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On a Three-Species Stochastic Hybrid Lotka-Volterra System with Distributed Delay and Lévy Noise

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Abstract. In this paper, a three-species stochastic hybrid Lotka-Volterra system with distributed delay and Lévy noise is proposed and studied by using stochastic analytical techniques. First, the existence and uniqueness of global positive solution with positive initial condition is proved. Then, sufficient conditions for persistence in mean and extinction of each species are established. Finally, some numerical simulations are provided to support our results.

1. Introduction

The relationship between predators and their preys has long been and will continue to be one of the dominant themes in ecology due to its ubiquity and importance ([1], [2], [3]). The most significant advance in population dynamics was the widely-accepted two-species Lotka-Volterra system ([4], [5]). However, several researchers found that numerous critical behaviors can only be exhibited by models with three or more species ([6]). The classical three-species food chain model can be expressed as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right], \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right], \\ \dfrac{\mathrm{d}x_3(t)}{\mathrm{d}t} = x_3(t)\left[-r_3 + a_{32}x_2(t) - a_{33}x_3(t)\right], \end{cases} \tag{1}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the population sizes of prey, intermediate predator and top predator, respectively. r_i and a_{ij} are positive constants.

However, the deterministic system has its limitation in mathematical modeling of ecosystems since the parameters involved in the system are unable to capture the influence of environmental noises ([7], [8]). Hence, it is important to consider the stochastic population systems. Assume that the growth rate and the death rates are affected by white noises, i.e., $r_1 \hookrightarrow r_1 + \sigma_1\dot{W}_1(t)$, $-r_2 \hookrightarrow -r_2 + \sigma_2\dot{W}_2(t)$, $-r_3 \hookrightarrow -r_3 + \sigma_3\dot{W}_3(t)$,

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where $W_i(t)$ are mutually independent standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and the three-species food chain model with white noises can be expressed as follows ([9]):

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t). \end{cases} \tag{2}$$

On the other hand, population system may be affected by telephone noises, which can cause the system to switch from one environmental regime to another ([10], [11], [12]). Some authors claimed that the regime switching can be described by a continuous-time Markov chain $\rho(t)$ with finite-state space ([11], [12], [13], [14], [15], [16], [17], [18]). System (2) under regime switching can be described by the following system:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1(\rho(t)) - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t + \sigma_1(\rho(t))x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2(\rho(t)) + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t + \sigma_2(\rho(t))x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3(\rho(t)) + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3(\rho(t))x_3(t)\mathrm{d}W_3(t). \end{cases} \tag{3}$$

Now, let us further improve system (3) by considering time-delay and another type of environmental noise-Lévy noise. On the one hand, "all species should exhibit time-delay" in the real world, and incorporating time-delays in biological systems makes the systems much more realistic than those without time-delays ([19], [20], [21]). As is known, systems with discrete time-delays and those with continuously distributed time-delays do not contain each other. However, systems with S-type distributed time-delays contain both ([22], [23]). On the other hand, some scholars pointed out that Lévy noise can be used to describe some sudden environmental perturbations, for instance, earthquakes and hurricanes ([24], [25], [26], [27], [28], [29], [30]). Recently, stochastic population systems have been received great attention ([31], [32], [33], [34], [35], [36], [37], [38], [39]). However, to the best of our knowledge, results about stochastic hybrid delay population systems with Lévy noise have rarely been reported. So, in this paper we concern the dynamics of the following stochastic hybrid three-species food chain model with distributed time-delays and Lévy noise:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(\rho(t)) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(\rho(t)) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(\rho(t)) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right], \end{cases} \tag{4}$$

where $\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$, $\mathcal{S}_i(t,\rho(t)) = \sigma_i(\rho(t))\mathrm{d}W_i(t) + \int_{\mathbb{Z}} \gamma_i(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)$, $\int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$ are Lebesgue-Stieltjes integrals, $\tau_{ji} > 0$ are time-delays, $\mu_{ji}(\theta)$ are nondecreasing bounded variation functions defined on $[-\tau, 0]$, $\tau = \max_{i,j=1,2,3}\left\{\tau_{ji}\right\}$, $\rho(t)$ is a right-continuous Markov chain, taking values in $\mathbb{S} = \{1,2,...,S\}$, N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $[0,+\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu) = N(\mathrm{d}t,\mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$, $\gamma_j(\mu,\rho(t)) > -1$ $(\mu \in \mathbb{Z})$ are bounded functions $(j = 1,2,3)$.

