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New Formulae of the Drazin Inverse of Anti-Triangular Complex Block Matrices

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Abstract. Let $E,F\in\mathbb{C}^{n\times n}$. If $EF^iE=0$ for all $i\in\mathbb{N}$, we give the explicit representation of the Drazin inverse of the block complex matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. We thereby solve a wider kind of singular differential equations posed by Campbell [S.L. Campbell, The Drazin inverse and systems of second order linear differential equations, Linear & Multilinear Algebra, 14(1983), 195–198].

1. Introduction

Let A be a $n\times n$ complex matrix. The Drazin inverse of A is the unique $n\times n$ matrix A^D satisfying the following equations

$$AA^D=A^DA, A^D=A^DAA^D, A^k=A^{k+1}A$$

for some $k\in\mathbb{N}$. The Drazin index $i(A)$ is the smallest $k\in\mathbb{N}$ satisfying $rank(A^k)=rank(A^{k+1})$. The Drazin inverse was extensively applied to many fields, such as Markov chains, differential equations, cryptography, control theory, etc (see [1–3, 7–9, 12, 13]).

Let E,F be $n\times n$ complex matrices and I be the identity matrix, and let $M=\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. As is well known, the solutions to singular systems of differential equations is determined by the representation of the Drazin inverse of the preceding matrix M (see [1, 2]). In [1, Theorem 3.2], Bu et al. investigated the Drazin inverse of the preceding matrix M under the condition $EF=FE$. Alteratively, Xu et al. obtained a new expression of the Drazin inverse M^D under the same condition (see [10, Theorem 3.8]. In [4, Theorem 2.2], Cvetković-Ilić considered the Drazin inverse of M under the condition $EFE=0, EF^2=0$. We refer the reader [7, 10, 11, 14] for much progress made in the representation of the block complex matrix M.

The motivation of this paper is to further study the representation of the Drazin inverse of this complex matrix M. If $EF^iE=0$ for all $i\in\mathbb{N}$, we shall give the explicit representation of the Drazin inverse of the block

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complex matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. Evidently, this provides a new class of singular differential equations which is solved.

Throughout the paper, $\mathbb{C}$ stands for the field of all complex numbers. Let $\mathbb{C}^{n\times n}$ be the algebra of all $n\times n$ complex matrices. $\mathbb{N}$ denotes the set of all natural numbers. Let A be a 2×2 block complex matrix. We always write it by (A_{ij}) where $i,j=1,2$.

2. Main results

We begin with the following elementary result.

Lemma 2.1. *Let*

$$M=\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \text{ or } \begin{pmatrix} B & C \\ 0 & A \end{pmatrix}\in \mathbb{C}^{n\times n}$$

Then

$$M^D=\begin{pmatrix} A^D & 0 \\ Z & B^D \end{pmatrix}, \text{ or } \begin{pmatrix} B^D & Z \\ 0 & A^D \end{pmatrix},$$

where

$$Z=\sum_{i=0}^{m}(B^D)^{i+2}CA^iA^{\pi}+\sum_{i=0}^{m}B^iB^{\pi}C(A^d)^{i+2}-B^DCA^D,$$

and $m\geq i(A)+i(B)-1$.

Proof. For any $s\geq i(A)$ and $t\geq i(B)$, we have $A^sA^{\pi}=0$ and $B^tB^{\pi}=0$. The result follows by [5, Lemma 1.1]. □

Lemma 2.2. *Let* $P,Q\in\mathbb{C}^{n\times n}$. *If* $PQ^iP=0$ *for* $i=1,2,\cdots,n$, *then*

$$\begin{aligned}(P+Q)^D &= (I+U)P^D+Q^D(I+U)+P^DVQ+Q^DWQ\\ &+VQQ^D+WQQ^D+UP^DVQ+\sum_{i=0}^{m}UP^iP^{\pi}(Q^D)^{i+1},\end{aligned}$$

where

$$\begin{aligned}U &= \sum_{i=0}^{m}(Q^D)^{i+1}P^{i+1}P^{\pi}+\sum_{i=0}^{m}Q^{i+1}Q^{\pi}(P^D)^{i+1}-QQ^DPP^D,\\ V &= \sum_{i=0}^{m}(P^D)^{i+1}Q^iQ^{\pi}+\sum_{i=0}^{m}P^{i+1}P^{\pi}(Q^D)^{i+2}-PP^DQ^D,\\ W &= UV+\sum_{i+l+j=m-2}\Big[(Q^D)^{k-i}P^{l+1}Q^jQ^{\pi}+Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{k+1-j}\Big]\\ &- \sum_{i=0}^{m}\Big[(Q^D)^{i+1}P^{i+1}(V+Q^D)+(U+QQ^D)P^{i+1}(Q^D)^{i+2}\Big],\end{aligned}$$

and $m\geq i(P)+2i(Q)$.

Proof. This is obvious by [6, Theorem 2.3.1]. □

We now come to the main result of this paper.

Theorem 2.3. *Let* $E,F\in\mathbb{C}^{n\times n}$ *and* $M=\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. *If* $FE^iF=0$ *for all* $i\in\mathbb{N}$, *then*

$$M^D=\begin{pmatrix} E\Gamma+\Lambda & E\Delta+\Xi \\ F\Gamma & F\Delta \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= F^D+(E^D)^2+\sum_{i=0}^{m}[E^{2i+2}E^{\pi}(F^D)^{i+2}+E^{2i+1}E^{\pi}(F^D)^{i+2}E]-EE^DF^D\\
&- E^DF^DE+\sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E]-(E^D)^2FF^D\\
&- (E^D)^3FF^DE+\sum_{i=0}^{m}(F^D)^{i+2}E^{2i+2}E^{\pi}-F^DEE^D\\
&+ \sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+4}-FF^D(E^D)^2+\eta+\kappa+\phi+\chi,\\
\Delta &= (E^D)^3+\sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+2}-E^DF^D+\sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1}\\
&- (E^D)^3FF^D+\sum_{i=0}^{m}\Big[(F^D)^{i+2}E^{2i+1}-(F^D)^{i+2}E^{2i+2}E^D\Big]-F^DE^D\\
&+ \sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+5}-FF^D(E^D)^3+\theta+\varsigma+\varphi+\omega,\\
\Lambda &= F^DE+\sum_{i=0}^{m}(F^D)^{i+2}E^{2i+3}E^{\pi}-F^DE^2E^D\\
&+ \sum_{i=0}^{m}\Big[F^{i+1}-F^{i+2}F^D\Big](E^D)^{2i+3}-FF^DE^D,\\
\Xi &= F^D+\sum_{i=0}^{m}\Big[(F^D)^{i+2}E^{2i+2}-(F^D)^{i+2}E^{2i+3}E^D\Big]-F^DEE^D\\
&+ \sum_{i=0}^{m}\Big[F^{i+1}-F^{i+2}F^D\Big](E^D)^{2i+4}-FF^D(E^D)^2;
\end{aligned}$$

