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A New Constraint Qualification for Optimality of Nonconvex Nonsmooth Optimization Problems

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Abstract. In this paper, we study the nonconvex nonsmooth optimization problem (P) of minimizing a tangentially convex function with inequality constraints where the constraint functions are tangentially convex. This is done by using the cone of tangential subdifferentials together with a new constraint qualification. Indeed, we present a new constraint qualification to guarantee that Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality of the problem (P). Moreover, various nonsmooth (generalized) constraint qualifications that are a modification of the well known constraint qualifications are investigated. Several illustrative examples are presented to clarify the connection between nonsmooth constraint qualifications and new constraint qualification.

1. Introduction

Consider the following general nonlinear programming:

$$\begin{array}{c} \min f(x), \\ \text{s.t.} \\ g_i(x) \leq 0,\ i = 1, 2, \dots, m, \\ x \in X_0, \end{array} \tag{NLP}$$

where $f, g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1, 2, \dots, m)$ are real valued functions and X_0 is a subset of $\mathbb{R}^n$. The most important subjects in optimization problems are of finding the optimal solutions and optimality conditions. Optimality conditions are the features that a feasible point of an optimization problem must be satisfied them when it is a candidate for an optimal solution. Accordingly, establishing optimality conditions for the problem (NLP) is one of the fundamentals in both the theory and practice. Although, an optimality condition that is both necessary and sufficient is preferred, but such kind of conditions may only be valid under some certain assumptions on the optimization problem, for example; convexity, differentiability.

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Therefore, in the past decades, a great deal of attention was given to establish the optimality conditions for scalar optimization problems (see, for example; [1, 3, 7, 11, 13, 16–18, 24–26, 29, 30]), and moreover, many classical results known for scalar optimization problems have been extended to vector-valued and set-valued optimization problems (see, for example; [2, 12, 14, 15, 19–23, 32]). One of the most common optimality conditions is Karush-Kuhn-Tucker (KKT) conditions [8, 10], and that the establishment of the (KKT) optimality conditions depends on the representation of the feasible set. As a general rule, to obtain a checkable necessary optimality condition for a given constrained optimization problem, one needs to assume some properties of the constraint system called constraint qualifications (CQs in short).

The subject of constraint qualifications is of significant importance in optimization. Indeed, constraint qualifications are corner stones for the study of the classical convex programming and they guarantee necessary and sufficient conditions for optimality. Thus, the (KKT) conditions and constraint qualifications play a key role in the study of optimization problems. Recently, the (KKT) conditions and constraint qualifications were studied by many authors for vector-valued and set-valued optimization problems and, in particular, for convex and nonconvex scalar optimization problems [1–3, 7, 11–23, 25, 27–30, 32]. Indeed, in 2010, Lasserre [25] considered an optimization problem, whose the objective function is differentiable and convex and constraint functions are differentiable but are not necessarily convex. In this case, it was shown that Slater's condition together with nondegeneracy condition ensures that the (KKT) optimality conditions are necessary and sufficient. Moreover, in 2013, Dutta and Lalitha [13] by using Clarke's subdifferential introduced a nonsmooth version of Lasserre's optimization problem, whose the objective function is convex but is not necessarily differentiable and constraint functions are locally Lipschitz. In continuation of the previous studies, in 2017, Chieu et al. [11], considered an optimization problem, whose the objective function is convex and constraint functions are differentiable but are not necessarily convex, and introduced the weakest constraint qualification that guarantees the (KKT) conditions are necessary for optimality. In this case, in 2019, for the latter problem it was given a new constraint qualification that is the weakest qualification for the (KKT) conditions to be necessary for optimality [7].

In this paper, our attention focuses on the class of tangentially convex functions. It should be noted that Pshenichnyi [31] introduced this class of functions and they were called "tangentially convex" by Lemaréchal [26]. The tangentially convex functions cover a broad class of functions, for example; this class contains Gâteaux differentiable functions, convex functions with open domain and locally Lipschitz regular functions. A few works have been done in optimizing the problems with such constraint functions, including: In 2015, Martínez-Legaz [29] presented necessary and sufficient optimality conditions for minimizing of pseudoconvex functions over convex feasible sets described by tangentially convex functions in terms of the notion of tangential subdifferentials. Actually, he has extended the results obtained in [13, 25]. Also, in [30] the necessary optimality condition for minimizing of convex functions over nonconvex feasible sets described by tangentially convex functions that are continuous at the feasible points was presented. Moreover, under an extra assumption (locally Lipschitzian of the constraint functions) the sufficient condition has been given for optimality. Although, in all of the above mentioned works, the constraint functions are not convex, but the feasible set and the objective function are convex, while in this paper, we go beyond this and remove the convexity of the feasible set and the objective function. We consider a nonconvex case of the problem (*NLP*), called the problem (*P*), whose the objective function is tangentially convex and active constraint functions are tangentially convex at a given feasible point but are not necessarily convex or differentiable, and moreover, the feasible set is not convex. Our aim is to present a condition on a nonconvex feasible set defined by tangentially convex functions and provide a new constraint qualification that guarantees the (KKT) conditions are necessary and sufficient for optimality of the problem (P). Our results recapture the corresponding known results of [7, 11, 13, 25, 29, 30].

The paper has the following structure: In Section 2, we provide basic concepts, notations and preliminary results related to nonconvex analysis and nonconvex geometry. Moreover, characterizing subdifferential cone and its interior, and to describe the connection between various types of cone of directions which are related to the negative polar of the subdifferential cone and also investigating various nonsmooth (generalized) constraint qualifications that are a modification of the well known constraint qualifications are given in Section 2. In Section 3, we present a new constraint qualification to guarantee that Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality of the problem (P). Several illustrative examples are given to clarify the connection between nonsmooth constraint qualifications and new constraint qualification.

2. Preliminaries

In this section, we provide some basic definitions, notations and results related to nonconvex analysis and nonconvex geometry. Throughout the paper, we assume that $\mathbb{R}^n$ is the Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is defined by:

$$\mathbb{R}^n_+ := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \dots, n\}.$$

The closed ball in $\mathbb{R}^n$ with center at $x \in \mathbb{R}^n$ and radius $r > 0$ is defined by

$$B(x, r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}.$$

For a nonempty set $X \subseteq \mathbb{R}^n$, we denote ([4, 9]) by *coneX* and *conicX* the cone generated by X and conical hull of X, respectively, and are defined as follow:

$$coneX := \mathbb{R}_+ X := \{\lambda x : \lambda \geq 0,\ x \in X\},$$

$$conicX := \Big\{ \sum_{j=1}^{n} \lambda_j x_j : \lambda_j \geq 0,\ x_j \in X,\ j = 1, 2, \dots, n,\ n \in \mathbb{N} \Big\}.$$

We also define the negative and positive polar cone of X, by

$$X^{\ominus} := \{u \in \mathbb{R}^n : \langle u, x\rangle \leq 0,\ \forall\, x \in X\},$$

and

$$X^{\oplus} := \{u \in \mathbb{R}^n : \langle u, x\rangle \geq 0,\ \forall\, x \in X\},$$

respectively. It is not difficult to see that polar cone of a set $X \subseteq \mathbb{R}^n$ is a closed convex cone (see, for example; [4, 9]).

