin{pmatrix} 0 & \lambda_j I & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \lambda_j I \\ 0 & \ddots & 0 & 0 \end{pmatrix} \text{ for } j = 1, \cdots, p.$$

Obviously we have that $\mathbf{S} = \mathbf{R} + \mathbf{J} = (R_1 + J_1, \cdots, R_p + J_p)$. Since $\mathbf{Q}$ is an $[m, C]$-isometric p-tuple, it follows that $\mathbf{R}$ is an $(m, C^{(r)})$-isometric p-tuple. However $\mathbf{J}$ is a r-nilpotent p-tuple. According to Theorem 2.18, $\mathbf{S}$ is an $(m + 2r - 2, C^{(r)})$-isometric tuple. □

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