$$\begin{aligned}
\eta &= \sum_{i+l+j=m-1}(Q^D)^{m+2-i}\Big[(E^D)^{2m-2i+4}F^{l+1}E^{2j+2}E^{\pi}+(E^D)^{2m-2i+5}F^{l+1}E^{2j+3}E^{\pi}\Big]\\
&- \sum_{i=0}^{m}\Big[\gamma_i+\varepsilon_i+(E^D)^{2i+4}F^{i+1}EE^D+(E^D)^{2i+5}F^{i+1}E^2E^D+(E^D)^2F^{i+1}(E^D)^{2i+2}\\
&+ (E^D)^3F^{i+1}(E^D)^{2i+1}\Big]+\alpha,\\
\theta &= \sum_{i+l+j=m-1}(Q^D)^{m+2-i}\Big[(E^D)^{2m-2i+4}F^{l+1}E^{2j+1}E^{\pi}+(E^D)^{2m-2i+5}F^{l+1}E^{2j+2}E^{\pi}\Big]\\
&- \sum_{i=0}^{m}\Big[\delta_i+\zeta_i+(E^D)^{2i+4}F^{i+1}E^D+(E^D)^{2i+5}F^{i+1}EE^D+(E^D)^2F^{i+1}(E^D)^{2i+3}\\
&+ (E^D)^3F^{i+1}(E^D)^{2i+2}\Big]+\beta;
\end{aligned}$$

$$\begin{aligned}
\alpha &= \Big[\sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E]-(E^D)^2FF^D-(E^D)^3FF^DE\Big]\\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big]\\
&+ \Big[\sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1}-(E^D)^3FF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+3}E^{\pi}+F^{i+1}(E^D)^{2i+1}\\
&- F^{i+2}F^D(E^D)^{2i+1}\Big)-FF^DE^2E^D\Big],\\
\beta &= \Big[\sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E]-(E^D)^2FF^D-(E^D)^3FF^DE\Big]\\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big]\\
&+ \Big[\sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1}-(E^D)^3FF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}-(F^D)^{i+1}E^{2i+3}E^D\\
&+ F^{i+1}(E^D)^{2i+2}-F^{i+2}F^D(E^D)^{2i+2}\Big)-FF^DEE^D\Big];
\end{aligned}$$

$$\begin{aligned}
\kappa &= \sum_{i+l+j=m-1}\Big[E^{2i+2}E^{\pi}F^{l+1}(E^D)^{2m+4-2j}+(E^{2i+1}-E^{2i+2}E^D)F^{l+1}(E^D)^{2m+3-2j}\Big]\\
&- \sum_{i=0}^{m}\Big[\sigma_i+\lambda F^{i+1}(E^D)^{2i+4}+\mu F^{i+1}(E^D)^{2i+3}+(E^D)^{2i+2}F^{i+1}(E^D)^2\\
&+ (E^D)^{2i+3}F^{i+1}E^D+EE^DF^{i+1}(E^D)^{2i+4}+E^DF^{i+1}(E^D)^{2i+3}\Big]+\xi,\\
\varsigma &= \sum_{i+l+j=m-1}\Big[E^{2i+2}E^{\pi}F^{l+1}(E^D)^{2m+5-2j}+(E^{2i+1}-E^{2i+2}E^D)F^{l+1}(E^D)^{2m+4-2j}\Big]\\
&- \sum_{i=0}^{m}\Big[\tau_i+\lambda F^{i+1}(E^D)^{2i+5}+\mu F^{i+1}(E^D)^{2i+4}+(E^D)^{2i+2}F^{i+1}(E^D)^3\\
&+ (E^D)^{2i+3}F^{i+1}(E^D)^2+EE^DF^{i+1}(E^D)^{2i+5}+E^DF^{i+1}(E^D)^{2i+4}+\rho;
\end{aligned}$$

$$\begin{aligned}
\gamma_i &= (E^D)^{2i+4}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+2}E^{\pi}+F^{j+1}F^{\pi}(E^D)^{2j+2}\Big)-FF^DEE^D\Big]\\
&+ (E^D)^{2i+5}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+3}E^{\pi}+F^{j+1}(E^D)^{2j+1}-F^{j+2}F^D(E^D)^{2j+1}\Big)\\
&- FF^DE^2E^D\Big],\\
\delta_i &= (E^D)^{2i+4}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+1}-(F^D)^{j+1}E^{2j+2}E^D+F^{j+1}F^{\pi}(E^D)^{2j+3}\Big)\\
&- FF^DE^D\Big]+(E^D)^{2i+5}F^{i+1}\Big[\sum_{j=0}^{m}\Big[(F^D)^{j+1}E^{2j+2}-(F^D)^{j+1}E^{2j+3}E^D\\
&+ F^{j+1}(E^D)^{2j+2}-F^{j+2}F^D(E^D)^{2j+2}\Big)-FF^DEE^D\Big];
\end{aligned}$$

$$\begin{aligned}
\varepsilon_i &= \Big[\sum_{j=0}^{m}(E^D)^{2j+4}F^j-(E^D)^2F^D\Big]F^{i+2}(E^D)^{2i+2}\\
&+ \Big[\sum_{j=0}^{m}(E^D)^{2j+5}F^j-(E^D)^3F^D\Big]F^{i+2}(E^D)^{2i+1},\\
\zeta_i &= \Big[\sum_{j=0}^{m}(E^D)^{2j+4}F^j-(E^D)^2F^D\Big]F^{i+2}(E^D)^{2i+3}\\
&+ \Big[\sum_{j=0}^{m}(E^D)^{2j+5}F^j-(E^D)^3F^D\Big]F^{i+2}(E^D)^{2i+2};
\end{aligned}$$

$$\begin{aligned}
\xi &= \lambda\Big[\sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+4}-FF^D(E^D)^2\Big]\\
&+ \mu\Big[\sum_{i=0}^{m}\Big(F^{i+1}-F^{i+2}F^D\Big)(E^D)^{2i+3}-FF^DE^D\Big]\\
\rho &= \lambda\Big[\sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+5}-FF^D(E^D)^3\Big]\\
&+ \mu\Big[\sum_{i=0}^{m}\Big(F^{i+1}-F^{i+2}F^D\Big)(E^D)^{2i+4}-FF^D(E^D)^2\Big];
\end{aligned}$$

$$\begin{aligned}
\lambda &= \sum_{i=0}^{m}\Big[(E^D)^{2i+2}F^{i+1}F^{\pi}+(E^D)^{2i+3}F^{i+1}E-(E^D)^{2i+3}F^{i+2}F^DE\Big]\\
&+ \sum_{i=0}^{m}E^{2i+2}E^{\pi}(F^D)^{i+1}-EE^DFF^D+\sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1}E-E^DFF^DE,\\
\mu &= \sum_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}+\sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1}-E^DFF^D;
\end{aligned}$$