The cone of feasible directions [3] of a nonempty set $X \subseteq \mathbb{R}^n$ at a point $x \in X$ is defined by

$$F_X(x) := \{d \in \mathbb{R}^n : \exists\, \delta > 0 \ \ni\ x + \lambda d \in X,\ \forall\, \lambda \in [0, \delta]\}.$$

Clearly, $F_X(x)$ is a cone containing the origin. It can be seen that $\operatorname{cl} F_X(x) \subseteq T_X(x)$, where $T_X(x)$ is the tangent cone of X at the point $x \in X$, and is defined by

$$T_X(x) := \{d \in \mathbb{R}^n : \exists\, t_k > 0,\ \exists\, d_k \in \mathbb{R}^n \ni t_k \downarrow 0, d_k \longrightarrow d, x + t_k d_k \in X,\ \forall\, k \geq 1\}.$$

It is clear that $T_X(x)$ is a closed cone. However, $T_X(x)$ is not necessarily convex (see [3, 8]). The normal cone of a convex set $X \subseteq \mathbb{R}^n$ at a point $x \in X$ is defined by $N_X(x) := (X - x)^{\ominus}$ (see [8]).

The cone of attainable directions [8] of a set $X \subseteq \mathbb{R}^n$ at a point $x \in X$ is defined by

$$A_X(x) := \Big\{d \in \mathbb{R}^n : \exists\, \delta > 0,\ \exists \text{ a function } \alpha : \mathbb{R} \longrightarrow \mathbb{R}^n \ni \alpha(t) \in X,\ \forall\, t \in (0, \delta),$$
$$\alpha(0) = x, \text{ and } d = \lim_{t \downarrow 0} \frac{\alpha(t) - \alpha(0)}{t} \Big\}.$$

In view of the definition of $F_X(x)$, $A_X(x)$ and $T_X(x)$, we have

$$F_X(x) \subseteq A_X(x) \subseteq T_X(x). \tag{1}$$

Moreover, since $T_X(x)$ is closed, we get

$$\operatorname{cl} F_X(x) \subseteq \operatorname{cl} A_X(x) \subseteq T_X(x). \tag{2}$$

The directional derivative [10] of a real valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given by the following if the limit exists.

$$f'(x, d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}. \tag{3}$$

The convex subdifferential [10] of a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq f(y) - f(x), \ \forall\, y \in \mathbb{R}^n\}.$$

For each $x \in \mathbb{R}^n$, one has $\partial f(x)$ is a nonempty compact convex subset of $\mathbb{R}^n$. Moreover,

$$f'(x, d) := \max_{\xi \in \partial f(x)} \langle \xi, d \rangle, \ d \in \mathbb{R}^n. \tag{4}$$

Definition 2.1. *[3] A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be locally Lipschitz at a point $x_0 \in \mathbb{R}^n$ if there exist positive real numbers $L > 0$ and $\delta > 0$ such that, for each $x, y \in B(x_0, \delta)$,*

$$|f(x) - f(y)| \leq L\|x - y\|.$$

The function f is said to be locally Lipschitz on $\mathbb{R}^n$ if f is locally Lipschitz at every point $x \in \mathbb{R}^n$.

Definition 2.2. *[3] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function on $\mathbb{R}^n$.*

(i) The Clarke directional derivative of f at a point $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ is defined by

$$f^\circ(x, v) := \limsup_{\substack{y \longrightarrow x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

(ii) The Clarke subdifferential of f at a point $x \in \mathbb{R}^n$, $\partial^\circ f(x)$, is defined by

$$\partial^\circ f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq f^\circ(x, v), \ \forall\, v \in \mathbb{R}^n\}.$$

Remark 2.1. *[3] For each $x \in \mathbb{R}^n$, the Clarke directional derivative, $f^\circ(x, \cdot)$, is a sublinear function with respect to the second variable (i.e., positively homogeneous and convex), and the Clarke subdifferential, $\partial^\circ f(x)$, is a nonempty compact convex subset of $\mathbb{R}^n$. Moreover, we have*

$$f^\circ(x, v) = \max_{\xi \in \partial^\circ f(x)} \langle \xi, v \rangle, \ v \in \mathbb{R}^n.$$

Definition 2.3. *[3] A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be regular at a point $x \in \mathbb{R}^n$, if f is locally Lipschitz at x and, for all $v \in \mathbb{R}^n$, the classical directional derivative $f'(x, v)$ exists and*

$$f^\circ(x, v) = f'(x, v), \ \forall\, v \in \mathbb{R}^n.$$

The function f is called regular on $\mathbb{R}^n$, if f is regular at every point $x \in \mathbb{R}^n$.

Definition 2.4. *[29] A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be tangentially convex at a given point $x \in \mathbb{R}^n$ if, for each $d \in \mathbb{R}^n$, the limit in (3) exists, finite and $f'(x, \cdot)$ is a convex function with respect to the second variable.*

Remark 2.2. *It should be noted that, in view of Remark 2.1 and Definition 2.3, if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a regular function at a point $x \in \mathbb{R}^n$, then, f is tangentially convex at x.*

Now, by using the concept of tangential convexity, we define the tangential subdifferential of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $x \in \mathbb{R}^n$ as follows.

Definition 2.5. *[29] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tangentially convex function at a point $x \in \mathbb{R}^n$. The tangential subdifferential of f at the point x is defined by*

$$\partial_T f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, d\rangle \leq f'(x,d),\ \forall\, d \in \mathbb{R}^n\}. \tag{5}$$

Remark 2.3. *[29] It should be noted that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a tangentially convex function at a point $x \in \mathbb{R}^n$, then the directional derivative of f at the point x in the direction $d \in \mathbb{R}^n$ is the support function of $\partial_T f(x)$, i.e., one has*

$$f'(x,d) = \max_{\xi \in \partial_T f(x)} \langle \xi, d\rangle,\ d \in \mathbb{R}^n. \tag{6}$$

Moreover, if the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex at a point $x \in \mathbb{R}^n$, then it is clear that $f'(x,\cdot)$ is a positively homogeneous function, and hence, in view of Definition 2.4, $f'(x,\cdot)$ is sublinear with respect to the second variable. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex at a point $x \in \mathbb{R}^n$, then $\partial_T f(x)$ is a nonempty compact convex set in $\mathbb{R}^n$.

