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On Univalent Log-Harmonic Mappings

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Abstract. We consider the class of univalent log-harmonic mappings on the unit disk. Firstly, we present general idea of constructing log-harmonic Koebe mappings, log-harmonic right half-plane mappings and log-harmonic two-slits mappings and then we show precise ranges of these mappings. Moreover, coefficient estimates for univalent log-harmonic starlike mappings are obtained. Growth and distortion theorems for certain special subclasses of log-harmonic mappings are studied. Finally, we propose two conjectures, namely, log-harmonic coefficient and log-harmonic covering conjectures.

1. Introduction and preliminary results

Let $\mathcal{A}$ be the linear space of all analytic functions defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{B}$ be the set of all functions $\mu \in \mathcal{A}$ such that $|\mu(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = \mu(z)\left(\frac{\overline{f(z)}}{f(z)}\right) f_z(z), \tag{1}$$

where μ is the second complex dilatation of f and $\mu \in \mathcal{B}$. Thus, the Jacobian J_f of f is given by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2(1 - |\mu|^2),$$

which is positive, and therefore, every non-constant log-harmonic mapping is sense-preserving and open in $\mathbb{D}$. If f is a non-vanishing log-harmonic mapping in $\mathbb{D}$, then f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

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where $h(z)$ and $g(z)$ are non-vanishing analytic functions in $\mathbb{D}$. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then such a mapping f admits the representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\operatorname{Re}\beta > -1/2$, $h, g \in \mathcal{A}$, $h(0) \neq 0$ and $g(0) = 1$ (cf. [4]). The class of all functions of this form has been widely studied. See, for example [3, 8, 15].

For simplicity, we set $\beta = 0$ and consider the class $\mathcal{S}_{Lh}$ of univalent log-harmonic mappings f of the form

$$f(z) = zh(z)\overline{g(z)},$$

where $h, g \in \mathcal{A}$ with the normalization $h(0) = g(0) = 1$, and such that

$$h(z) = \exp\left(\sum_{n=1}^{\infty} a_n z^n\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{n=1}^{\infty} b_n z^n\right). \tag{2}$$

A complex-valued function $f : \Omega \to \mathbb{C}$ is said to belong to the class $C^1(\Omega)$ (resp. $C^2(\Omega)$) if $\operatorname{Re} f$ and $\operatorname{Im} f$ have continuous first order (resp. second order) partial derivatives in Ω. For $f \in C^1(\Omega)$, consider the complex linear differential operator Df defined on $C^1(\Omega)$ by

$$Df = zf_z - \overline{z}f_{\overline{z}}.$$

In view of the Riemann mapping theorem, it suffices to consider the case when $\Omega = \mathbb{D}$.

Definition 1.1. *Let $\alpha \in [0, 1)$. A univalent function $f \in C^1(\mathbb{D})$ with $f(0) = 0$ is called* starlike of order α, *denoted by $\mathcal{FS}^*(\alpha)$, if*

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) = \operatorname{Re}\left(\frac{Df(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) > \alpha,$$

for all $z = re^{i\theta} \in \mathbb{D}\backslash\{0\}$. We set $\mathcal{FS}^*(0) =: \mathcal{FS}^*$, and functions in $\mathcal{FS}^*$ are C^1-(fully) starlike in $\mathbb{D}$. If $f \in C^1(\mathbb{D})$ is replaced by $f \in \mathcal{S}$ (the class of normalized analytic univalent mappings of the unit disk), then $\mathcal{FS}^*(\alpha)$ coincides with the class $\mathcal{S}^*(\alpha)$, of the class of all normalized analytic starlike functions of order α.

We denote by $\mathcal{S}^*_H(\alpha)$ and $\mathcal{S}^*_{Lh}(\alpha)$ the set of all starlike harmonic functions of order α and starlike log-harmonic functions of order α, respectively. If $\alpha = 0$, then we simply denote these classes by $\mathcal{S}^*_H$ and $\mathcal{S}^*_{Lh}$, respectively. Thus, $\mathcal{S}^*_H$ and $\mathcal{S}^*_{Lh}$ are called the set of all starlike harmonic functions and starlike log-harmonic functions, respectively. These classes are investigated in detail by a number of authors. See for example [1, 3].

The following theorem establishes a link between the classes $\mathcal{S}^*_{Lh}(\alpha)$ and $\mathcal{S}^*(\alpha)$.

Theorem A. ([5, Lemma 2.4] or [1, Theorem 2.1]) *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping on $\mathbb{D}$, $0 \notin (hg)(\mathbb{D})$. Then $f \in \mathcal{S}^*_{Lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in \mathcal{S}^*(\alpha)$.*

In [12], Li et al. proved the following result.

Theorem B. *Let $f(z) = \varphi(z)|z|^{2(p-1)}$ $(p \geq 1)$, where $\varphi \in C^1(\mathbb{D})$ is starlike (not necessarily harmonic) in $\mathbb{D}$. Then $f \in C^1(\mathbb{D})$ is starlike and univalent in $\mathbb{D}$.*

We state the following simple result which generalizes both Theorems A and B.

Proposition 1.2. *Let $f(z) = \varphi(z)|g(z)|^2$ be a complex-valued function on $\mathbb{D}$, where $\varphi, g \in \mathcal{A}$ such that φ and g are non-vanishing in $\mathbb{D}\backslash\{0\}$. Then $f \in \mathcal{FS}^*(\alpha)$ if and only if $\varphi \in \mathcal{FS}^*(\alpha)$.*

Proof. A simply calculation shows that

$$zf_z(z) = z\varphi_z(z)|g(z)|^2 + \varphi(z)zg'(z)\overline{g(z)} = \left(\frac{z\varphi_z(z)}{\varphi(z)} + \frac{zg'(z)}{g(z)}\right)f(z)$$

and similarly

$$\overline{z}f_{\overline{z}}(z) = \left(\frac{\overline{z}\varphi_{\overline{z}}(z)}{\varphi(z)} + \overline{\left(\frac{zg'(z)}{g(z)}\right)}\right)f(z),$$

which clearly implies that

$$\mathrm{Re}\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{z\varphi_z(z) - \overline{z}\varphi_{\overline{z}}(z)}{\varphi(z)}\right).$$