$$\begin{aligned}
\phi &= \Big[\sum_{i=0}^{m}\Big(E^{2i+2}E^{\pi}(F^D)^{i+2}+E^{2i+1}E^{\pi}(F^D)^{i+2}E\Big)-EE^DF^D-E^DF^DE\Big] \\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}E^{\pi}+F^{i+1}F^{\pi}(E^D)^{2i+2}\Big)-FF^DEE^D\Big] \\
&+ \Big[\sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+2}-E^DF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+3}E^{\pi}+F^{i+1}(E^D)^{2i+1} \\
&- F^{i+2}F^D(E^D)^{2i+1}\Big)-FF^DE^2E^D\Big], \\
\varphi &= \Big[\sum_{i=0}^{m}(E^{2i+2}E^{\pi}(F^D)^{i+2}+E^{2i+1}E^{\pi}(F^D)^{i+2}E)-EE^DF^D-E^DF^DE\Big] \\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big] \\
&+ \Big[\sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+2}-E^DF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}-(F^D)^{i+1}E^{2i+3}E^D \\
&+ F^{i+1}(E^D)^{2i+2}-F^{i+2}F^D(E^D)^{2i+2}\Big)-FF^DEE^D\Big];
\end{aligned}$$

$$\begin{aligned}
\chi &= \Big[\sum_{i=0}^{m}\Big((E^D)^{2i+2}F^{i+1}F^{\pi}+(E^D)^{2i+3}F^{i+1}E-(E^D)^{2i+3}F^{i+2}F^DE\Big)\Big] \\
&\quad \Big[(E^D)^2+\sum_{i=1}^{m}F^i(E^D)^{2i+2}\Big]+\Big[\sum_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}\Big]\Big[\sum_{i=1}^{m}F^i(E^D)^{2i+1}\Big], \\
\omega &= \Big[\sum_{i=0}^{m}\Big((E^D)^{2i+2}F^{i+1}F^{\pi}+(E^D)^{2i+3}F^{i+1}E-(E^D)^{2i+3}F^{i+2}F^DE\Big)\Big] \\
&\quad \Big[(E^D)^3+\sum_{i=1}^{m}F^i(E^D)^{2i+3}\Big]+\Big[\sum_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}\Big]\Big[\sum_{i=1}^{m}F^i(E^D)^{2i+2}\Big];
\end{aligned}$$

$$m=i(E)+2i(F)+1.$$

Proof. Clearly, we have

$$M^2=\begin{pmatrix} E^2+F & E \\ FE & F\end{pmatrix}=P+Q,$$

where

$$P=\begin{pmatrix} F & 0 \\ FE & F\end{pmatrix}, Q=\begin{pmatrix} E^2 & E \\ 0 & 0\end{pmatrix}.$$

In light of Lemma 2.1, we have

$$P^D=\begin{pmatrix} F^D & 0 \\ X & F^D\end{pmatrix}, P^{\pi}=\begin{pmatrix} F^{\pi} & 0 \\ -FX & F^{\pi}\end{pmatrix};$$
$$Q^D=\begin{pmatrix} (E^D)^2 & (E^D)^3 \\ 0 & 0\end{pmatrix}, Q^{\pi}=\begin{pmatrix} E^{\pi} & -E^D \\ 0 & I\end{pmatrix}.$$

where

$$\begin{aligned}
X &= \sum_{i=0}^{2i(F)-1}[F^{\pi}F^{i+1}E(F^D)^{i+2}+(F^D)^{i+1}EF^iF^{\pi}] \\
&= F^DE.
\end{aligned}$$

For any $i\geq 2$, we have

$$P^i=\begin{pmatrix} F^i & 0 \\ X_i & F^i\end{pmatrix}, X_1=FE, X_i=X_{i-1}F+F^iE;$$
$$(P^D)^i=\begin{pmatrix} (F^D)^i & 0 \\ Y_i & (F^D)^i\end{pmatrix}, Y_1=F^DE, Y_i=F^DY_{i-1};$$
$$Q^i=\begin{pmatrix} E^{2i} & E^{2i-1} \\ 0 & 0\end{pmatrix}, (Q^D)^i=\begin{pmatrix} (E^D)^{2i} & (E^D)^{2i+1} \\ 0 & 0\end{pmatrix}.$$

Since $FE^iF = 0$ for all $i \in \mathbb{N}$, we verify that

$$X_i = F^iE, Y_i = (F^D)^iE.$$

We easily check that

$$\begin{array}{rcl} PQ^iP & = & \begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}\begin{pmatrix} E^{2i} & E^{2i-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & 0 \\ FE & F \end{pmatrix} \\ & = & \begin{pmatrix} FE^{2i} & FE^{2i-1} \\ FE^{2i+1} & FE^{2i} \end{pmatrix}\begin{pmatrix} F & 0 \\ FE & F \end{pmatrix} \\ & = & 0. \end{array}$$

for all $i \in \mathbb{N}$. In view of Lemma 2.2,

$$\begin{array}{rcl} (M^2)^D & = & (P+Q)^D \\ & = & P^D + Q^D + UP^D + Q^DU + P^DVQ + Q^DWQ \\ & + & VQQ^D + WQQ^D + UP^DVQ + \sum\limits_{i=0}^{m} UP^iP^{\pi}(Q^D)^{i+1}, \end{array}$$

where

$$\begin{array}{rcl} U & = & \sum\limits_{i=0}^{m}\Big[(Q^D)^{i+1}P^{i+1}P^{\pi} + Q^{i+1}Q^{\pi}(P^D)^{i+1}\Big] - QQ^DPP^D, \\ V & = & \sum\limits_{i=0}^{m}\Big[(P^D)^{i+1}Q^iQ^{\pi} + P^{i+1}P^{\pi}(Q^D)^{i+2}\Big] - PP^DQ^D, \\ W & = & UV + \sum\limits_{i+l+j=m-1}\Big[(Q^D)^{m+1-i}P^{l+1}Q^jQ^{\pi} + Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j}\Big] \\ & - & \sum\limits_{i=0}^{m}\Big[(Q^D)^{i+1}P^{i+1}(V+Q^D) + (U+QQ^D)P^{i+1}(Q^D)^{i+2}\Big], \\ m & = & i(E) + 2i(F) + 1 \geq i(P) + 2i(Q). \end{array}$$