We now give the following lemma which has a crucial role for proving the main results and it has been proved in [6].

Lemma 2.1. *[6] Let X be a subset of $\mathbb{R}^n$, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tangentially convex function at a point $x^* \in X$. Assume that f is continuous at x^* and x^* is a minimizer of f over X. Then*

$$T_X(x^*) \cap \{d \in \mathbb{R}^n : f'(x^*,d) < 0\} = \emptyset. \tag{7}$$

We now consider the following nonconvex nonsmooth constrained optimization problem:

$$\begin{array}{c} \min f(x), \\ \text{s.t.} \\ x \in X, \end{array} \tag{P}$$

with the constraint set X is defined by:

$$X := C \cap K, \tag{8}$$

where

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1,2,\ldots,m\}, \tag{9}$$

C is a subset of $\mathbb{R}^n$ such that $C \cap K \neq \emptyset$, and $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1,2,\ldots,m)$ is a tangentially convex function at a given point $x^* \in X$, and moreover, $f, g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1,2,\ldots,m)$ are continuous functions at the point x^*.

The set of active indices at a point $x \in X$ is defined by

$$I(x) := \{i \in \{1,2,\ldots,m\} : g_i(x) = 0\}. \tag{10}$$

In the following, we give some results related to the problem (P).

Definition 2.6. (**Cone of Tangential Subdifferentials**) *Consider the problem (P). The cone of tangential subdifferentials at a point $x \in X$ is defined by*

$$M(x) := \Big\{ \sum_{i \in I(x)} \lambda_i \partial_T g_i(x) : \lambda_i \geqslant 0,\ i \in I(x)\Big\}. \tag{11}$$

The next lemma states that $M(x)$ is a convex cone. However, $M(x)$ is not necessarily closed.

Lemma 2.2. *[6] $M(x)$ is a convex cone.*

Remark 2.4. *It is easy to check that for each $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ with $\lambda_i g_i(x) = 0,\ i = 1, 2, \cdots, m$, we have*

$$\sum_{i=1}^{m} \lambda_i \partial_T g_i(x) \subseteq M(x). \tag{12}$$

Moreover,

$$\text{cone}\Big(\partial_T g_i(x)\Big) \subseteq \text{cone}\Big(\bigcup_{i\in I(x)} \partial_T g_i(x)\Big) \subseteq M(x),\ i \in I(x), \tag{13}$$

and

$$M(x) = conic\Big(\bigcup_{i\in I(x)} \partial_T g_i(x)\Big). \tag{14}$$

Definition 2.7. *Consider the problem (P). We define the set $M_0(x)$ at a point $x \in X$ by*

$$M_0(x) := \{d \in \mathbb{R}^n : \langle d, \xi_i\rangle < 0,\ \forall\ \xi_i \in \partial_T g_i(x),\ \forall\ i \in I(x)\}. \tag{15}$$

It is not difficult to show that $M_0(x)$ is an open convex set in $\mathbb{R}^n$. For an easy reference, we now present the following lemmas (without proof) which have been proved in [6].

Lemma 2.3. *Consider the problem (P) with $x^* \in X$. Let $M(x^*)$ and $M_0(x^*)$ be given as in* (11) *and* (15)*, respectively. Then the following assertions hold.*

(i) $M(x^*)^\ominus = \{d \in \mathbb{R}^n : \langle d, \xi_i\rangle \leq 0,\ \forall\ \xi_i \in \partial_T g_i(x^*),\ \forall\ i \in I(x^*)\}$.

(ii) *If* $0 \notin \bigcup_{i\in I(x^*)} \partial_T g_i(x^*)$, *then* $\text{int}\, M(x^*)^\ominus = M_0(x^*)$.

(iii) *If* $0 \notin conv\Big(\bigcup_{i\in I(x^*)} \partial_T g_i(x^*)\Big)$, *then* $M(x^*)$ *is closed, and* $M_0(x^*) \neq \emptyset$.

(iv) $\text{cl}\, M_0(x^*) = M(x^*)^\ominus$ *if and only if* $M_0(x^*) \neq \emptyset$.

Lemma 2.4. *Consider the problem (P) with $x^* \in X$. Let $M(x^*)$ and $M_0(x^*)$ be given as in* (11) *and* (15)*, respectively. Then the following assertions hold.*

(i)

$$M(x^*)^\ominus = \bigcap_{i\in I(x^*)} [\partial_T g_i(x^*)]^\ominus = \Big\{d \in \mathbb{R}^n : g_i'(x^*, d) \leq 0,\ \forall\ i \in I(x^*)\Big\}. \tag{16}$$

(ii)

$$M_0(x^*) = \Big\{d \in \mathbb{R}^n : g_i'(x^*, d) < 0,\ \forall\ i \in I(x^*)\Big\}. \tag{17}$$

Lemma 2.5. *Consider the problem (P) with $x^* \in X$. Let $M(x^*)$ and $M_0(x^*)$ be given as in* (11) *and* (15)*, respectively. Then, $T_X(x^*) \subseteq M(x^*)^\ominus$. In addition, if in the problem (P) the set $C \subseteq \mathbb{R}^n$ is open, then*

$$M_0(x^*) \subseteq F_X(x^*).$$

Remark 2.5. *Consider the problem* (P) *with* $x^* \in X$. *Therefore, by* (2) *and Lemma 2.5, it can easily be verified that the following inclusions are satisfied at the point* $x^* \in X$.

$$\mathrm{cl}\, F_X(x^*) \subseteq \mathrm{cl}\, A_X(x^*) \subseteq T_X(x^*) \subseteq M(x^*)^{\ominus}. \tag{18}$$

Moreover, if in the problem (P) *the set* $C \subseteq \mathbb{R}^n$ *is open, then by* (18) *and Lemma 2.5, the following inclusions hold.*

$$\mathrm{cl}\, M_0(x^*) \subseteq \mathrm{cl}\, F_X(x^*) \subseteq \mathrm{cl}\, A_X(x^*) \subseteq T_X(x^*) \subseteq M(x^*)^{\ominus}, \tag{19}$$

while the converse inclusions may not hold, in general.

In the sequel, we recall that the Karush-Kuhn-Tucker (KKT) conditions have been developed and used by many authors under various constraint qualifications [1–3, 7, 11–23, 25, 27–30, 32]. Over the years, many constraint qualifications have been presented by researchers. In this paper, based on the cone of tangential subdifferentials, we give various nonsmooth (generalized) constraint qualifications that are generalizations of the well known constraint qualifications which have been used for optimality of convex or nonconvex programs. We start with the definition of the Karush-Kuhn-Tucker (KKT) conditions in nonsmooth case.