The desired conclusion follows. □

Definition 1.3. *Let $\alpha \in [0,1)$. A function $f \in C^2(\mathbb{D})$ with $f(0) = 0$ and $\frac{\partial}{\partial\theta}f(re^{i\theta}) \neq 0$, $0 < r < 1$, is called a* fully convex of order α, *denoted by $\mathcal{FC}(\alpha)$, if*

$$\frac{\partial}{\partial\theta}\left(\arg\frac{\partial}{\partial\theta}f(re^{i\theta})\right) = \mathrm{Re}\left(\frac{D^2f(z)}{Df(z)}\right) > \alpha$$

for $z = re^{i\theta} \in \mathbb{D}\backslash\{0\}$ (see also [9, 12] in order to distinguish convexity in the analytic and the harmonic cases), where

$$D^2f = z(Df)_z - \overline{z}(Df)_{\overline{z}}.$$

We see that

$$D^2f = zf_z(z) + \overline{z}f_{\overline{z}}(z) - 2|z|^2f_{z\overline{z}}(z) + z^2f_{zz}(z) + \overline{z}^2f_{\overline{z}\overline{z}}(z).$$

Set $\mathcal{FC}(0) =: \mathcal{FC}$, the class of fully convex *(univalent) on $\mathbb{D}$. In the analytic case, $\mathcal{FC}(\alpha)$ coincides with the class $C(\alpha)$ of convex functions of order α.*

In particular, we denote by $\mathcal{FC}_H^0(\alpha)$ and $\mathcal{FC}_{Lh}(\alpha)$ the set of all fully convex harmonic functions of order α and fully convex log-harmonic functions of order α, respectively. If $\alpha = 0$, we denote these classes by $\mathcal{FC}_H^0$ and $\mathcal{FC}_{Lh}$, the set of all fully convex harmonic functions and fully convex log-harmonic functions, respectively. Throughout the paper, we treat fully starlike (or convex) mappings as starlike (or convex) mappings although this is not the case in strict sense.

We remark that the function $f(z) = \varphi(z)|g(z)|^2$ is not necessarily convex in $\mathbb{D}$ even if $\varphi(z)$ is a complex-valued convex mapping in $\mathbb{D}$. For example, consider

$$\varphi(z) = \frac{z}{1-z}, \quad g(z) = \exp\left(\frac{1}{4}\log\left(\frac{1-z}{1+z}\right)\right)\exp\left(\frac{z}{2(1-z)}\right)$$

and therefore, by (1), f defined by

$$f(z) = \varphi(z)|g(z)|^2 = \frac{z}{1-z}\left|\frac{1-z}{1+z}\right|^{1/2}\exp\left(\mathrm{Re}\left(\frac{z}{1-z}\right)\right)$$

is a univalent log-harmonic mapping in $\mathbb{D}$ but is not convex in $\mathbb{D}$ although $\varphi(z) = z/(1-z)$ is convex in $\mathbb{D}$. The function f is indeed belongs to $\mathcal{FS}^*$, according to Proposition 1.2. The image of $\mathbb{D}$ under $f(z)$ is shown in Figure 1.

For an analytic proof of this fact, we need to show that

$$V(r,\theta) := \mathrm{Re}\left\{\frac{D^2f\left(re^{i\theta}\right)}{Df\left(re^{i\theta}\right)}\right\} < 0$$

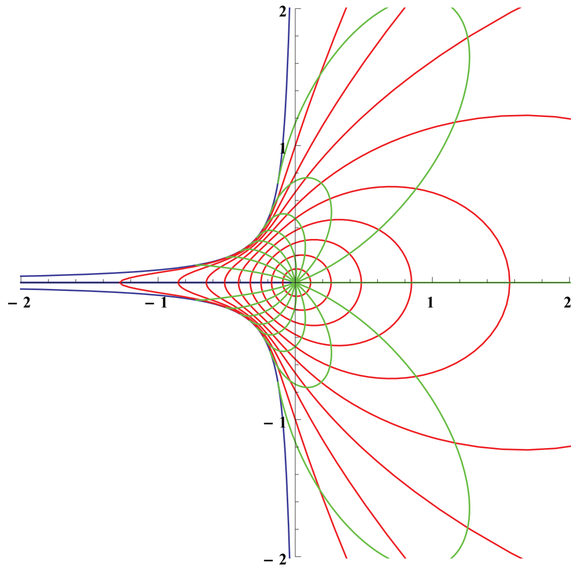


Figure 1: Image of $\mathbb{D}$ under $f(z)$

for some $0 < r < 1$ and $0 \le \theta < 2\pi$. Since $f = \varphi |g|^2$, by direct calculation, we find that

$$Df = z\varphi' g\overline{g} + z\varphi g'\overline{g} - \varphi g \overline{z}\overline{g'}$$

and

$$D^2 f = z\varphi' g\overline{g} + z^2\varphi'' g\overline{g} + 2z^2\varphi' g'\overline{g} + z\varphi g'\overline{g} + z^2\varphi g''\overline{g} - 2z\varphi' g\overline{z}\,\overline{g} - 2z\varphi g'\overline{z}\overline{g'} + \varphi g\overline{z}^2\overline{g''}.$$

Using this, we see that $V(r,\theta) < 0$ for some values of r and θ. To avoid the technical details, we compute $V(r,\theta)$ for $r = 8/9$, and $\theta = \pi/3,\ \pi/4,\ 2\pi/3$ with the help of *Mathematica* (see Table 1). According to this, we obtain that $f(z)$ is not convex in $\mathbb{D}$.

Table 1: $V(r,\theta)$ for values of r and θ

r	θ	$V(r,\theta)$
8/9	$\pi/4$	−0.284821
8/9	$\pi/3$	−0.447807
8/9	$2\pi/3$	−0.510244

However, with $g(z) = z^{p-1}$ $(p \ge 1)$, we can obtain a necessary and sufficient condition for the function f of the form $f(z) = \varphi(z)|z|^{2(p-1)}$ to be convex.