We compute that

$$UP^D = \sum_{i=0}^{m} Q^{i+1}Q^{\pi}(P^D)^{i+2} - QQ^DP^D,$$

where

$$\begin{array}{rl} & Q^{i+1}Q^{\pi}(P^D)^{i+2} \\ = & \begin{pmatrix} E^{2i+2} & E^{2i+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E^{\pi} & -E^D \\ 0 & I \end{pmatrix}\begin{pmatrix} (F^D)^{i+2} & 0 \\ (F^D)^{i+2}E & (F^D)^{i+2} \end{pmatrix} \\ = & \begin{pmatrix} E^{2i+2}E^{\pi}(F^D)^{i+2} + E^{2i+1}E^{\pi}(F^D)^{i+2}E & E^{2i+1}E^{\pi}(F^D)^{i+2} \\ 0 & 0 \end{pmatrix}, \\ & QQ^DP^D \\ = & \begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (E^D)^2 & (E^D)^3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^D & 0 \\ F^DE & F^D \end{pmatrix} \\ = & \begin{pmatrix} EE^DF^D + E^DF^DE & E^DF^D \\ 0 & 0 \end{pmatrix}. \end{array}$$

Hence,

$$\begin{array}{rcl} (UP^D)_{11} & = & \sum\limits_{i=0}^{m}[E^{2i+2}E^{\pi}(F^D)^{i+2} + E^{2i+1}E^{\pi}(F^D)^{i+2}E] - EE^DF^D - E^DF^DE, \\ (UP^D)_{12} & = & \sum\limits_{i=0}^{m} E^{2i+1}E^{\pi}(F^D)^{i+2} - E^DF^D, \\ (UP^D)_{21} & = & 0, \\ (UP^D)_{22} & = & 0. \end{array}$$

Also we see that

$$Q^DU = \sum_{i=0}^{m}(Q^D)^{i+2}P^{i+1}P^{\pi} - Q^DPP^D,$$

where

$$\begin{array}{rl} & (Q^D)^{i+2}P^{i+1}P^{\pi} \\ = & \begin{pmatrix} (E^D)^{2i+4} & (E^D)^{2i+5} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1}, \end{pmatrix}\begin{pmatrix} F^{\pi} & 0 \\ -FF^DE & F^{\pi} \end{pmatrix} \\ = & \begin{pmatrix} (E^D)^{2i+4}F^{i+1} + (E^D)^{2i+5}F^{i+1}E & (E^D)^{2i+5}F^{i+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^{\pi} & 0 \\ -FF^DE & F^{\pi} \end{pmatrix} \\ = & \begin{pmatrix} (E^D)^{2i+4}F^{i+1} + (E^D)^{2i+5}F^{i+1}E & (E^D)^{2i+5}F^{i+1} \\ 0 & 0 \end{pmatrix}, \\ & Q^DPP^D \\ = & \begin{pmatrix} (E^D)^2 & (E^D)^3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}\begin{pmatrix} F^D & 0 \\ F^DE & F^D \end{pmatrix} \\ = & \begin{pmatrix} (E^D)^2F + (E^D)^3FE & (E^D)^3F \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^D & 0 \\ F^DE & F^D \end{pmatrix} \\ = & \begin{pmatrix} (E^D)^2FF^D + (E^D)^3FF^DE & (E^D)^3FF^D \\ 0 & 0 \end{pmatrix}. \end{array}$$

Hence,

$$\begin{array}{rcl} (Q^DU)_{11} & = & \sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1} + (E^D)^{2i+5}F^{i+1}E] - (E^D)^2FF^D - (E^D)^3FF^DE, \\ (Q^DU)_{12} & = & \sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1} - (E^D)^3FF^D, \\ (Q^DU)_{21} & = & 0, \\ (Q^DU)_{22} & = & 0. \end{array}$$

Clearly, we have

$$P^DVQ = \sum_{i=0}^{m}(P^D)^{i+2}Q^{i+1}Q^{\pi} - P^DQQ^D,$$

where

$$\begin{array}{rl} & (P^D)^{i+2}Q^{i+1}Q^{\pi} \\ = & \begin{pmatrix} (F^D)^{i+2} & 0 \\ (F^D)^{i+2}E & (F^D)^{i+2} \end{pmatrix}\begin{pmatrix} E^{2i+2} & E^{2i+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E^{\pi} & -E^D \\ 0 & I \end{pmatrix} \\ = & \begin{pmatrix} (F^D)^{i+2} & 0 \\ (F^D)^{i+2}E & (F^D)^{i+2} \end{pmatrix}\begin{pmatrix} E^{2i+2}E^{\pi} & E^{2i+1} - E^{2i+2}E^D \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} (F^D)^{i+2}E^{2i+2}E^{\pi} & (F^D)^{i+2}E^{2i+1} - (F^D)^{i+2}E^{2i+2}E^D \\ (F^D)^{i+2}E^{2i+3}E^{\pi} & (F^D)^{i+2}E^{2i+2} - (F^D)^{i+2}E^{2i+3}E^D \end{pmatrix} \\ & P^DQQ^D \\ = & \begin{pmatrix} F^D & 0 \\ F^DE & F^D \end{pmatrix}\begin{pmatrix} EE^D & E^D \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} F^DEE^D & F^DE^D \\ F^DE^2E^D & F^DEE^D \end{pmatrix}. \end{array}$$

Hence,

$$\begin{array}{rcl} (P^DVQ)_{11} & = & \sum_{i=0}^{m}(F^D)^{i+2}E^{2i+2}E^{\pi} - F^DEE^D, \\ (P^DVQ)_{12} & = & \sum_{i=0}^{m}\left[(F^D)^{i+2}E^{2i+1} - (F^D)^{i+2}E^{2i+2}E^D\right] - F^DE^D, \\ (P^DVQ)_{21} & = & \sum_{i=0}^{m}(F^D)^{i+2}E^{2i+3}E^{\pi} - F^DE^2E^D, \\ (P^DVQ)_{22} & = & \sum_{i=0}^{m}\left[(F^D)^{i+2}E^{2i+2} - (F^D)^{i+2}E^{2i+3}E^D\right] - F^DEE^D. \end{array}$$

We observe that

$$\begin{array}{rcl} Q^DWQ & = & Q^DUVQ + \sum\limits_{i+l+j=m-1}(Q^D)^{m+2-i}P^{l+1}Q^{j+1}Q^{\pi} - \sum\limits_{i=0}^{m}[(Q^D)^{i+2}P^{i+1}VQ \\ & + & Q^DUP^{i+1}(Q^D)^{i+1} + (Q^D)^{i+2}P^{i+1}QQ^D + Q^DP^{i+1}(Q^D)^{i+1}], \end{array}$$

$$VQ = \sum_{i=0}^{m}\Big[(P^D)^{i+1}Q^{i+1}Q^{\pi} + P^{i+1}P^{\pi}(Q^D)^{i+1}\Big] - PP^DQQ^D,$$

$$\begin{array}{rl} & (P^D)^{i+1}Q^{i+1}Q^{\pi} \\ = & \begin{pmatrix} (F^D)^{i+1} & 0 \\ (F^D)^{i+1}E & (F^D)^{i+1} \end{pmatrix}\begin{pmatrix} E^{2i+2} & E^{2i+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E^{\pi} & -E^D \\ 0 & I \end{pmatrix} \\ = & \begin{pmatrix} (F^D)^{i+1}E^{2i+2}E^{\pi} & (F^D)^{i+1}E^{2i+1} - (F^D)^{i+1}E^{2i+2}E^D \\ (F^D)^{i+1}E^{2i+3}E^{\pi} & (F^D)^{i+1}E^{2i+2} - (F^D)^{i+1}E^{2i+3}E^D \end{pmatrix} \end{array}$$