Definition 2.8. ((*KKT*) **Conditions**). *Consider the problem* (P). *Let* $x^* \in X$ *be given. We say that the* (*KKT*) *conditions hold at* x^*, *if there exist* $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ *with* $\lambda_i g_i(x^*) = 0,\ i = 1, 2, \cdots, m$, *such that*

$$0 \in \partial_T f(x^*) + \sum_{i=1}^{m} \lambda_i \partial_T g_i(x^*) + (C - x^*)^{\ominus}. \tag{20}$$

Condition (20) *is called the Karush-Kuhn-Tucker* (*KKT*) *conditions, and* λ_i, $i = 1, 2, \cdots, m$, *in* (20) *are called Lagrange multipliers at* x^*, *and* x^* *is called a* (*KKT*) *point for the problem* (P).

Furthermore, the (*KKT*) *conditions at the point* $x^* \in X$ *in terms of the cone* $M(x^*)$ *are given as follow:*

$$0 \in \partial_T f(x^*) + M(x^*) + (C - x^*)^{\ominus}, \tag{21}$$

or equivalently,

$$\partial_T f(x^*) \bigcap \Big(-M(x^*) + (C - x^*)^{\oplus} \Big) \neq \emptyset. \tag{22}$$

We now present a nonsmooth (generalized) version of the well known constraint qualifications.

Definition 2.9. *Consider the problem* (P) *with* $x^* \in X$. *We say that*

(*i*) *Nonsmooth Abadie's constraint qualification* (*NACQ*) *is satisfied at* x^* *for the problem* (P) *if* $M(x^*)^{\ominus} \subseteq T_X(x^*)$.

(*ii*) *Nonsmooth Guignard's constraint qualification* (*NGCQ*) *is satisfied at* x^* *for the problem* (P) *if* $M(x^*)^{\ominus} \subseteq \mathrm{cl}\,(conv(T_X(x^*)))$.

(*iii*) *Nonsmooth Cottle's constraint qualification* (*NCCQ*) *is satisfied at* x^* *for the problem* (P) *if* $M(x^*)^{\ominus} = \mathrm{cl}\, M_0(x^*)$.

(*iv*) *Nonsmooth Zangwill's constraint qualification* (*NZCQ*) *is satisfied at* x^* *for the problem* (P) *if* $M(x^*)^{\ominus} \subseteq \mathrm{cl}\, F_X(x^*)$.

(*v*) *Nonsmooth Kuhn-Tucker's constraint qualification* (*NKTCQ*) *is satisfied at* x^* *for the problem* (P) *if* $M(x^*)^{\ominus} \subseteq \mathrm{cl}\, A_X(x^*)$.

(*vi*) *Nonsmooth Mangasarian-Fromovitz constraint qualification* (*NMFCQ*) *is satisfied at* x^* *for the problem* (P) *if there exists* $0 \neq d \in \mathbb{R}^n$ *such that*

$$\langle d, \xi_i \rangle < 0,\ \forall\ \xi_i \in \partial_T g_i(x^*),\ \forall\ i \in I(x^*).$$

(*vii*) *Slater's constraint qualification* (*SCQ*) *holds for the problem* (P) *if there exists* $x_0 \in C$ *such that* $g_i(x_0) < 0$ *for all* $i = 1, 2, \ldots, m$.

Remark 2.6. *It should be noted that by Remark 2.5* (18)*, in all statements* (*i*)*,* (*ii*)*,* (*iv*) *and* (*v*) *of Definition 2.9, the equality holds.*

Remark 2.7. *Obviously, the definitions of nonsmooth constraint qualifications which are given by Definition 2.9 reduce to their counterparts in the case of differentiability [5, 9]. Clearly, in view of Remark 2.5* (18) *and Definition 2.9 the following implications hold.*

$$(NZCQ) \Longrightarrow (NKTCQ) \Longrightarrow (NACQ) \Longrightarrow (NGCQ). \tag{23}$$

Moreover, if in the problem (*P*) *the set* $C \subseteq \mathbb{R}^n$ *is open, then in view of Remark 2.5* (19) *and Definition 2.9 the following implications hold.*

$$(NMFCQ) \Longrightarrow (NCCQ) \Longrightarrow (NZCQ) \Longrightarrow (NKTCQ) \Longrightarrow (NACQ) \Longrightarrow (NGCQ). \tag{24}$$

It is worth noting that $(NMFCQ) \Longrightarrow (SCQ)$.

The following example shows that nonsmooth Guignard's constraint qualification is the weakest among the others.

Example 2.1. *Let* $g_j : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $(j = 1, 2, 3)$ *be defined by*

$$\begin{aligned} g_1(x_1, x_2) &:= |x_1| - x_1, \\ g_2(x_1, x_2) &:= -x_2, \\ g_3(x_1, x_2) &:= x_1 x_2, \end{aligned}$$

for all $(x_1, x_2) \in \mathbb{R}^2$*. It is easy to see that all* g_j*'s* $(j = 1, 2, 3)$ *are not smooth functions, and*

$$\begin{aligned} K &= \{(x_1, x_2) \in \mathbb{R}^2 : g_j(x_1, x_2) \leq 0, \ \forall \ j = 1, 2, 3\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, \ x_2 \geq 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 = 0\}. \end{aligned}$$

So, K is not convex. Let $C := \mathbb{R} \times (-1, +\infty)$*. Thus, C is an open set in* $\mathbb{R}^2$*, and we have* $X := C \cap K = K$*, which is not convex. Let* $x^* := (0, 0) \in X$*. Then,* $g_j(x^*) = 0$ *for all* $j = 1, 2, 3$*, and so,* $I(x^*) = \{1, 2, 3\}$*. It is not difficult to check that*

$$g_1'(x^*, t) = |t_1| - t_1, \ g_2'(x^*, t) = -t_2, \ g_3'(x^*, t) = 0, \ \forall \ t := (t_1, t_2) \in \mathbb{R}^2.$$

Clearly, g_1, g_2, g_3 *are tangentially convex at* x^**. Moreover, one can see that*

$$\begin{aligned} \partial_T g_1(x^*) &= \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \leq 0, \ u_2 = 0\}, \\ \partial_T g_2(x^*) &= \{(0, -1)\}, \\ \partial_T g_3(x^*) &= \{(0, 0)\}. \end{aligned}$$