Theorem 1.4. *Let $f(z) = \varphi(z)|z|^{2(p-1)}$ $(p \ge 1)$. Then $f \in \mathcal{FC}(\alpha)$ if and only if $\varphi \in \mathcal{FC}(\alpha)$.*

Proof. We compute

$$zf_z(z) = |z|^{2(p-1)}\left[z\varphi_z(z) + (p-1)\varphi(z)\right]$$

and similarly,

$$\overline{z}f_{\overline{z}}(z) = |z|^{2(p-1)}\left[\overline{z}\varphi_{\overline{z}}(z) + (p-1)\varphi(z)\right].$$

Thus, we have

$$Df = zf_z - \overline{z}f_{\overline{z}} = |z|^{2(p-1)}(z\varphi_z - \overline{z}\varphi_{\overline{z}}),$$

which may be conveniently written as $F(z) = |z|^{2(p-1)}\Phi(z)$. Thus, to compute $D^2 f = D(Df) = DF$, we first find that

$$z\Phi_z(z) - \overline{z}\Phi_{\overline{z}}(z) = z\left(\varphi_z(z) + z\varphi_{zz}(z) - \overline{z}\varphi_{\overline{z}z}(z)\right) - \overline{z}\left(z\varphi_{z\overline{z}}(z) - \varphi_z(z) - \overline{z}\varphi_{\overline{z}\,\overline{z}}(z)\right),$$

so that

$$D^2 f = |z|^{2(p-1)} (z\Phi_z - \bar{z}\Phi_{\bar{z}}) = |z|^{2(p-1)} \left(z\varphi_z + \bar{z}\varphi_{\bar{z}} - 2|z|^2\varphi_{z\bar{z}} + z^2\varphi_{zz} + \bar{z}^2\varphi_{\bar{z}\bar{z}} \right).$$

Finally, we see that

$$\mathrm{Re}\left(\frac{D^2 f}{Df}\right) = \mathrm{Re}\left(\frac{z\varphi_z + \bar{z}\varphi_{\bar{z}} - 2|z|^2\varphi_{z\bar{z}} + z^2\varphi_{zz} + \bar{z}^2\varphi_{\bar{z}\bar{z}}}{z\varphi_z - \bar{z}\varphi_{\bar{z}}}\right) = \mathrm{Re}\left(\frac{D^2\varphi}{D\varphi}\right),$$

and this completes the proof. □

Example 1.5. *Set $\varphi(z) = z - \lambda|z|^2$, where $0 < |\lambda| \le 1/2$. It is easy to see that $\varphi(z)$ is log-harmonic mapping in $\mathbb{D}$, as a solution of (1) with the dilatation as*

$$\mu(z) = \frac{\bar{\lambda}z}{1 + \bar{\lambda}z}$$

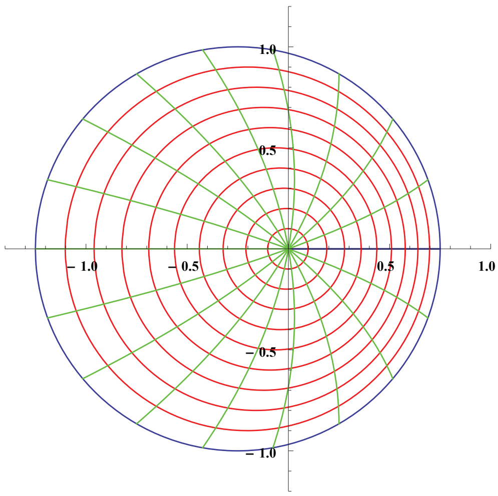
which is analytic in $\mathbb{D}$ and $|\mu(z)| < 1$ in $\mathbb{D}$ (as $0 < |\lambda| \le 1/2$). Simple calculation shows that

$$D\varphi(z) = z = D^2\varphi(z)$$

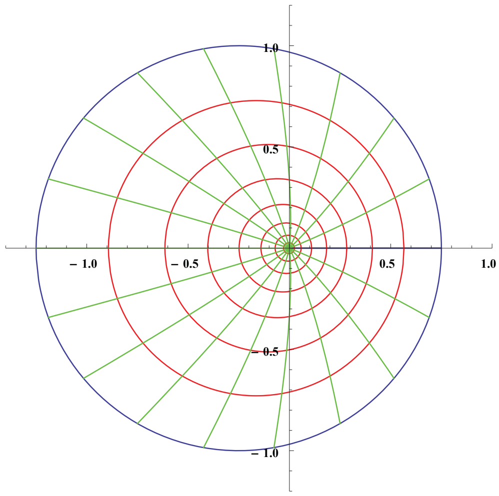
and, for $0 < |\lambda| \le 1/2$,

$$J_\varphi(z) = |1 - \lambda\bar{z}|^2 - |\lambda z|^2 = 1 - 2\,\mathrm{Re}(\lambda\bar{z}) > 1 - 2|\lambda| \ge 0.$$

Consequently, φ is convex and sense preserving in $\mathbb{D}$, and by Theorem 1.4, we find that $f(z) = \varphi(z)|z|^{2(p-1)}$ is log-p-*harmonic convex in $\mathbb{D}$. For more details about* log-p-*harmonic mappings, see [12] or [13]. The images of $\mathbb{D}$ under $\varphi(z)$ and $f(z) = \varphi(z)|z|^{2(p-1)}$ for $\lambda = 1/4$ and $p = 2$ are shown in Figure 2.*



(a) Graph of $\varphi(z) = z - \frac{1}{4}|z|^2$



(b) Graph of $f(z) = |z|^2(z - \frac{1}{4}|z|^2)$

Figure 2: Images of $\mathbb{D}$ under log-harmonic mapping $\varphi(z) = z - \lambda|z|^2$ and log-p-harmonic mapping $f_1(z) = \varphi(z)|z|^{2(p-1)}$ for $\lambda = 1/4$ and $p = 2$.

The rest of the paper is arranged as follows: In Section 2, we construct some interesting univalent log-harmonic mappings and show ranges of these mappings. In Section 3, we obtain sharp coefficient estimates for log-harmonic starlike mappings. In Section 4, we study the growth and distortion theorems for the special subclass of $\mathcal{S}_{Lh}$. Finally, in Section 5, we propose a couple of conjectures concerning coefficients and covering theorem for log-harmonic univalent mappings, as an analog of Bieberbach conjecture and the corresponding covering theorem.

2. Construction of univalent log-harmonic mappings

Clunie and Sheil-Small [10] established a method of constructing a harmonic mapping onto a domain convex in one direction by *"shearing"* a given conformal mapping convex in the same direction. In [2, 5], a method was introduced for constructing univalent log-harmonic mappings $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$ from the unit disk onto a strictly starlike domain Ω.