$$\begin{array}{rl} & P^{i+1}P^{\pi}(Q^D)^{i+1} \\ = & \begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1} \end{pmatrix}\begin{pmatrix} F^{\pi} & 0 \\ -FF^DE & F^{\pi} \end{pmatrix}\begin{pmatrix} (E^D)^{2i+2} & (E^D)^{2i+3} \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1} \end{pmatrix}\begin{pmatrix} F^{\pi}(E^D)^{2i+2} & F^{\pi}(E^D)^{2i+3} \\ -FF^D(E^D)^{2i+1} & -FF^D(E^D)^{2i+2} \end{pmatrix} \\ = & \begin{pmatrix} F^{i+1}F^{\pi}(E^D)^{2i+2} & F^{i+1}F^{\pi}(E^D)^{2i+3} \\ F^{i+1}(E^D)^{2i+1} - F^{i+2}F^D(E^D)^{2i+1} & F^{i+1}(E^D)^{2i+2} - F^{i+2}F^D(E^D)^{2i+2} \end{pmatrix}, \end{array}$$

$$\begin{array}{rl} & PP^DQ^D \\ = & \begin{pmatrix} FF^D & 0 \\ FF^DE & FF^D \end{pmatrix}\begin{pmatrix} (E^D)^2 & (E^D)^3 \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} FF^D(E^D)^2 & FF^D(E^D)^3 \\ FF^DE^D & FF^D(E^D)^2 \end{pmatrix}, \end{array}$$

$$\begin{array}{rl} & PP^DQQ^D \\ = & \begin{pmatrix} FF^D(E^D)^2 & FF^D(E^D)^3 \\ FF^DE^D & FF^D(E^D)^2 \end{pmatrix}\begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} FF^DEE^D & FF^DE^D \\ FF^DE^2E^D & FF^DEE^D \end{pmatrix}. \end{array}$$

Therefore

$$\begin{array}{rcl} (VQ)_{11} & = & \sum\limits_{i=0}^{m}\Big[(F^D)^{i+1}E^{2i+2}E^{\pi} + F^{i+1}F^{\pi}(E^D)^{2i+2}\Big] - FF^DEE^D, \\ (VQ)_{12} & = & \sum\limits_{i=0}^{m}\Big[(F^D)^{i+1}E^{2i+1} - (F^D)^{i+1}E^{2i+2}E^D + F^{i+1}F^{\pi}(E^D)^{2i+3}\Big] - FF^DE^D, \\ (VQ)_{21} & = & \sum\limits_{i=0}^{m}\Big[(F^D)^{i+1}E^{2i+3}E^{\pi} + F^{i+1}(E^D)^{2i+1} - F^{i+2}F^D(E^D)^{2i+1}\Big] - FF^DE^2E^D, \\ (VQ)_{22} & = & \sum\limits_{i=0}^{m}\Big[(F^D)^{i+1}E^{2i+2} - (F^D)^{i+1}E^{2i+3}E^D + F^{i+1}(E^D)^{2i+2} - F^{i+2}F^D(E^D)^{2i+2}\Big] \\ & - & FF^DEE^D. \end{array}$$

Hence

$$Q^DUVQ = (Q^DU)(VQ) = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\alpha &= \Big[\sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E]-(E^D)^2FF^D-(E^D)^3FF^DE\Big] \\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big] \\
&+ \Big[\sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1}-(E^D)^3FF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+3}E^{\pi}+F^{i+1}(E^D)^{2i+1} \\
&- F^{i+2}F^D(E^D)^{2i+1}\Big)-FF^DE^2E^D\Big], \\
\beta &= \Big[\sum_{i=0}^{m}[(E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E]-(E^D)^2FF^D-(E^D)^3FF^DE\Big] \\
&\quad \Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big] \\
&+ \Big[\sum_{i=0}^{m}(E^D)^{2i+5}F^{i+1}-(E^D)^3FF^D\Big]\Big[\sum_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}-(F^D)^{i+1}E^{2i+3}E^D \\
&+ F^{i+1}(E^D)^{2i+2}-F^{i+2}F^D(E^D)^{2i+2}\Big)-FF^DEE^D\Big];
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\Big((Q^D)^{m+2-i}P^{l+1}Q^{j+1}Q^{\pi}\Big)_{11} &= (E^D)^{2m-2i+4}F^{l+1}E^{2j+2}E^{\pi}+(E^D)^{2m-2i+5}F^{l+1}E^{2j+3}E^{\pi}, \\
\Big((Q^D)^{m+2-i}P^{l+1}Q^{j+1}Q^{\pi}\Big)_{12} &= (E^D)^{2m-2i+4}F^{l+1}E^{2j+1}E^{\pi}+(E^D)^{2m-2i+5}F^{l+1}E^{2j+2}E^{\pi}, \\
\Big((Q^D)^{m+2-i}P^{l+1}Q^{j+1}Q^{\pi}\Big)_{21} &= 0, \\
\Big((Q^D)^{m+2-i}P^{l+1}Q^{j+1}Q^{\pi}\Big)_{22} &= 0.
\end{aligned}$$

We easily see that

$$\begin{aligned}
(Q^D)^{i+2}P^{i+1} &= \begin{pmatrix} (E^D)^{2i+4} & (E^D)^{2i+5} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1} \end{pmatrix} \\
&= \begin{pmatrix} (E^D)^{2i+4}F^{i+1}+(E^D)^{2i+5}F^{i+1}E & (E^D)^{2i+5}F^{i+1} \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
P^{i+1}(Q^D)^{i+1} &= \begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1} \end{pmatrix}\begin{pmatrix} (E^D)^{2i+2} & (E^D)^{2i+3} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} F^{i+1}(E^D)^{2i+2} & F^{i+1}(E^D)^{2i+3} \\ F^{i+1}(E^D)^{2i+1} & F^{i+1}(E^D)^{2i+2} \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Big((Q^D)^{i+2}P^{i+1}QQ^D\Big)_{11} &= (E^D)^{2i+4}F^{i+1}EE^D+(E^D)^{2i+5}F^{i+1}E^2E^D, \\
\Big((Q^D)^{i+2}P^{i+1}QQ^D\Big)_{12} &= (E^D)^{2i+4}F^{i+1}E^D+(E^D)^{2i+5}F^{i+1}EE^D, \\
\Big((Q^D)^{i+2}P^{i+1}QQ^D\Big)_{21} &= 0, \\
\Big((Q^D)^{i+2}P^{i+1}QQ^D\Big)_{22} &= 0;
\end{aligned}$$

$$\begin{aligned}
\Big(Q^DP^{i+1}(Q^D)^{i+1}\Big)_{11} &= (E^D)^2F^{i+1}(E^D)^{2i+2}+(E^D)^3F^{i+1}(E^D)^{2i+1}, \\
\Big(Q^DP^{i+1}(Q^D)^{i+1}\Big)_{12} &= (E^D)^2F^{i+1}(E^D)^{2i+3}+(E^D)^3F^{i+1}(E^D)^{2i+2}, \\
\Big(Q^DP^{i+1}(Q^D)^{i+1}\Big)_{21} &= 0, \\
\Big(Q^DP^{i+1}(Q^D)^{i+1}\Big)_{22} &= 0.
\end{aligned}$$