Therefore, by Remark 2.4 (14)*, we have*

$$M(x^*) = conic\Big(\bigcup_{j \in I(x^*)} \partial_T g_j(x^*) \Big) = \mathbb{R}^2_-,$$

which is closed. Obviously, we obtain that $M(x^*)^\ominus = \mathbb{R}^2_+$*. On the other hand, one can show that*

$$T_X(x^*) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \geq 0, \ u_2 \geq 0, \ u_1 u_2 = 0\},$$

which is not convex. Hence, $M(x^*)^\ominus = \mathbb{R}^2_+ = \mathrm{cl}\,(conv(T_X(x^*)))$*, while* $M(x^*)^\ominus \neq T_X(x^*)$*, i.e.,* (*NGCQ*) *holds at* x^**, but* (*NACQ*) *does not hold at* x^**, and hence* (*note that C is open*)*, in view of Remark 2.7* (23) *and* (24)*, nonsmooth constraint qualifications* (*NMFCQ*)*,* (*NCCQ,*) (*NZCQ*) *and* (*NKTCQ*) *are not satisfied at* x^**. Clearly, Slater's constraint qualification* (*SCQ*) *does not hold. Consequently,* (*NGCQ*) *holds at* x^**, while* (*SCQ*)*,* (*NMFCQ*)*,* (*NCCQ*)*,* (*NZCQ*)*,* (*NKTCQ*) *and* (*NACQ*) *do not hold at* x^**.*

In the following, we give a definition of pseudoconvexity for tangentially convex functions, which was given in [29].

Definition 2.10. *[29] Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tangentially convex function at a given point $x \in \mathbb{R}^n$. We say that g is pseudoconvex at x, if*

$$y \in \mathbb{R}^n, \ \ g(y) < g(x) \implies g'(x, y - x) < 0.$$

3. Necessary and sufficient conditions for optimality of the problem (P)

In this section, we first present a new constraint qualification which provides characterizations for optimal solutions of the problem (P). Moreover, we show that under this constraint qualification, the (KKT) conditions are necessary and sufficient for optimality of the problem (P). We now introduce a new constraint qualification called "tangentially constraint qualification" ((TCQ) in short).

Definition 3.1. *Consider the problem (P), and let $x^* \in X = C \cap K$ be given. We say that "tangentially constraint qualification" ((TCQ) in short) holds at the point x^*, if*

$$C - x^* \subseteq T_X(x^*). \tag{25}$$

In the following, we show that tangentially constraint qualification (TCQ) is independent of the other nonsmooth constraint qualifications. We first give a sufficient condition under which a point $x^* \in X$ is a global minimizer of the problem (P) over X.

Theorem 3.1. (Karush-Kuhn-Tucker Sufficient Conditions). *Consider the problem (P) with the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex and pseudoconvex at a given point $x^* \in X$. Assume that x^* is a (KKT) point for the problem (P). If $X - x^* \subseteq T_X(x^*)$, then, x^* is a global minimizer of the problem (P) over X.*

Proof. Suppose that x^* is a (KKT) point for problem (P). Therefore, by Definition 2.8, there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ with $\lambda_i g_i(x^*) = 0, \ i = 1, 2, \ldots, m$, such that

$$0 \in \partial_T f(x^*) + \sum_{i=1}^{m} \lambda_i \partial_T g_i(x^*) + (C - x^*)^{\ominus}.$$

Then there exist $\xi^* \in \partial_T f(x^*)$, $\xi_i^* \in \partial_T g_i(x^*), \ i = 1, 2, \ldots, m$, and $u \in (C - x^*)^{\ominus}$ such that

$$0 = \xi^* + \sum_{i=1}^{m} \lambda_i \xi_i^* + u.$$

Using $g_i(x^*) < 0$ for each $i \notin I(x^*)$ and the fact that $\lambda_i g_i(x^*) = 0$ for all $i = 1, 2, \ldots, m$, we get $\lambda_i = 0$ for all $i \notin I(x^*)$. So,

$$0 = \xi^* + \sum_{i \in I(x^*)} \lambda_i \xi_i^* + u.$$

Let $x \in X$ be arbitrary. Since $u \in (C - x^*)^{\ominus}$ and $X \subseteq C$, we conclude from the later equality and Remark 2.3

that

$$
\begin{aligned}
0 &= \langle \xi^* + \sum_{i\in I(x^*)} \lambda_i \xi_i^* + u, x - x^* \rangle \\
&= \langle \xi^*, x - x^* \rangle + \sum_{i\in I(x^*)} \lambda_i \langle \xi_i^*, x - x^* \rangle + \langle u, x - x^* \rangle \\
&\le \max_{\xi\in\partial_T f(x^*)} \langle \xi, x - x^* \rangle + \sum_{i\in I(x^*)} \lambda_i \max_{\xi_i\in\partial_T g_i(x^*)} \langle \xi_i, x - x^* \rangle + \langle u, x - x^* \rangle \\
&= f'(x^*, x - x^*) + \sum_{i\in I(x^*)} \lambda_i g_i'(x^*, x - x^*) + \langle u, x - x^* \rangle \\
&\le f'(x^*, x - x^*) + \sum_{i\in I(x^*)} \lambda_i g_i'(x^*, x - x^*).
\end{aligned}
$$

Thus,

$$
f'(x^*, x - x^*) + \sum_{i\in I(x^*)} \lambda_i g_i'(x^*, x - x^*) \ge 0, \ \forall\, x \in X. \tag{26}
$$

Due to Lemma 2.5, one has $T_X(x^*) \subseteq M(x^*)^{\ominus}$. So, in view of the hypothesis that $X - x^* \subseteq T_X(x^*)$, it follows that $X - x^* \subseteq M(x^*)^{\ominus}$, and hence, by Lemma 2.4 (*i*), we obtain that

$$
g_i'(x^*, x - x^*) \le 0, \ \forall\, x \in X, \ \forall\, i \in I(x^*).
$$

This together with (26) implies that

$$
f'(x^*, x - x^*) \ge 0, \ \forall\, x \in X. \tag{27}
$$

Now, by the pseudoconvexity of f at the point x^*, we conclude from (27) and Definition 2.10 that $f(x^*) \le f(x)$ for all $x \in X$. This means that x^* is a global minimizer of the problem (P) over X. □

We now show that under new constraint qualification (TCQ), the (KKT) conditions are necessary and sufficient for optimality of the problem (P).