Following the method of shearing-construction by Clunie and Shell-Small, we now present a method of log-harmonic mappings with prescribed dilatation $\mu(z)$ with $\mu(0) = 0$.

Algorithm 2.1. *Let $f(z) = zh(z)\overline{g(z)}$ be a sense-preserving log-harmonic mapping, where h and g are non-vanishing analytic functions in $\mathbb{D}$, normalized by $h(0) = g(0) = 1$. Then the dilatation μ, defined by*

$$\mu(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} \cdot \frac{f(z)}{\overline{f(z)}} = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)}, \tag{3}$$

is analytic with $|\mu(z)| < 1$ in $\mathbb{D}$. According to the definition of log-harmonic mapping, the construction of log-harmonic mappings proceeds by letting $zh(z)/g(z) = \varphi(z)$, where φ is analytic satisfying $\varphi(0) = \varphi'(0) - 1 = 0$, and $\varphi(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. This gives the pair of nonlinear differential equations

$$\frac{zh(z)}{g(z)} = \varphi(z) \quad \textit{and} \quad \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = \mu(z), \tag{4}$$

which may be equivalently written as

$$z\,(\log h)'\,(z) - z\,(\log g)'\,(z) = \frac{z\varphi'(z)}{\varphi(z)} - 1 \quad \textit{and} \quad z\,(\log g)'\,(z) = \mu(z)\Big(1 + z\,(\log h)'\,(z)\Big).$$

Solving these two equations yield

$$(\log g)'\,(z) = \frac{\mu(z)}{1 - \mu(z)} \cdot \frac{\varphi'(z)}{\varphi(z)}.$$

Integrating with the normalization $g(0) = 1$, we arrive at

$$g(z) = \exp\left(\int_0^z \left(\frac{\mu(s)}{1 - \mu(s)} \cdot \frac{\varphi'(s)}{\varphi(s)}\right) ds\right). \tag{5}$$

In this way, we obtain the log-harmonic mapping f defined by

$$f(z) = zh(z)\overline{g(z)} = \frac{zh(z)}{g(z)}|g(z)|^2 = \varphi(z)\exp\left(2\,\mathrm{Re}\int_0^z \left(\frac{\mu(s)}{1 - \mu(s)} \cdot \frac{\varphi'(s)}{\varphi(s)}\right) ds\right). \tag{6}$$

For the construction of the univalent log-harmonic mappings f of the form $f(z) = zh(z)\overline{g(z)}$, the following steps may be used:

(a) Choose an arbitrary $\varphi \in \mathcal{S}^*$ and an arbitrary analytic function $\mu : \mathbb{D} \to \mathbb{D}$ with $\mu(0) = 0$.
(b) Establish the pair of equations given by (4).
(c) Solving for $(\log g)'\,(z)$, and then integrating with normalization $g(0) = 1$ yields (5).
(d) The desired univalent log-harmonic mapping $f \in \mathcal{S}_{Lh}$ is then given by (6).

Various choices of the conformal mapping φ and the dilatation μ produce a number of univalent log-harmonic mappings as demonstrated below.

Example 2.2. *Let $f_\alpha(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$ be a log-harmonic mapping in $\mathbb{D}$ and satisfy the pair of equations*

$$\varphi_\alpha(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad \mu(z) = \frac{zg'(z)/g(z)}{1+zh'(z)/h(z)} = z, \tag{7}$$

where $0 \le \alpha < 1$. The resulting nonlinear system

$$\begin{cases} z\,(\log h)'\,(z) - z\,(\log g)'\,(z) = \dfrac{2(1-\alpha)z}{1-z} \\ (\log g)'\,(z) - z\,(\log h)'\,(z) = 1 \end{cases}$$

has the solution

$$(\log g)'\,(z) = \frac{1-z+2(1-\alpha)z}{(1-z)^2},$$

which by integration gives

$$\log g(z) = 2(1-\alpha)\frac{z}{1-z} + (1-2\alpha)\log(1-z).$$

Thus, we have

$$\begin{cases} g(z) = (1-z)^{1-2\alpha}\exp\left(2(1-\alpha)\dfrac{z}{1-z}\right), \\ h(z) = \dfrac{g(z)}{(1-z)^{2(1-\alpha)}} = \dfrac{1}{1-z}\exp\left(2(1-\alpha)\dfrac{z}{1-z}\right), \end{cases} \tag{8}$$

and hence, $f_\alpha(z)$ has the form

$$\begin{aligned} f_\alpha(z) &= \varphi_a(z)|g(z)|^2 \\ &= \frac{z}{(1-z)^{2(1-\alpha)}}\left|(1-z)^{1-2\alpha}\exp\left(2(1-\alpha)\frac{z}{1-z}\right)\right|^2. \end{aligned} \tag{9}$$

Note that φ_α is starlike of order α. By Theorem A, we know that the log-harmonic mapping f_α is also starlike of order α in $\mathbb{D}$. Now we discuss two special cases: $\alpha = 0, 1/2$.

Case 1. *In the case of $\alpha = 0$, we have the Koebe function*

$$\varphi_0(z) = \frac{z}{(1-z)^2}$$

and thus (8) takes the form

$$\begin{cases} g(z) = (1-z)\exp\left(\dfrac{2z}{1-z}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)z^n\right), \\ h(z) = \dfrac{1}{1-z}\exp\left(\dfrac{2z}{1-z}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)z^n\right). \end{cases} \tag{10}$$

Finally (9) gives

$$f_0(z) = zh(z)\overline{g(z)} = \frac{z}{(1-z)^2}|1-z|^2\exp\left(\mathrm{Re}\left(\frac{4z}{1-z}\right)\right), \tag{11}$$

*which is the well-known univalent **log-harmonic Koebe function**. It is known that $f_0(z)$ maps $\mathbb{D}$ onto the slit plane $f_0(\mathbb{D}) = \mathbb{C}\backslash\{u+iv : u \le -1/e^2, v = 0\}$. This fact is known in [8], but the details were not given. For the sake of completeness, we include the details below for the benefit of readers.*