Moreover, we have

$$(Q^D)^{i+2}P^{i+1}VQ=\Big[(Q^D)^{i+2}P^{i+1}\Big][VQ]=\begin{pmatrix} \gamma_i & \delta_i \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\gamma_i &= (E^D)^{2i+4}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+2}E^{\pi} + F^{j+1}F^{\pi}(E^D)^{2j+2}\Big) - FF^D EE^D\Big] \\
&+ (E^D)^{2i+5}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+3}E^{\pi} + F^{j+1}(E^D)^{2j+1} - F^{j+2}F^D(E^D)^{2j+1}\Big) \\
&- FF^D E^2 E^D\Big], \\
\delta_i &= (E^D)^{2i+4}F^{i+1}\Big[\sum_{j=0}^{m}\Big((F^D)^{j+1}E^{2j+1} - (F^D)^{j+1}E^{2j+2}E^D + F^{j+1}F^{\pi}(E^D)^{2j+3}\Big) \\
&- FF^D E^D\Big] + (E^D)^{2i+5}F^{i+1}\Big[\sum_{j=0}^{m}\Big[(F^D)^{j+1}E^{2j+2} - (F^D)^{j+1}E^{2j+3}E^D \\
&+ F^{j+1}(E^D)^{2j+2} - F^{j+2}F^D(E^D)^{2j+2}\Big) - FF^D EE^D\Big];
\end{aligned}$$

Also we have

$$Q^D UP^{i+1}(Q^D)^{i+1} = [Q^D U][P^{i+1}(Q^D)^{i+1}] = \begin{pmatrix} \varepsilon_i & \zeta_i \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\varepsilon_i &= \Big[\sum_{j=0}^{m}(E^D)^{2j+4}F^j - (E^D)^2F^D\Big]F^{i+2}(E^D)^{2i+2} \\
&+ \Big[\sum_{j=0}^{m}(E^D)^{2j+5}F^j - (E^D)^3F^D\Big]F^{i+2}(E^D)^{2i+1}, \\
\zeta_i &= \Big[\sum_{j=0}^{m}(E^D)^{2j+4}F^j - (E^D)^2F^D\Big]F^{i+2}(E^D)^{2i+3} \\
&+ \Big[\sum_{j=0}^{m}(E^D)^{2j+5}F^j - (E^D)^3F^D\Big]F^{i+2}(E^D)^{2i+2};
\end{aligned}$$

Therefore we compute that

$$Q^D WQ = \begin{pmatrix} \eta & \theta \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\eta &= \sum_{i+l+j=m-1}(Q^D)^{m+2-i}\Big[(E^D)^{2m-2i+4}F^{l+1}E^{2j+2}E^{\pi} + (E^D)^{2m-2i+5}F^{l+1}E^{2j+3}E^{\pi}\Big] \\
&- \sum_{i=0}^{m}\Big[\gamma_i + \varepsilon_i + (E^D)^{2i+4}F^{i+1}EE^D + (E^D)^{2i+5}F^{i+1}E^2E^D + (E^D)^2F^{i+1}(E^D)^{2i+2} \\
&+ (E^D)^3F^{i+1}(E^D)^{2i+1}\Big] + \alpha, \\
\theta &= \sum_{i+l+j=m-1}(Q^D)^{m+2-i}\Big[(E^D)^{2m-2i+4}F^{l+1}E^{2j+1}E^{\pi} + (E^D)^{2m-2i+5}F^{l+1}E^{2j+2}E^{\pi}\Big] \\
&- \sum_{i=0}^{m}\Big[\delta_i + \zeta_i + (E^D)^{2i+4}F^{i+1}E^D + (E^D)^{2i+5}F^{i+1}EE^D + (E^D)^2F^{i+1}(E^D)^{2i+3} \\
&+ (E^D)^3F^{i+1}(E^D)^{2i+2}\Big] + \beta;
\end{aligned}$$

Further, we get

$$\begin{aligned}
(VQQ^D)_{11} &= \sum_{i=0}^{m} F^{i+1}F^{\pi}(E^D)^{2i+4} - FF^D(E^D)^2, \\
(VQQ^D)_{12} &= \sum_{i=0}^{m} F^{i+1}F^{\pi}(E^D)^{2i+5} - FF^D(E^D)^3, \\
(VQQ^D)_{21} &= \sum_{i=0}^{m}\Big[F^{i+1} - F^{i+2}F^D\Big](E^D)^{2i+3} - FF^D E^D, \\
(VQQ^D)_{22} &= \sum_{i=0}^{m}\Big[F^{i+1} - F^{i+2}F^D\Big](E^D)^{2i+4} - FF^D(E^D)^2.
\end{aligned}$$

We now compute

$$\begin{aligned}
WQQ^D &= UVQQ^D + \sum_{i+l+j=m-1} Q^{i+1}Q^{\pi}P^{l+1}Q(Q^D)^{m+3-j} \\
&- \sum_{i=0}^{m}\Big[(Q^D)^{i+1}P^{i+1}(V+Q^D)QQ^D + (U+QQ^D)P^{i+1}(Q^D)^{i+2}\Big] \\
&= UVQQ^D + \sum_{i+l+j=m-1} Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j} - \sum_{i=0}^{m}\Big[(Q^D)^{i+1}P^{i+1}VQQ^D \\
&+ UP^{i+1}(Q^D)^{i+2} + (Q^D)^{i+1}P^{i+1}Q^D + QQ^DP^{i+1}(Q^D)^{i+2}\Big].
\end{aligned}$$

One checks that $UP^{\pi} = \sum_{i=0}^{m}(Q^D)^{i+1}P^{i+1}P^{\pi}$ and so

$$\begin{aligned}
(UP^{\pi})_{11} &= \sum_{i=0}^{m}\Big[(E^D)^{2i+2}F^{i+1}F^{\pi} + (E^D)^{2i+3}F^{i+1}E - (E^D)^{2i+3}F^{i+2}F^DE\Big], \\
(UP^{\pi})_{12} &= \sum_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}, \\
(UP^{\pi})_{21} &= 0, \\
(UP^{\pi})_{22} &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
(UPP^D)_{11} &= \sum_{i=0}^{m}E^{2i+2}E^{\pi}(F^D)^{i+1} - EE^DFF^D + \sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1}E - E^DFF^DE, \\
(UPP^D)_{12} &= \sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1} - E^DFF^D, \\
(UPP^D)_{21} &= 0, \\
(UPP^D)_{22} &= 0.
\end{aligned}$$