Theorem 3.2. (Karush-Kuhn-Tucker Necessary and Sufficient Conditions). *Consider the problem (P), and let $x^* \in X = C \cap K$ be given. Assume that constraint qualification (TCQ) holds at x^*.*

(*i*) *Suppose that the set C is convex and the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex at the point x^*. If x^* is a global minimizer of the problem (P) over X, then, x^* is a (KKT) point for the problem (P).*

(*ii*) *If the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is tangentially convex and pseudoconvex at x^*, and x^* is a (KKT) point for the problem (P), then, x^* is a global minimizer of the problem (P) over X.*

Proof. (*i*). Suppose that x^* is a global minimizer of the problem (P) over X. Then, by Lemma 2.1, $f'(x^*, d) \ge 0$ for all $d \in T_X(x^*)$. This together with Remark 2.3 implies that $\max_{\xi\in\partial_T f(x^*)} \langle \xi, d \rangle \ge 0$ for all $d \in T_X(x^*)$. Therefore,

$$
\inf_{d\in T_X(x^*)} \max_{\xi\in\partial_T f(x^*)} \langle \xi, d \rangle \ge 0. \tag{28}
$$

Since, by the hypothesis, constraint qualification (TCQ) holds at x^*, it follows that $C - x^* \subseteq T_X(x^*)$. So, we conclude from (28) that

$$
\inf_{x\in C} \max_{\xi\in\partial_T f(x^*)} \langle \xi, x - x^* \rangle \ge 0. \tag{29}
$$

By Remark 2.3, $\partial_T f(x^*)$ is compact convex and C is also convex, thus by Sion's minimax theorem in [24], we get

$$\max_{\xi \in \partial_T f(x^*)} \inf_{x \in C} \langle \xi, x - x^* \rangle \geq 0. \tag{30}$$

This implies that there exists $\bar{\xi} \in \partial_T f(x^*)$ such that $\langle \bar{\xi}, x - x^* \rangle \geq 0$ for all $x \in C$. So, there exists $\bar{\xi} \in \partial_T f(x^*)$ such that $-\bar{\xi} \in (C - x^*)^{\ominus}$. Therefore,

$$0 \in \partial_T f(x^*) + (C - x^*)^{\ominus}. \tag{31}$$

In view of Definition 2.6, one has $0 \in M(x^*)$, and so, it follows from (31) that

$$0 \in \partial_T f(x^*) + M(x^*) + (C - x^*)^{\ominus}.$$

This together with Definition 2.6 implies that there exist $\bar{\lambda}_i \geq 0$ and $\bar{\xi}_i \in \partial_T g_i(x^*)$ $(i \in I(x^*))$ such that

$$0 \in \partial_T f(x^*) + \sum_{i \in I(x^*)} \bar{\lambda}_i \bar{\xi}_i + (C - x^*)^{\ominus}.$$

Put $\bar{\lambda}_i = 0$ for each $i \notin I(x^*)$. Since, $g_i(x^*) = 0$ for all $i \in I(x^*)$, we deduce that $\bar{\lambda}_i g_i(x^*) = 0$ for all $i = 1, 2, \ldots, m$. Hence,

$$0 \in \partial_T f(x^*) + \sum_{i=1}^{m} \bar{\lambda}_i \partial_T g_i(x^*) + (C - x^*)^{\ominus},$$

where $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m \geq 0$ and $\bar{\lambda}_i g_i(x^*) = 0,\ i = 1, 2, \ldots, m$. Thus, x^* is a (KKT) point for the problem (P).

(*ii*). Suppose that f is tangentially convex and pseudoconvex at the point x^*, and x^* is a (KKT) point for the problem (P). In view of the fact that $X - x^* \subseteq C - x^*$, and by the hypothesis constraint qualification (TCQ) holds at x^*, i.e., $C - x^* \subseteq T_X(x^*)$, we obtain that $X - x^* \subseteq T_X(x^*)$. Hence, we conclude from Theorem 3.1 that x^* is a global minimizer of the problem (P) over X. □

Corollary 3.1. *Consider the problem* (*P*) *with C is convex and the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Gâteaux differentiable and pseudoconvex at a given point $x^* \in X$. Assume that constraint qualification* (*TCQ*) *holds at x^*. Then, x^* is a global minimizer of the problem* (*P*) *over X if and only if x^* is a* (*KKT*) *point for the problem* (*P*).

Proof. This is an immediate consequence of Theorem 3.2, because every Gâteaux differentiable function at the point x^* is tangentially convex at x^*. Note that, in this case, we have $\partial_T f(x^*) = \{\nabla f(x^*)\}$. □

Corollary 3.2. *Consider the problem* (*P*) *with C is convex and the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex over X, and let $x^* \in X$ be given. Assume that constraint qualification* (*TCQ*) *holds at the point x^*. Then, x^* is a global minimizer of the problem* (*P*) *over X if and only if x^* is a* (*KKT*) *point for the problem* (*P*).

Proof. This is an immediate consequence of Theorem 3.2, because every convex function f defined on X is tangentially convex at every point $x \in X$, and f is also pseudoconvex at every point $x \in X$. Note that, in this case, one has $\partial f(x^*) = \partial_T f(x^*)$. □

Corollary 3.3. *Consider the problem* (*P*) *with C is convex and the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is regular and pseudoconvex at a given point $x^* \in X$. Assume that constraint qualification* (*TCQ*) *holds at x^*. Then, x^* is a global minimizer of the problem* (*P*) *over X if and only if x^* is a* (*KKT*) *point for the problem* (*P*).

Proof. This is an immediate consequence of Theorem 3.2 because, in view of Remark 2.2, every regular function at the point x^* is tangentially convex at x^*. Moreover, in this case, we conclude from Definition 2.3 that $\partial^\circ f(x^*) = \partial_T f(x^*)$. □

In view of Theorem 3.2 and its corollaries, one can see that under constraint qualification (TCQ), the (KKT) conditions are necessary and sufficient for optimality of the problem (P) with various objective functions. The obtained results recapture the corresponding known results of [7, 11, 13, 25, 29, 30]. The following examples illustrate Theorem 3.2 and show that constraint qualification (TCQ) is independent of the other nonsmooth constraint qualifications which have been given in Section 2.

Example 3.1. *Consider the problem* (*P*) *with the following objective and constraint functions.*