Set $z = e^{i\theta}$, $0 < \theta < 2\pi$. A straightforward but tedious calculations show that

$$\mathrm{Re}\left\{f_0(e^{i\theta})\right\} = -\frac{1}{e^2} \quad \textit{and} \quad \mathrm{Im}\left\{f_0(e^{i\theta})\right\} = 0,$$

so that $f_0(z) = -\frac{1}{e^2}$ on the unit circle $|z| = 1$ except at the point $z = 1$. Now, consider the Möbius transformation

$$w = \frac{1+z}{1-z} = u + iv,$$

which maps $\mathbb{D}$ onto the right half-plane $\mathrm{Re}\, w = u > 0$. Calculations show that

$$\begin{aligned} f_0(z) &= (w^2-1)\frac{1}{|w+1|^2}\exp\left(\mathrm{Re}\left(2(w-1)\right)\right) \\ &= \frac{u^2-v^2-1+i2uv}{(u+1)^2+v^2}\exp(2(u-1)), \quad u > 0. \end{aligned}$$

Now we observe that:

(a) *Each point z such that $|z| = 1$ ($z \neq 1$) is carried onto a point w on the imaginary axis so that $u = 0$ and $f_0(z) = -1/e^2$.*

(b) *If $uv = 0$, we observe that the positive real axis*

$$\{w = u + iv : \ u > 0, v = 0\}$$

is mapped monotonically onto the real interval $(-1/e^2, \infty)$.

(c) *Finally, each hyperbola $uv = c$, where c is a non-zero real constant, is carried univalently onto the set*

$$\left\{ w_1 = \frac{u^2 - \left(\frac{c}{u}\right)^2 - 1}{(u+1)^2}\exp(2(u-1)) + i2c\exp(2(u-1)) : u > 0 \right\},$$

which is the entire line $\{w_1 = u_1 + iv_1 : \ -\infty < u_1 < \infty\}$.

Thus, the log-harmonic Koebe mapping $f_0(z)$ is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the entire plane minus the real interval $(-\infty, -1/e^2]$. Figure 3 shows that the image of $\mathbb{D}$ under Koebe mapping and log-harmonic Koebe mapping.

Case 2. *For $\alpha = 1/2$, by (7), we have*

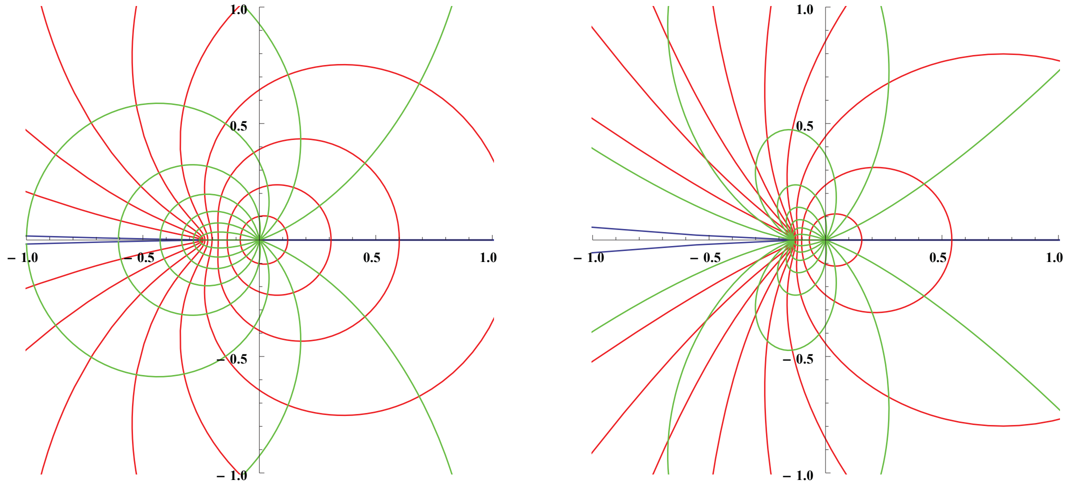
$$\varphi_{\frac{1}{2}}(z) = \frac{z}{1-z} = l(z)$$

and from (8), we obtain

$$\begin{cases} g(z) = \exp\left(\dfrac{z}{1-z}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty} z^n\right), \\ h(z) = \dfrac{1}{1-z}\exp\left(\dfrac{z}{1-z}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)z^n\right). \end{cases}$$

*From (9), we obtain the univalent **log-harmonic right half-plane mapping***

$$f_{\frac{1}{2}}(z) = \varphi(z)|g(z)|^2 = \frac{z}{1-z}\exp\left(\mathrm{Re}\left(\frac{2z}{1-z}\right)\right). \tag{12}$$



(a) The Koebe function $\frac{z}{(1-z)^2}$ (b) The log-harmonic Koebe mapping $f_0(z)$

Figure 3: Images of $\mathbb{D}$ under Koebe mapping and log-harmonic Koebe mapping

We claim now that $f_{\frac{1}{2}}(\mathbb{D}) = \left\{w : \operatorname{Re} w > -\frac{1}{2e}\right\}$. *In order to prove this fact, we set* $z = e^{i\theta}$ $(0 < \theta < 2\pi)$ *into* (12). *Then a straightforward calculation shows that*

$$\operatorname{Re}\left\{f_{\frac{1}{2}}(e^{i\theta})\right\} = -\frac{1}{2e} \quad \textit{and} \quad \operatorname{Im}\left\{f_{\frac{1}{2}}(e^{i\theta})\right\} = \frac{1}{2e}\cot\frac{\theta}{2} \in \mathbb{R},$$

so that $f_{\frac{1}{2}}(z)$ *maps the unit circle* $|z| = 1$ $(z \neq 1)$ *onto the line* $u = -\frac{1}{2e}$. *With* $\zeta = \frac{z}{1-z} = a + ib$, *where* $a > -1/2$, $-\infty < b < \infty$ *for* $z \in \mathbb{D}$, *the log-harmonic mapping* $f_{\frac{1}{2}}(z)$ *takes the form*

$$f_{\frac{1}{2}}(z) = (a + ib)\exp(2a), \quad z = l^{-1}(\zeta) = \frac{\zeta}{1+\zeta}.$$

This shows that $f_{\frac{1}{2}} \circ l^{-1}$ *maps each vertical line*

$$\zeta = a_0 + ib, \quad a_0 > -1/2, \, -\infty < b < \infty,$$

monotonically onto

$$\{w = u + iv : \, u = a_0 \exp(2a_0) > -\frac{1}{2e}, \, -\infty < v = b\exp(2a_0) < \infty\},$$