Hence,

$$U = UP^{\pi} + UPP^D = \begin{pmatrix} \lambda & \mu \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\lambda &= \sum_{i=0}^{m}\Big[(E^D)^{2i+2}F^{i+1}F^{\pi} + (E^D)^{2i+3}F^{i+1}E - (E^D)^{2i+3}F^{i+2}F^DE\Big] \\
&+ \sum_{i=0}^{m}E^{2i+2}E^{\pi}(F^D)^{i+1} - EE^DFF^D + \sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1}E - E^DFF^DE, \\
\mu &= \sum_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi} + \sum_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+1} - E^DFF^D.
\end{aligned}$$

It follows that

$$UVQQ^D = U(VQQ^D) = \begin{pmatrix} \xi & \rho \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\xi &= \lambda\Big[\sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+4} - FF^D(E^D)^2\Big] + \mu\Big[\sum_{i=0}^{m}\Big(F^{i+1} - F^{i+2}F^D\Big)(E^D)^{2i+3} - FF^DE^D\Big] \\
\rho &= \lambda\Big[\sum_{i=0}^{m}F^{i+1}F^{\pi}(E^D)^{2i+5} - FF^D(E^D)^3\Big] + \mu\Big[\sum_{i=0}^{m}\Big(F^{i+1} - F^{i+2}F^D\Big)(E^D)^{2i+4} - FF^D(E^D)^2\Big].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\Big(Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j}\Big)_{11} = E^{2i+2}E^{\pi}F^{l+1}(E^D)^{2m+4-2j} + (E^{2i+1} - E^{2i+2}E^D)F^{l+1}(E^D)^{2m+3-2j}, \\
&\Big(Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j}\Big)_{12} = E^{2i+2}E^{\pi}F^{l+1}(E^D)^{2m+5-2j} + (E^{2i+1} - E^{2i+2}E^D)F^{l+1}(E^D)^{2m+4-2j}, \\
&\Big(Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j}\Big)_{21} = 0, \\
&\Big(Q^{i+1}Q^{\pi}P^{l+1}(Q^D)^{m+2-j}\Big)_{22} = 0,
\end{aligned}$$

$$\begin{aligned}(Q^D)^{i+1}P^{i+1} &= \begin{pmatrix} (E^D)^{2i+2} & (E^D)^{2i+3} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^{i+1} & 0 \\ F^{i+1}E & F^{i+1} \end{pmatrix} \\ &= \begin{pmatrix} (E^D)^{2i+2}F^{i+1} + (E^D)^{2i+3}F^{i+1}E & (E^D)^{2i+3}F^{i+1} \\ 0 & 0 \end{pmatrix},\end{aligned}$$

$$(Q^D)^{i+1}P^{i+1}VQQ^D = [(Q^D)^{i+1}P^{i+1}][VQQ^D] = \begin{pmatrix} \sigma_i & \tau_i \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}\sigma_i &= (E^D)^{2i+2}F^{i+2}\Big[\sum_{j=0}^{m} F^jF^\pi(E^D)^{2j+4} - F^D(E^D)^2\Big] + (E^D)^{2i+3}F^{i+2}\Big[\sum_{j=0}^{m}\Big(F^j - F^{j+1}F^D\Big)(E^D)^{2j+3} - F^DE^D\Big],\\ \tau_i &= (E^D)^{2i+2}F^{i+2}\Big[\sum_{j=0}^{m} F^jF^\pi(E^D)^{2j+5} - F^D(E^D)^3\Big] + (E^D)^{2i+3}F^{i+2}\Big[\sum_{j=0}^{m}\Big(F^j - F^{j+1}F^D\Big)(E^D)^{2j+4} - F^D(E^D)^2\Big],\end{aligned}$$

$$\begin{aligned}\Big(UP^{i+1}(Q^D)^{i+2}\Big)_{11} &= \lambda F^{i+1}(E^D)^{2i+4} + \mu F^{i+1}(E^D)^{2i+3},\\ \Big(UP^{i+1}(Q^D)^{i+2}\Big)_{12} &= \lambda F^{i+1}(E^D)^{2i+5} + \mu F^{i+1}(E^D)^{2i+4},\\ \Big(UP^{i+1}(Q^D)^{i+2}\Big)_{21} &= 0,\\ \Big(UP^{i+1}(Q^D)^{i+2}\Big)_{22} &= 0;\end{aligned}$$

$$\begin{aligned}\Big((Q^D)^{i+1}P^{i+1}Q^D\Big)_{11} &= (E^D)^{2i+2}F^{i+1}(E^D)^2 + (E^D)^{2i+3}F^{i+1}E^D,\\ \Big((Q^D)^{i+1}P^{i+1}Q^D\Big)_{12} &= (E^D)^{2i+2}F^{i+1}(E^D)^3 + (E^D)^{2i+3}F^{i+1}(E^D)^2,\\ \Big((Q^D)^{i+1}P^{i+1}Q^D\Big)_{21} &= 0,\\ \Big((Q^D)^{i+1}P^{i+1}Q^D\Big)_{22} &= 0;\end{aligned}$$

$$\begin{aligned}\Big(QQ^DP^{i+1}(Q^D)^{i+2}\Big)_{11} &= EE^DF^{i+1}(E^D)^{2i+4} + E^DF^{i+1}(E^D)^{2i+3},\\ \Big(QQ^DP^{i+1}(Q^D)^{i+2}\Big)_{12} &= EE^DF^{i+1}(E^D)^{2i+5} + E^DF^{i+1}(E^D)^{2i+4},\\ \Big(QQ^DP^{i+1}(Q^D)^{i+2}\Big)_{21} &= 0,\\ \Big(QQ^DP^{i+1}(Q^D)^{i+2}\Big)_{22} &= 0.\end{aligned}$$

Therefore we obtain

$$WQQ^D = \begin{pmatrix} \kappa & \varsigma \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}\kappa &= \sum_{i+l+j=m-1}\Big[E^{2i+2}E^\pi F^{l+1}(E^D)^{2m+4-2j} + (E^{2i+1} - E^{2i+2}E^D)F^{l+1}(E^D)^{2m+3-2j}\Big]\\ &- \sum_{i=0}^{m}\Big[\sigma_i + \lambda F^{i+1}(E^D)^{2i+4} + \mu F^{i+1}(E^D)^{2i+3} + (E^D)^{2i+2}F^{i+1}(E^D)^2\\ &+ (E^D)^{2i+3}F^{i+1}E^D + EE^DF^{i+1}(E^D)^{2i+4} + E^DF^{i+1}(E^D)^{2i+3}\Big] + \xi,\\ \varsigma &= \sum_{i+l+j=m-1}\Big[E^{2i+2}E^\pi F^{l+1}(E^D)^{2m+5-2j} + (E^{2i+1} - E^{2i+2}E^D)F^{l+1}(E^D)^{2m+4-2j}\Big]\\ &- \sum_{i=0}^{m}\Big[\tau_i + \lambda F^{i+1}(E^D)^{2i+5} + \mu F^{i+1}(E^D)^{2i+4} + (E^D)^{2i+2}F^{i+1}(E^D)^3\\ &+ (E^D)^{2i+3}F^{i+1}(E^D)^2 + EE^DF^{i+1}(E^D)^{2i+5} + E^DF^{i+1}(E^D)^{2i+4} + \rho.\end{aligned}$$