$$\min\ f(x),$$
$$s.t.$$
$$x \in X := C \cap K,$$

where $f(x_1, x_2) := x_1^3 + x_1$,

$$K := \{(x_1, x_2) \in \mathbb{R}^2 : g_i(x_1, x_2) \le 0,\ i = 1, 2, 3\},$$

with $g_1(x_1, x_2) := 3x_2^3 - x_1$, $g_2(x_1, x_2) := -\sin(x_1) - 2x_2^2$ *and* $g_3(x_1, x_2) := x_2 - x_1^2$. *Let* $C := \mathbb{R}_+ \times \{0\}$. *We see that* $X = C \cap K$ *is not convex and* $I(x^*) = \{1, 2, 3\}$ *whenever* $x^* := (0, 0) \in X$. *It is easy to check that the function* f *is pseudoconvex on* $\mathbb{R}^2$, *and so at* x^*, *and moreover,* $f'(x^*, t) = t_1$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$. *So,* f *is tangentially convex at* x^*. *Furthermore, we have* $g_1'(x^*, t) = -t_1$, $g_2'(x^*, t) = -t_1$, $g_3'(x^*, t) = t_2$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$. *It is clear that the functions* g_1, g_2 *and* g_3 *are tangentially convex at* x^*. *Also, one can see that* $\partial_T g_1(x^*) = \{(-1, 0)\}$, $\partial_T g_2(x^*) = \{(-1, 0)\}$, $\partial_T g_3(x^*) = \{(0, 1)\}$ *and* $\partial_T f(x^*) = \{(1, 0)\}$. *Thus,* $M(x^*) = \mathbb{R}_- \times \mathbb{R}_+$ *and* $M(x^*)^\ominus = \mathbb{R}_+ \times \mathbb{R}_-$. *It is easy to show that* $T_X(x^*) = \mathbb{R}_+ \times \{0\} = C - x^*$, *and so, constraint qualification* (*TCQ*) *holds at* x^*. *Hence, all hypotheses of Theorem 3.2 are satisfied. Clearly,* x^* *is a global minimizer of the problem* (*P*) *over* X, *and by choosing* $\lambda_1, \lambda_2 \ge 0$ *with* $\lambda_1 + \lambda_2 \le 1$ *and* $\lambda_3 = 0$ (*for example,* $\lambda_1 := \frac{1}{2}$, $\lambda_2 := \frac{1}{2}$ *and* $\lambda_3 := 0$), *it follows from Theorem 3.2*(*i*) *that* x^* *is a* (*KKT*) *point for the problem* (*P*).

Moreover, note that $M(x^*)^\ominus = \mathbb{R}_+ \times \mathbb{R}_- \ne \mathbb{R}_+ \times \{0\} = \mathrm{cl}\,(conv(T_X(x^*)))$ (*note that* $T_X(x^*)$ *is closed and convex*), *and hence, nonsmooth constraint qualification* (*NGCQ*) *does not hold at* x^*. *Therefore, in view of* (23), *nonsmooth constraint qualifications* (*NZCQ*), (*NKTCQ*) *and* (*NACQ*) *do not hold at* x^*. *Consequently, nonsmooth constraint qualifications* (*NZCQ*), (*NKTCQ*), (*NACQ*) *and* (*NGCQ*) *do not hold at* x^*, *while constraint qualification* (*TCQ*) *holds at* x^*.

Example 3.2. *Let* $K := \left\{(x_1, x_2) \in \mathbb{R}^2 : g_1(x_1, x_2) := |x_2| - x_1 \le 0\right\}$ *and* $C := \mathbb{R}_+ \times \{0\}$. *Then,* $X = C \cap K = C$, *which is a convex set. So,* $T_X(x^*) = \mathbb{R}_+ \times \{0\}$ *whenever* $x^* := (0, 0) \in X$. *Moreover, we have* $g_1(x^*) = 0$. *Also,* $g_1'(x^*, t) = |t_2| - t_1$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$, *and thus,* g_1 *is tangentially convex at* x^*. *Then,* $\partial_T g_1(x^*) = conv\,\{(-1, -1), (-1, 1)\}$, *and hence,* $M(x^*) = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| + x_1 \le 0\right\}$. *Thus,*

$$M(x^*)^\ominus = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \le x_1\right\}.$$

Clearly, $C - x^* = T_X(x^*)$, $M(x^*)^\ominus \ne T_X(x^*)$ *and* $M(x^*)^\ominus \ne \mathrm{cl}\,(conv(T_X(x^*)))$. *Therefore, constraint qualification* (*TCQ*) *holds at* x^*, *while constraint qualification* (*NGCQ*), *and hence, in view of* (23), *the other nonsmooth constraint qualifications do not hold at* x^*.

Example 3.3. *Let* $K := \{(x_1, x_2) \in \mathbb{R}^2 : g_i(x_1, x_2) \le 0,\ i = 1, 2\}$ *with* $g_1(x_1, x_2) := 1 - (x_1 - 3)^2 - x_2^2$ *and* $g_2(x_1, x_2) := |x_2| - x_1$ *for all* $(x_1, x_2) \in \mathbb{R}^2$. *Let* $C := \mathbb{R}_+ \times \mathbb{R}$. *We see that* $X = C \cap K$ *is a nonconvex set. It is easy to check that* $g_1(x^*) = -8 \ne 0$ *and* $g_2(x^*) = 0$ *whenever* $x^* := (0, 0) \in X$. *Thus,* $I(x^*) = \{2\}$. *The functions* g_1 *and* g_2 *are tangentially convex at* x^* *because* $g_1'(x^*, t) = 6t_1$ *and* $g_2'(x^*, t) = |t_2| - t_1$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$. *Moreover, we obtain that* $\partial_T g_2(x^*) = conv\,\{(-1, -1), (-1, 1)\}$, *and so,* $M(x^*) = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| + x_1 \le 0\right\}$. *Therefore,*

$$M(x^*)^\ominus = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \le x_1\right\}.$$

It is easy to show that $T_X(x^*) = \left\{(t_1, t_2) \in \mathbb{R}^2 : |t_2| \le t_1\right\}$, *and hence,* $C - x^* \nsubseteq T_X(x^*)$. *Thus, constraint qualification* (*TCQ*) *does not hold at* x^*, *while* $T_X(x^*) = M(x^*)^\ominus$, *and so,* $M(x^*)^\ominus = \mathrm{cl}\,(conv(T_X(x^*)))$ *because* $T_X(x^*)$ *is closed and convex. Thus nonsmooth constraint qualifications* (*NACQ*) *and* (*NGCQ*) *hold at* x^*.

Example 3.4. *Let*

$$K := \left\{(x_1, x_2) \in \mathbb{R}^2 : g_i(x_1, x_2) \leq 0,\ i = 1, 2\right\} = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\},$$

where $g_1(x_1, x_2) := 1 - (x_1 + 1)^2 - x_2^2$ *and* $g_2(x_1, x_2) := |x_1| - x_2$ *for all* $(x_1, x_2) \in \mathbb{R}^2$. *Let*

$$C := (\mathbb{R}_- \times \mathbb{R}_+) \cup \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}.$$