The structure of this paper is as follows. In Section 2, we show the existence and uniqueness of global positive solution. In Section 3, we obtain sufficient conditions for persistence in mean and extinction of each species. In Section 4, some numerical simulations are provided to verify the correctness of the theoretical results.

2. Existence and uniqueness of global positive solution

Throughout this paper, the generator $\Gamma = (\gamma_{ij})_{S\times S}$ of $\rho(t)$ is given by

$$P\left\{\rho(t+\varsigma) = j|\rho(t) = i\right\} = \begin{cases} \gamma_{ij}\varsigma + o(\varsigma), & i \neq j, \\ 1 + \gamma_{ij}\varsigma + o(\varsigma), & i = j, \end{cases} \tag{5}$$

where $\varsigma > 0$. Here γ_{ij} represents the transition rate from i to j and $\gamma_{ij} \geq 0$ if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Assume that $\rho(t)$, $W_1(t)$, $W_2(t)$, $W_3(t)$ and N are mutually independent and $\rho(t)$ is irreducible. Hence, system (4) can switch from any regime to any other regime and $\rho(t)$ has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, ..., \pi_S) \in \mathbb{R}^{1\times S}$ which can be determined by solving $\pi\Gamma = 0$, subject to $\sum_{i=1}^{S}\pi_i = 1$ and $\pi_i > 0$, $\forall i \in \mathbb{S}$. Denote

$$\begin{cases} B_1(\cdot) = r_1(\cdot) - \dfrac{\sigma_1^2(\cdot)}{2} - \displaystyle\int_{\mathbb{Z}} [\gamma_1(\mu,\cdot) - \ln(1+\gamma_1(\mu,\cdot))]\,\lambda(\mathrm{d}\mu), \\ B_j(\cdot) = r_j(\cdot) + \dfrac{\sigma_j^2(\cdot)}{2} + \displaystyle\int_{\mathbb{Z}} \left[\gamma_j(\mu,\cdot) - \ln(1+\gamma_j(\mu,\cdot))\right]\lambda(\mathrm{d}\mu),\ (j=2,3), \\ A_{ij} = a_{ij} + \displaystyle\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta)\ \left(i,j=1,2,3,\ (i,j)\neq(1,3),\ (i,j)\neq(3,1)\right), \\ \Sigma_1 = \displaystyle\sum_{i=1}^{S}\pi_i B_1(i),\ \Sigma_2 = -\sum_{i=1}^{S}\pi_i B_2(i) + \frac{A_{21}}{A_{11}}\Sigma_1,\ \Sigma_3 = -\sum_{i=1}^{S}\pi_i B_3(i) + \frac{A_{32}}{A_{22}}\Sigma_2, \\ M_{33}^{|A|} = \begin{vmatrix} A_{11} & A_{12} \\ -A_{21} & A_{22}\end{vmatrix},\ M_{33}^{|A_1|} = \begin{vmatrix} \Sigma_1 & A_{12} \\ \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 & A_{22}\end{vmatrix},\ M_{33}^{|A_2|} = \begin{vmatrix} A_{11} & \Sigma_1 \\ -A_{21} & \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1\end{vmatrix}, \\ A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ -A_{21} & A_{22} & A_{23} \\ 0 & -A_{32} & A_{33}\end{pmatrix},\ A_1 = \begin{pmatrix} \Sigma_1 & A_{12} & 0 \\ \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 & A_{22} & A_{23} \\ \Sigma_3 - \frac{A_{32}}{A_{22}}\Sigma_2 & -A_{32} & A_{33}\end{pmatrix}, \\ A_2 = \begin{pmatrix} A_{11} & \Sigma_1 & 0 \\ -A_{21} & \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 & A_{23} \\ 0 & \Sigma_3 - \frac{A_{32}}{A_{22}}\Sigma_2 & A_{33}\end{pmatrix},\ A_3 = \begin{pmatrix} A_{11} & A_{12} & \Sigma_1 \\ -A_{21} & A_{22} & \Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 \\ 0 & -A_{32} & \Sigma_3 - \frac{A_{32}}{A_{22}}\Sigma_2\end{pmatrix}. \end{cases}$$

In this paper, we impose the following assumptions:
(H_1) $r_j(i) > 0$, $a_{jk} > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu,i) \leq \gamma_j^*(i)$ ($\mu \in \mathbb{Z}$), $\forall i \in \mathbb{S}$, $j,k = 1,2,3$.
(H_2) $A_{22}A_{33}M_{33}^{|A|} > A_{12}A_{21}A_{23}A_{32}$.