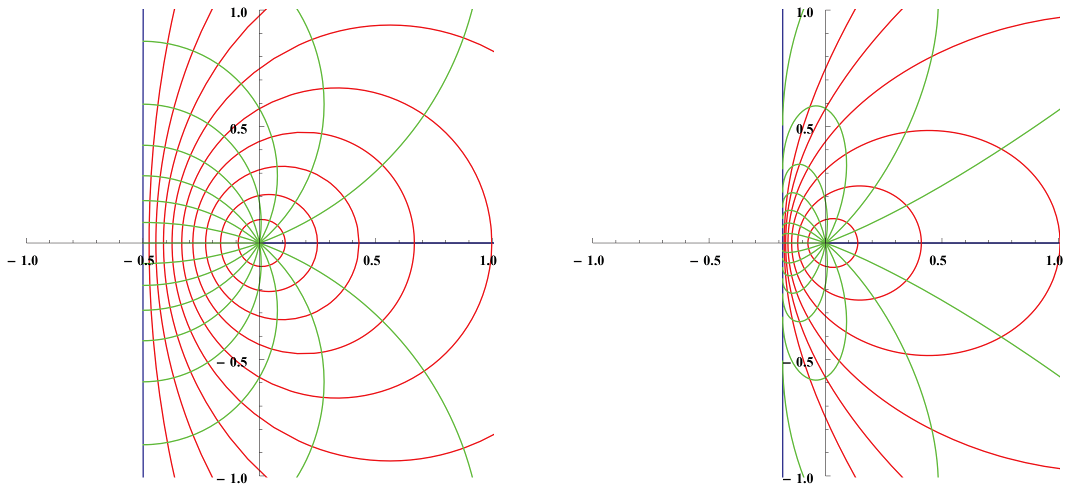
where these lines correspond to circles in the unit disk. This shows that the mapping $w = f_{\frac{1}{2}}(z)$ *sends* $\mathbb{D}$ *univalently onto the right half-plane* $\operatorname{Re} w > -\frac{1}{2e}$. *The images under* $f_{\frac{1}{2}}(z)$ *of concentric circles and radial segments are shown in Figure 4(b). For a comparison we include the images of analytic right half-plane mapping and the log-harmonic right half-plane mapping. See Figure 4(a).*

Example 2.3. (**Log-harmonic two-slits mapping**) *We know that* $s(z) = z/(1-z^2)$ *maps* $\mathbb{D}$ *onto* $\mathbb{C}\backslash\{u + iv : \, u = 0, |v| \geq 1/2\}$. *Let us now construct a log-harmonic two-slits mapping. Consider* $LS(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$ *with*

$$\varphi(z) = \frac{zh(z)}{g(z)} = s(z) = \frac{z}{1-z^2} \quad \text{and} \quad \mu(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = z^2.$$

Thus, as before, we have

$$z\,(\log h)'\,(z) - z\,(\log g)'\,(z) = \frac{2z^2}{1-z^2} \quad \text{and} \quad (\log g)'\,(z) - z^2\,(\log h)'\,(z) = z.$$



(a) The right half-plane mapping $\frac{z}{1-z}$ (b) The log-harmonic right half-plane mapping $f_{\frac{1}{2}}(z)$

Figure 4: Images of $\mathbb{D}$ under right half-plane mapping and log-harmonic right half-plane mapping

Solving for the solution yields

$$(\log g)'(z) = \frac{z(1+z^2)}{(1-z^2)^2}.$$

Integrating with the normalization $h(0) = g(0) = 1$, *we arrive at*

$$\begin{cases} g(z) = \sqrt{1-z^2}\exp\left(\dfrac{z^2}{1-z^2}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)z^{2n}\right), \\ h(z) = \dfrac{g(z)}{1-z^2} = \dfrac{1}{\sqrt{1-z^2}}\exp\left(\dfrac{z^2}{1-z^2}\right) = \exp\left(\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)z^{2n}\right), \end{cases}$$

and thus,

$$LS(z) = \varphi(z)|g(z)|^2 = \frac{z}{1-z^2}|1-z^2|\exp\left(\mathrm{Re}\left(\frac{2z^2}{1-z^2}\right)\right). \tag{13}$$

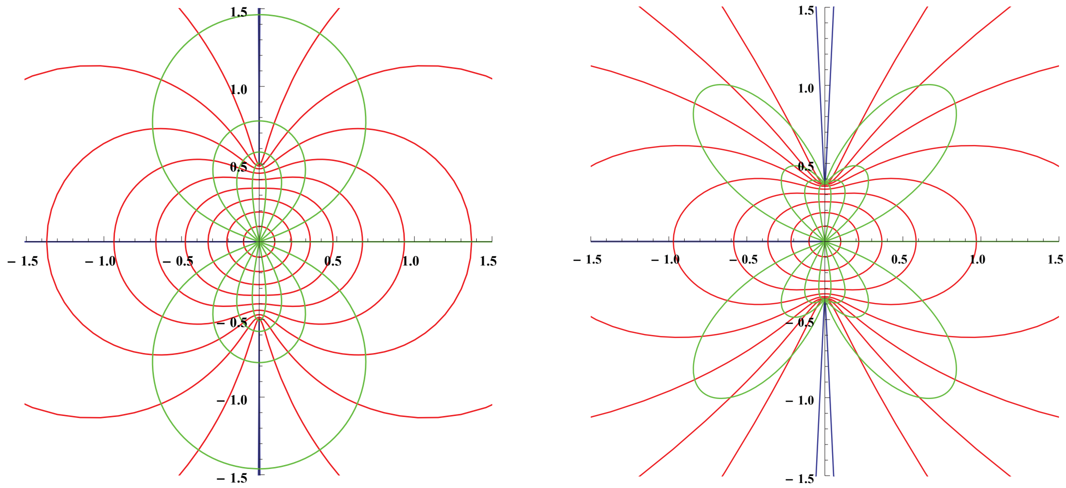
By Theorem A, we know that $LS(z)$ *is univalent and starlike in* $\mathbb{D}$. *We now claim that* $LS(z)$ *maps* $\mathbb{D}$ *onto the two-slits plane* $\mathbb{C}\backslash\{u+iv : |v| \geq 1/e\}$. *The images of* $\mathbb{D}$ *under* $s(z) = z/(1-z^2)$ *and log-harmonic two-slits mapping* $LS(z)$ *are shown in Figures 5(a) and 5(b), respectively.*

In order to prove that $LS(\mathbb{D}) = \mathbb{C}\backslash\{u+iv : |v| \geq 1/e, u = 0\}$, *set* $z = e^{i\theta}$ $(\theta \in (0,\pi)\cup(\pi,2\pi))$ *into* (13). *We have*

$$LS(e^{i\theta}) = i\frac{\sin\theta}{|\sin\theta|}e^{-1} = \begin{cases} i/e & \text{for} \quad 0 < \theta < \pi, \\ -i/e & \text{for} \quad \pi < \theta < 2\pi, \end{cases}$$

which shows that $LS(z) = \pm i/e$ *on the unit circle except at the points* $z = \pm 1$. *The argument similar to the analysis of Example 2.2 gives the desired claim and we omit the details.*

Remark 2.4. *The above three univalent log-harmonic mappings play the role of extremal functions for many extremal problems over the subclasses of* $\mathcal{S}_{Lh}$.