Since $g_1'(x^*, t) = -2t_1$ *and* $g_2'(x^*, t) = |t_1| - t_2$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$, *it follows that the functions* g_1 *and* g_2 *are tangentially convex at* x^* *whenever* $x^* := (0, 0) \in X = C \cap K = K$. *Note that* X *is a convex set because* K *is convex. Moreover,* $g_1(x^*) = g_2(x^*) = 0$, *and so,* $I(x^*) = \{1, 2\}$. *Therefore,* $\partial_T g_1(x^*) = \{(-2, 0)\}$ *and* $\partial_T g_2(x^*) = conv\{(-1, -1), (1, -1)\}$. *Thus,* $M(x^*) = \mathbb{R}^2_- \cup \{(t_1, t_2) \in \mathbb{R}^2 : 0 \leq t_1 \leq -t_2\}$, *and hence,* $M(x^*)^\ominus = \{(t_1, t_2) \in \mathbb{R}^2 : 0 \leq t_1 \leq t_2\}$. *It is not difficult to show that* $T_X(x^*) = \left\{(t_1, t_2) \in \mathbb{R}^2 : 0 \leq t_1 \leq t_2\right\}$. *Therefore, it is obvious that* $M(x^*)^\ominus = T_X(x^*)$, $M(x^*)^\ominus = \mathrm{cl}\,(conv(T_X(x^*)))$ *and* $C - x^* \nsubseteq T_X(x^*)$. *Hence, constraint qualification* (*TCQ*) *does not hold at* x^*, *but nonsmooth constraint qualifications* (*NACQ*) *and* (*NGCQ*) *hold at* x^*. *Moreover, by a simple calculation one can show that* $\mathrm{cl}\, M_0(x^*) = \mathrm{cl}\, F_X(x^*) = \mathrm{cl}\, A_X(x^*) = T_X(x^*) = M(x^*)^\ominus$. *Also, for* $0 \neq d := (4, 1) \in \mathbb{R}^n$, *we have* $\langle d, \xi_i \rangle < 0$ *for all* $\xi_i \in \partial_T g_i(x^*)$ *and all* $i \in I(x^*)$. *Consequently, the other nonsmooth constraint qualifications* (*NMFCQ*), (*NCCQ*), (*NZCQ*) *and* (*NKTCQ*), *also hold at* x^*, *while constraint qualification* (*TCQ*) *does not hold at* x^*.

Remark 3.1. *In view of Example 3.1, Example 3.2, Example 3.3 and Example 3.4, we see that in any case that the constraint set* X *is convex or nonconvex, the constraint qualification* (*TCQ*) *is independent of the other nonsmooth constraint qualifications. Note that in the case that the set* $C := \mathbb{R}^n$, *constraint qualification* (*TCQ*) *implies* (*NGCQ*). *Indeed, if* $C := \mathbb{R}^n$, *then,* $C - x^* = \mathbb{R}^n$ (*for any* $x^* \in X$). *Now, we assume that* (*TCQ*) *holds at* x^*. *Thus,* $\mathbb{R}^n = C - x^* \subseteq T_X(x^*)$, *and so,* $T_X(x^*) = \mathbb{R}^n$. *Hence,* $M(x^*)^\ominus \subseteq \mathbb{R}^n = \mathrm{cl}\,(conv(T_X(x^*)))$, *i.e.,* (*NGCQ*) *holds at* x^*.

Therefore, as a consequence, it should be noted that in addition to the easiness of using the constraint qualification (*TCQ*), *an important advantage of* (*TCQ*), *is that* (*TCQ*) *is a constraint qualification under which* (*KKT*) *conditions are necessary and sufficient for optimality of the nonconvex nonsmooth optimization problem* (*P*) *without any further assumption* (*see Theorem 3.2 and its corollaries*), *while the other nonsmooth constraint qualifications together with a further assumption* (*closedness assumption, i.e.,* $M(x^*)$ *is closed*) *implies that* (*KKT*) *conditions are only "necessary" for optimality of the problem* (*P*) (*see [6, Theorem 3.1 and Corollary 3.1]*).

Finally, we investigate the connection between the constraint qualification (*TCQ*) and Slater's constraint qualification (*SCQ*).

Example 3.5. *Let* $g_1(x_1, x_2) := |x_2| - x_1$ *and* $g_2(x_1, x_2) := -\sin(x_1) - 2x_2^3$ *for all* $(x_1, x_2) \in \mathbb{R}^2$. *Let* $K = \left\{(x_1, x_2) \in \mathbb{R}^2 : g_i(x_1, x_2) \leq 0,\ i = 1, 2\right\}$, *and let*

$$C := \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| - \frac{3}{10}x_1 \leq 0\}.$$

We see that $X := C \cap K$ *is not convex. It is easy to check that* $g_1(x^*) = g_2(x^*) = 0$ *whenever* $x^* := (0, 0) \in X$. *Thus,* $I(x^*) = \{1, 2\}$. *Moreover, we have* $g_1'(x^*, t) = |t_2| - t_1$ *and* $g_2'(x^*, t) = -t_1$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$. *It is clear that the functions* g_1 *and* g_2 *are tangentially convex at* x^*. *Clearly,* (*SCQ*) *holds at the point* $x_0 := (10, 1) \in C$ *because* $g_1(x_0) < 0$ *and* $g_2(x_0) < 0$. *Moreover, we have* $C - x^* = \mathbb{R}_+ \times \mathbb{R} \nsubseteq \mathbb{R}_+ \times \mathbb{R}_- = T_X(x^*)$, *i.e.,* (*TCQ*) *does not hold at* x^*. *Hence,* (*SCQ*) *does not imply* (*TCQ*).

The following example also shows that (*TCQ*) does not imply (*SCQ*).

Example 3.6. *Let* $g_1(x_1, x_2) := -sin(x_1) - 2x_2^2$, $g_2(x_1, x_2) := |x_2| + 2x_2$ *and* $g_3(x_1, x_2) := |x_2| - x_1$ *for all* $(x_1, x_2) \in \mathbb{R}^2$. *Let* $K := \left\{(x_1, x_2) \in \mathbb{R}^2 : g_i(x_1, x_2) \leq 0,\ i = 1, 2, 3\right\}$, *and let* $C := \mathbb{R}_+ \times \{0\}$. *Clearly,* $X = C \cap K$ *is not a convex set, and also,* $g_1(x^*) = g_2(x^*) = g_3(x^*) = 0$ *whenever* $x^* := (0, 0) \in X$. *Therefore,* $I(x^*) = \{1, 2, 3\}$, $g_1'(x^*, t) = -t_1$, $g_2'(x^*, t) = 2t_2 + |t_2|$ *and* $g_3'(x^*, t) = |t_2| - t_1$ *for all* $t := (t_1, t_2) \in \mathbb{R}^2$. *Then the functions* g_1, g_2 *and* g_3 *are tangentially convex at* x^*. *Therefore,* $C - x^* = \mathbb{R}_+ \times \{0\} = T_X(x^*)$, *and so,* (*TCQ*) *holds at* x^*. *On the other hand, one has* $g_2(x) = 0$ *for all* $x \in C$, *i.e.,* (*SCQ*) *does not hold.*

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