Theorem 2.1. *For any initial condition $\left(\phi, \rho(0)\right) \in C([-\tau,0],\mathbb{R}_+^3)\times\mathbb{S}$, system (4) has a unique global positive solution on $t \in [-\tau,+\infty)$ a.s.*

Proof. Since the coefficients of system (4) are locally Lipschitz continuous, from [40] and [41] we observe that system (4) admits a unique local solution $\mathbf{x}^{loc}(t)$ on $t \in [-\tau, \tau_e)$ a.s., where τ_e is the explosion time. To prove $\tau_e = +\infty$ a.s., consider the following stochastic hybrid delay differential equation:

$$\begin{cases} \mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}X_2(t) = X_2(t)\left[(-r_2(\rho(t)) + \mathcal{D}_{21}(X_1)(t) - \mathcal{D}_{22}(X_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ \mathrm{d}X_3(t) = X_3(t)\left[(-r_3(\rho(t)) + \mathcal{D}_{32}(X_2)(t) - \mathcal{D}_{33}(X_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right]. \end{cases} \tag{6}$$

Thanks to Theorem 2.1 in [42] and the stochastic comparison theorem, we deduce that system (6) admits a unique global positive solution on $t \in [-\tau,+\infty)$ a.s. By the stochastic comparison theorem, $x_i(t) \leq X_i(t)$ a.s., $t \in [0,+\infty)$ $(i = 1,2,3)$, which implies $\tau_e = +\infty$ a.s. The proof is complete. □

3. Extinction and persistence in mean

Lemma 3.1. *([43]) Let $Z(t) \in C(\Omega\times[0,+\infty),\mathbb{R}_+)$ and $f(t)$ be two stochastic processes satisfying $\lim_{t\to+\infty} f(t)/t = 0$ a.s..*

(i) *If there exist positive constants T and δ_0 such that for all $t \geq T$,*

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + f(t), \tag{7}$$

then

$$\begin{cases} \limsup\limits_{t\to+\infty} t^{-1} \int_0^t Z(s)\mathrm{d}s \leq \dfrac{\delta}{\delta_0} \quad a.s. & if \quad \delta \geq 0; \\ \lim\limits_{t\to+\infty} Z(t) = 0 \ a.s. & if \quad \delta < 0. \end{cases} \tag{8}$$

(ii) *If there exist positive constants T, δ and δ_0 such that for all $t \geq T$,*

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + f(t), \tag{9}$$

then

$$\liminf_{t\to+\infty} t^{-1} \int_0^t Z(s)\mathrm{d}s \geq \frac{\delta}{\delta_0} \quad a.s. \tag{10}$$

Lemma 3.2. *For system (6):*
(i) *If $\Sigma_1 < 0$, then $\lim_{t\to+\infty} X_i(t) = 0$ a.s. $(i = 1, 2, 3)$.*
(ii) *If $\Sigma_1 \geq 0$, $\Sigma_2 < 0$, then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_1(s)\mathrm{d}s = \frac{\Sigma_1}{A_{11}}, \ \lim_{t\to+\infty} X_i(t) = 0 \quad a.s. \ (i = 2, 3). \tag{11}$$

(iii) *If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 < 0$, then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_i(s)\mathrm{d}s = \frac{\Sigma_i}{A_{ii}}, \ \lim_{t\to+\infty} X_3(t) = 0 \ a.s. \ (i = 1, 2). \tag{12}$$

(iv) *If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 \geq 0$, then*

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_i(s)\mathrm{d}s = \frac{\Sigma_i}{A_{ii}} \ a.s. \ (i = 1, 2, 3). \tag{13}$$

Proof. Consider $\mathrm{d}X_1(t) = X_1(t)\left[\left(r_1(\rho(t)) - \mathcal{D}_{11}(X_1)(t)\right)\mathrm{d}t + \mathcal{S}_1\left(t, \rho(t)\right)\right]$. Similar to the proof of Lemma 2.3 in [44], we have

$$\begin{cases} \lim\limits_{t\to+\infty} X_1(t) = 0 \quad a.s. & (\Sigma_1 < 0); \\ \lim\limits_{t\to+\infty} t^{-1} \int_0^t X_1(s)\mathrm{d}s = \dfrac{\Sigma_1}{A_{11}} \quad a.s. & (\Sigma_1 \geq 0). \end{cases} \tag{14}$$