(a) The two-slits function $\frac{z}{1-z^2}$ (b) The log-harmonic two-slits function $LS(z)$

Figure 5: Images of $\mathbb{D}$ under $s(z)$ and $LS(z)$

3. Coefficients estimate for log-harmonic starlike mappings

Let $s_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $s_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is *subordinate* to $s_2(z)$ (written by $s_1(z) \prec s_2(z)$ or simply by $s_1 \prec s_2$) if

$$s_1(z) = s_2(\omega(z))$$

for some analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$. Then, by the Schwarz lemma, $|\omega(z)| \leq |z|$ and $|\omega'(0)| \leq 1$ so that $|s_1'(0)| \leq |s_2'(0)|$. For additional details on subordination classes, see for example [11, Chapter 6] or [16, p. 35].

Lemma 3.1. ([11, Theorem 6.4]) *If $s_1(z) \prec s_2(z)$, where $s_i(0) = 0$ and $s_i'(0) = 1$ $(i = 1, 2)$, and*

(a) *if $s_2 \in \mathcal{C}$, then $|a_n| \leq 1$ for $n = 2, 3, \cdots$;*
(b) *if $s_2 \in \mathcal{S}^*$, then $|a_n| \leq n$ for $n = 2, 3, \cdots$.*

Theorem 3.2. *Let $f(z) = zh(z)\overline{g(z)}$ belong to $\mathcal{S}^*_{Lh}(\alpha)$ $(0 \leq \alpha < 1)$, where $h(z)$ and $g(z)$ are given by* (2). *Then for all $n \geq 1$,*

$$|a_n - b_n| \leq \frac{2(1-\alpha)}{n}. \tag{14}$$

Moreover,

(a) $|a_n| \leq 2(1-\alpha) + \frac{1}{n}$;
(b) $|b_n| \leq 2(1-\alpha) + \frac{2\alpha-1}{n}$.

Equality holds if $f(z) = f_\alpha(z)$ or one of its rotation, where $f_\alpha(z)$ is given by (9).

Proof. Let $f(z) = zh(z)\overline{g(z)}$ belong to $\mathcal{S}^*_{Lh}(\alpha)$. Then we have

$$\alpha < \mathrm{Re}\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) = \mathrm{Re}\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right), \quad z \in \mathbb{D}.$$

By (2) and Theorem A, we obtain

$$1+\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}\prec\frac{1+(1-2\alpha)z}{1-z},\quad z\in\mathbb{D},\tag{15}$$

which is equivalent to

$$\frac{1}{2(1-\alpha)}\left(\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}\right)=\sum_{n=1}^{\infty}\frac{n(a_n-b_n)}{2(1-\alpha)}z^n\prec\frac{z}{1-z},\quad z\in\mathbb{D}.$$

Also, since $z/(1-z)$ is convex in $\mathbb{D}$, by Lemma 3.1, we get

$$\frac{n\,|a_n-b_n|}{2(1-\alpha)}\leq 1\quad\text{for}\quad n\geq 1,$$

and (14) follows.

The coefficient estimate inequalities (a) and (b) of Theorem 3.2 were obtained in [14, Theorem 2.6]. Finally, it is evident that the equalities are attained by a suitable rotation of $h(z)$ and $g(z)$ given by (9). The proof is complete. □

For $\alpha = 0$, one obtains the coefficients estimate of the starlike log-harmonic mappings $f\in\mathcal{S}^*_{Lh}$, $f(z) = zh(z)\overline{g(z)}$.

Theorem 3.3. *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping in $\mathbb{D}$, where $h(z)$ and $g(z)$ are given by* (2)*, and satisfy the condition*

$$\sum_{n=1}^{\infty}n\,|a_n-b_n|\leq 1-\alpha\tag{16}$$

for some $\alpha\in[0,1)$. Then $f\in\mathcal{S}^*_{Lh}(\alpha)$.

Proof. By using the series representation of $h(z)$ and $g(z)$ given by (2), we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}\right)&=\mathrm{Re}\left(1+\sum_{n=1}^{\infty}n(a_n-b_n)z^n\right)\\&\geq 1-\left|\sum_{n=1}^{\infty}n(a_n-b_n)z^n\right|\\&>1-\sum_{n=1}^{\infty}n\,|a_n-b_n|\geq\alpha.\end{aligned}$$

By (16), the desired conclusion follows. □

In particular, $\alpha = 0$ in Theorem 3.3 provides a sufficient coefficient condition for the log-harmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ to be starlike in $\mathbb{D}$.

4. Growth and Distortion Theorem

In this section, we introduce the subclass C_{Lh} of $\mathcal{S}_{Lh}$, which yields sharp growth and distortion estimates, where C_{Lh} is defined by

$$C_{Lh}=\left\{f\in\mathcal{S}_{Lh}:\ f(z)=zh(z)\overline{g(z)},\ \frac{zh(z)}{g(z)}=\frac{z}{1-z}\right\}.$$

The following two theorems are the growth theorem and distortion theorem for the class C_{Lh}. We remark that Theorem 4.1 below follows from [1, Theorem 3.1] and [14, Theorem 2.4] because of the fact that C_{Lh} is the subclass of $\mathcal{S}^*_{Lh}(\alpha)$ for $\alpha = 1/2$.