Further, we obtain

$$UP^DVQ = (UP^D)(VQ) = \begin{pmatrix} \phi & \varphi \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{array}{rcl}\phi & = & \Big[\sum\limits_{i=0}^{m}\Big(E^{2i+2}E^{\pi}(F^D)^{i+2}+E^{2i+1}E^{\pi}(F^D)^{i+2}E\Big)-EE^DF^D-E^DF^DE\Big]\\ & & \Big[\sum\limits_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}E^{\pi}+F^{i+1}F^{\pi}(E^D)^{2i+2}\Big)-FF^DEE^D\Big]\\ & + & \Big[\sum\limits_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+2}-E^DF^D\Big]\Big[\sum\limits_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+3}E^{\pi}+F^{i+1}(E^D)^{2i+1}\\ & - & F^{i+2}F^D(E^D)^{2i+1}\Big)-FF^DE^2E^D\Big],\\ \varphi & = & \Big[\sum\limits_{i=0}^{m}(E^{2i+2}E^{\pi}(F^D)^{i+2}+E^{2i+1}E^{\pi}(F^D)^{i+2}E)-EE^DF^D-E^DF^DE\Big]\\ & & \Big[\sum\limits_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+1}-(F^D)^{i+1}E^{2i+2}E^D+F^{i+1}F^{\pi}(E^D)^{2i+3}\Big)-FF^DE^D\Big]\\ & + & \Big[\sum\limits_{i=0}^{m}E^{2i+1}E^{\pi}(F^D)^{i+2}-E^DF^D\Big]\Big[\sum\limits_{i=0}^{m}\Big((F^D)^{i+1}E^{2i+2}-(F^D)^{i+1}E^{2i+3}E^D\\ & + & F^{i+1}(E^D)^{2i+2}-F^{i+2}F^D(E^D)^{2i+2}\Big)-FF^DEE^D\Big].\end{array}$$

Obviously, we have

$$\sum_{i=0}^{m}UP^iP^{\pi}(Q^D)^{i+1}=(UP^{\pi})\Big[Q^D+\sum_{i=1}^{m}P^i(Q^D)^{i+1}\Big].$$

Also we see that

$$\begin{array}{rcl}Q^D+\sum\limits_{i=1}^{m}P^i(Q^D)^{i+1} & = & \begin{pmatrix}(E^D)^2 & (E^D)^3\\ 0 & 0\end{pmatrix}+\sum\limits_{i=1}^{m}\begin{pmatrix}F^i & 0\\ F^iE & F^i,\end{pmatrix}\begin{pmatrix}(E^D)^{2i+2} & (E^D)^{2i+3}\\ 0 & 0\end{pmatrix}\\ & = & \begin{pmatrix}(E^D)^2+\sum\limits_{i=1}^{m}F^i(E^D)^{2i+2} & (E^D)^3+\sum\limits_{i=1}^{m}F^i(E^D)^{2i+3}\\ \sum\limits_{i=1}^{m}F^i(E^D)^{2i+1} & \sum\limits_{i=1}^{m}F^i(E^D)^{2i+2}\end{pmatrix}.\end{array}$$

Then

$$\sum_{i=0}^{m}UP^iP^{\pi}(Q^D)^{i+1}=\begin{pmatrix}\chi & \omega\\ 0 & 0\end{pmatrix},$$

where

$$\begin{array}{rcl}\chi & = & \Big[\sum\limits_{i=0}^{m}\Big((E^D)^{2i+2}F^{i+1}F^{\pi}+(E^D)^{2i+3}F^{i+1}E-(E^D)^{2i+3}F^{i+2}F^DE\Big)\Big]\\ & & \Big[(E^D)^2+\sum\limits_{i=1}^{m}F^i(E^D)^{2i+2}\Big]+\Big[\sum\limits_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}\Big]\Big[\sum\limits_{i=1}^{m}F^i(E^D)^{2i+1}\Big],\\ \omega & = & \Big[\sum\limits_{i=0}^{m}\Big((E^D)^{2i+2}F^{i+1}F^{\pi}+(E^D)^{2i+3}F^{i+1}E-(E^D)^{2i+3}F^{i+2}F^DE\Big)\Big]\\ & & \Big[(E^D)^3+\sum\limits_{i=1}^{m}F^i(E^D)^{2i+3}\Big]+\Big[\sum\limits_{i=0}^{m}(E^D)^{2i+3}F^{i+1}F^{\pi}\Big]\Big[\sum\limits_{i=1}^{m}F^i(E^D)^{2i+2}\Big].\end{array}$$

Accordingly, we derive

$$(M^2)^D=\begin{pmatrix}\Gamma & \Delta\\ \Lambda & \Xi\end{pmatrix},$$

where Γ,Δ,Λ and Ξ are given as in (∗) by direct computation. Therefore

$$\begin{array}{rcl}M^D & = & M(M^D)^2\\ & = & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}(M^2)^D\\ & = & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}\begin{pmatrix}\Gamma & \Delta\\ \Lambda & \Xi\end{pmatrix}\\ & = & \begin{pmatrix}E\Gamma+\Lambda & E\Delta+\Xi\\ F\Gamma & F\Delta\end{pmatrix},\end{array}$$

as asserted. □

This following example illustrates that Theorem 2.3 is valid even $EF^2 \neq 0$. (see [4, Theorem 2.2]).

Example 2.4. *Let* $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix} \in \mathbb{C}^{16\times 16}$, *where*

$$E = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}, F = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \in \mathbb{C}^{8\times 8}.$$

Then $FEF = 0$ *and* $FE^2 = 0$, *while* $FE \neq 0$ *and* $EF^2 \neq 0$. *Construct P and Q as in Theorem 2.3. Since* $E^2 = 0$ *and* $F^3 = 0$, *we prove that P and Q are nilpotent. We check that*

$$PQ^2 = \begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}\begin{pmatrix} E^4 & E^3 \\ 0 & 0 \end{pmatrix} = 0,$$
$$PQP = \begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}\begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & 0 \\ FE & F \end{pmatrix} = 0.$$

In light of Lemma 2.2, $M^2 = P + Q$ *is nilpotent. Therefore M is nilpotent, i.e.,* $M^D = 0$.

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