By Lemma 3.1 in [24] and the strong law of large numbers,

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1} \int_0^t \sigma_j(\rho(s))\mathrm{d}W_j(s) = 0 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1} \int_0^t \int_{\mathbb{Z}} \ln\left(1 + \gamma_j(\mu, \rho(s))\right) \widetilde{N}(\mathrm{d}s, \mathrm{d}\mu) = 0 \quad a.s. \end{cases} \tag{15}$$

By Itô's formula and (15),

$$\begin{cases} \ln X_1(t) = \int_0^t B_1(\rho(s))\mathrm{d}s - A_{11}\int_0^t X_1(s)\mathrm{d}s - \mathcal{T}_{11}(X_1)(t) + o(t), \\ \ln X_2(t) = -\int_0^t B_2(\rho(s))\mathrm{d}s + A_{21}\int_0^t X_1(s)\mathrm{d}s - A_{22}\int_0^t X_2(s)\mathrm{d}s + \mathcal{T}_{21}(X_1)(t) - \mathcal{T}_{22}(X_2)(t) + o(t), \\ \ln X_3(t) = -\int_0^t B_3(\rho(s))\mathrm{d}s + A_{32}\int_0^t X_2(s)\mathrm{d}s - A_{33}\int_0^t X_3(s)\mathrm{d}s + \mathcal{T}_{32}(X_2)(t) - \mathcal{T}_{33}(X_3)(t) + o(t), \end{cases} \tag{16}$$

where $\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^0 \int_\theta^0 X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta) - \int_{-\tau_{ji}}^0 \int_{t+\theta}^t X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta)$.

$\mathfrak{Case}$ (i) : $\Sigma_1 < 0$. Then $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Therefore, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_2(t) \leq \left(-\sum_{i=1}^{S} \pi_i B_2(i) + \epsilon\right) t - a_{22}\int_0^t X_2(s)\mathrm{d}s. \tag{17}$$

Thus, $\lim_{t\to+\infty} X_2(t) = 0$ a.s. Similarly, $\lim_{t\to+\infty} X_3(t) = 0$ a.s.

$\mathfrak{Case}$ (ii) : $\Sigma_1 \geq 0$. Then,

$$\lim_{t\to+\infty} t^{-1}\int_0^t X_1(s)\mathrm{d}s = \frac{\Sigma_1}{A_{11}} \quad a.s. \tag{18}$$

Consider the following auxiliary function:

$$\mathrm{d}\widetilde{X_2(t)} = \widetilde{X_2(t)}\left[\left(-r_2(\rho(t)) + \mathcal{D}_{21}(X_1)(t) - a_{22}\widetilde{X_2(t)}\right)\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right]. \tag{19}$$

Then $X_2(t) \leq \widetilde{X_2(t)}$ a.s. By Itô's formula and (18),

$$\ln\widetilde{X_2(t)} = -\int_0^t B_2(\rho(s))\mathrm{d}s + A_{21}\int_0^t X_1(s)\mathrm{d}s - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s + o(t). \tag{20}$$

Thanks to (18) and (20), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln\widetilde{X_2(t)} \leq (\Sigma_2 + \epsilon)\, t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s, \\ \ln\widetilde{X_2(t)} \geq (\Sigma_2 - \epsilon)\, t - a_{22}\int_0^t \widetilde{X_2(s)}\mathrm{d}s. \end{cases} \tag{21}$$

In view of Lemma 3.1, (21) and the arbitrariness of ϵ, we obtain:

⟨1⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 < 0$, then $\lim_{t\to+\infty} \widetilde{X_2(t)} = 0$ a.s.

⟨2⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1}\int_0^t \widetilde{X_2(s)}\mathrm{d}s = \frac{\Sigma_2}{a_{22}} \quad a.s. \tag{22}$$

Therefore, for arbitrary $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1,2). \tag{23}$$

According to (23) and system (16), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln X_2(t) \leq (\Sigma_2 + \epsilon)\, t - A_{22}\int_0^t X_2(s)\mathrm{d}s, \\ \ln X_2(t) \geq (\Sigma_2 - \epsilon)\, t - A_{22}\int_0^t X_2(s)\mathrm{d}s. \end{cases} \tag{24}$$

Thanks to Lemma 3.1 and the arbitrariness of ϵ, we obtain:
⟨3⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 < 0$, then $\lim_{t\to+\infty} X_2(t) = 0$ a.s.
⟨4⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_2(s)\mathrm{d}s = \frac{\Sigma_2}{A_{22}} \quad a.s. \tag{25}$$

Therefore, the desired assertion (ii) follows from combining (18) with ⟨3⟩.