Theorem 4.1. *Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{C}_{Lh}$. Then for $z \in \mathbb{D}$ we have*

(a) $\frac{1}{1+|z|}\exp\left(\frac{-|z|}{1+|z|}\right) \le |h(z)| \le \frac{1}{1-|z|}\exp\left(\frac{|z|}{1-|z|}\right)$;

(b) $\exp\left(\frac{-|z|}{1+|z|}\right) \le |g(z)| \le \exp\left(\frac{|z|}{1-|z|}\right)$;

(c) $\frac{|z|}{1+|z|}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |f(z)| \le \frac{|z|}{1-|z|}\exp\left(\frac{2|z|}{1-|z|}\right)$.

The equalities occur if and only if $f(z)$ is of the form $\overline{\eta} f_{\frac{1}{2}}(\eta z)$, $|\eta| = 1$, where $f_{\frac{1}{2}}(z)$ is given by (12).

Theorem 4.2. *Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{C}_{Lh}$. Then for $z \in \mathbb{D}$ we have*

(a) $\frac{1}{(1+|z|)^3}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |f_z(z)| \le \frac{1}{(1-|z|)^3}\exp\left(\frac{2|z|}{1-|z|}\right)$;

(b) $\frac{|z|}{(1+|z|)^3}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |f_{\overline{z}}(z)| \le \frac{|z|}{(1-|z|)^3}\exp\left(\frac{2|z|}{1-|z|}\right)$;

(c) $\frac{|z|(1-|z|)}{(1+|z|)^3}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |Df(z)| \le \frac{|z|(1+|z|)}{(1-|z|)^3}\exp\left(\frac{2|z|}{1-|z|}\right)$.

The equalities occur if and only if $f(z)$ is of the form $\overline{\eta} f_{\frac{1}{2}}(\eta z)$, $|\eta| = 1$, where $f_{\frac{1}{2}}(z)$ is given by (12).

Proof. Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{C}_{Lh}$. In view of (12), we find that

$$f_z(z) = (h(z) + zh'(z))\overline{g(z)} = \left(1 + \frac{zh'(z)}{h(z)}\right)\frac{h(z)}{g(z)}|g(z)|^2.$$

Now the relations

$$\frac{h(z)}{g(z)} = \frac{1}{1-z} \quad \text{and} \quad 1 - \mu(z) = \frac{1 + zh'(z)/h(z) - zg'(z)/g(z)}{1 + zh'(z)/h(z)}$$

give

$$1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = \frac{1}{1-z}$$

and

$$1 + \frac{zh'(z)}{h(z)} = \frac{1}{(1-\mu(z))(1-z)},$$

so that $f_z(z)$ takes the form

$$f_z(z) = \frac{1}{(1-\mu(z))(1-z)^2}|g(z)|^2,$$

where $\mu \in \mathcal{B}$ with $\mu(0) = 0$. Similarly, we see that

$$f_{\overline{z}}(z) = zh(z)\overline{g'(z)} = \overline{\left(\frac{\mu(z)/z}{(1-\mu(z))(1-z)}\right)} \cdot \frac{z}{1-z}|g(z)|^2.$$

For $|z| = r$, we have

$$\left|\frac{1}{1-\mu(z)}\right| \le \frac{1}{1-r} \quad \text{and} \quad \left|\frac{\mu(z)/z}{1-\mu(z)}\right| \le \frac{1}{1-r}$$

by (a) and (b) in Theorem 4.1, so that

$$\frac{1}{(1+r)^3}\exp\left(\frac{-2r}{1+r}\right) \le |f_z(z)| \le \frac{1}{(1-r)^3}\exp\left(\frac{2r}{1-r}\right)$$

and

$$\frac{r}{(1+r)^3}\exp\left(\frac{-2r}{1+r}\right) \le |f_{\overline{z}}(z)| \le \frac{r}{(1-r)^3}\exp\left(\frac{2r}{1-r}\right),$$

which prove (a) and (b) of Theorem 4.2. Using these two inequalities, we obtain that

$$|Df(z)| = |zf_z(z) - \overline{z}f_{\overline{z}}(z)| \leq |z|\left(|f_z(z)| + |f_{\overline{z}}(z)|\right) \leq \frac{r(1+r)}{(1-r)^3}\exp\left(\frac{2r}{1-r}\right)$$

and

$$|Df(z)| = |zf_z(z) - \overline{z}f_{\overline{z}}(z)| \geq |z|\left(|f_z(z)| - |f_{\overline{z}}(z)|\right) \geq \frac{r(1-r)}{(1+r)^3}\exp\left(\frac{-2r}{1+r}\right),$$

which proves (3) of Theorem 4.2.

Finally, equalities occur if and only if $\mu(z) = \eta z$, $|\eta| = 1$ which leads to $f(z) = \overline{\eta} f_{\frac{1}{2}}(\eta z)$. □

5. Open problems

The function f_0 given by (11) plays the role of the Koebe mapping in the set of log-harmonic mappings (see also [4]). As an analog of analytic and harmonic Bieberbach conjectures, it is natural to propose the following:

Conjecture 5.1. (**Log-harmonic Coefficient Conjecture**) *Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$, where h and g are given by* (2). *Then for all $n \geq 1$,*

(a) $|a_n| \leq 2 + \frac{1}{n}$;
(b) $|b_n| \leq 2 - \frac{1}{n}$;
(c) $|a_n - b_n| \leq \frac{2}{n}$.

We remark that if (b) and (c) hold in Conjecture 5.1, then, by (c), we can obtain

$$|a_n| \leq \frac{2}{n} + |a_n| \leq \frac{2}{n} + 2 - \frac{1}{n} = 2 + \frac{1}{n}.$$

Thus, it suffices to prove (b) and (c). It is worth pointing out that Conjecture 5.1 is true for starlike log-harmonic mappings, see [6, Theorem 3.3] and the case $\alpha = 0$ of Theorem 3.2.

Conjecture 5.2. (**Log-harmonic $1/e^2$-Covering Conjecture**) *For $f \in \mathcal{S}_{Lh}$, we conjecture that $f(\mathbb{D})$ covers the disk $\{w \in \mathbb{C} : |w| < 1/e^2\}$.*

For log-harmonic Koebe mapping $f_0(z)$ defined by (11), the constant $1/e^2$ (see Example 2.2) cannot be improved. In [4], it was shown that the range of $f \in \mathcal{S}_{Lh}$ covers the disk $\{w \in \mathbb{C} : |w| < 1/16\}$.

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