$\mathfrak{Case}$ (iii) : $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$. Consider the following stochastic differential equation:

$$\mathrm{d}\widetilde{X_3(t)} = \widetilde{X_3(t)}\left[\left(-r_3(\rho(t)) + \mathcal{D}_{32}(X_2)(t) - a_{33}\widetilde{X_3(t)}\right)\mathrm{d}t + \mathcal{S}_3\left(t, \rho(t)\right)\right]. \tag{26}$$

By Itô's formula and (25),

$$\ln \widetilde{X_3(t)} = -\int_0^t B_3(\rho(s))\mathrm{d}s + A_{32}\int_0^t X_2(s)\mathrm{d}s - a_{33}\int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t). \tag{27}$$

Thanks to (25) and (27), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln \widetilde{X_3(t)} \leq (\Sigma_3 + \epsilon)\, t - a_{33}\displaystyle\int_0^t \widetilde{X_3(s)}\mathrm{d}s, \\ \ln \widetilde{X_3(t)} \geq (\Sigma_3 - \epsilon)\, t - a_{33}\displaystyle\int_0^t \widetilde{X_3(s)}\mathrm{d}s. \end{cases} \tag{28}$$

Thanks to Lemma 3.1, (28) and the arbitrariness of ϵ, we deduce:
⟨5⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 < 0$, then $\lim_{t\to+\infty} \widetilde{X_3(t)} = 0$ a.s.
⟨6⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t \widetilde{X_3(s)}\mathrm{d}s = \frac{\Sigma_3}{a_{33}} \quad a.s. \tag{29}$$

Hence, for arbitrary $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1} \int_{t-\gamma}^t X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2, 3). \tag{30}$$

Combining (30) with system (16) yields that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln X_3(t) \leq (\Sigma_3 + \epsilon)\, t - A_{33}\displaystyle\int_0^t X_3(s)\mathrm{d}s, \\ \ln X_3(t) \geq (\Sigma_3 - \epsilon)\, t - A_{33}\displaystyle\int_0^t X_3(s)\mathrm{d}s. \end{cases} \tag{31}$$

Thanks to Lemma 3.1, (31) and the arbitrariness of ϵ, we deduce:
⟨7⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 < 0$, then $\lim_{t\to+\infty} X_3(t) = 0$ a.s.
⟨8⟩ If $\Sigma_1 \geq 0$, $\Sigma_2 \geq 0$, $\Sigma_3 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_3(s)\mathrm{d}s = \frac{\Sigma_3}{A_{33}} \quad a.s. \tag{32}$$

In view of (18), ⟨4⟩ and ⟨7⟩, we obtain (iii). And ($\mathfrak{iv}$) follows from combining (18), ⟨4⟩ with ⟨8⟩. The proof is complete. □

Lemma 3.3. *For system (4):*
(i) $\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le 0$ *a.s.* $(i = 1, 2, 3)$.
(ii) $\lim_{t\to+\infty} x_i(t) = 0 \Longrightarrow \lim_{t\to+\infty} x_j(t) = 0$ *a.s.* $(1 \le i < j \le 3)$.

Proof. An application of (30) and Lemma 3.2 in system (16) yields

$$\limsup_{t\to+\infty} t^{-1}\ln X_i(t) \le 0 \quad a.s. \quad (i = 1, 2, 3). \tag{33}$$

Hence, the desired assertion (i) follows from (33). The proof of (ii) is similar to that of Lemma 3.2 (a) and here is omitted. □

Theorem 3.4. *For system (4):*
(i) *if* $|A_3| > 0$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{|A_i|}{|A|} \;\; a.s. \;\; (i = 1, 2, 3). \tag{34}$$

(ii) *if* $M_{33}^{|A_2|} > 0 > |A_3|$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\mathrm{d}s = \frac{M_{33}^{|A_i|}}{M_{33}^{|A|}}, \quad \lim_{t\to+\infty} x_3(t) = 0 \;\; a.s. \;\; (i = 1, 2). \tag{35}$$

(iii) *if* $\Sigma_1 > 0 > M_{33}^{|A_2|}$, *then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Sigma_1}{A_{11}} \quad \lim_{t\to+\infty} x_i(t) = 0 \;\; a.s. \;\; (i = 2, 3). \tag{36}$$

(iv) *if* $0 > \Sigma_1$, *then* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.* $(i = 1, 2, 3)$.

Proof. Compute $|A_3| < A_{32}M_{33}^{|A_2|} < A_{21}A_{32}\Sigma_1$. By (30), for any $\gamma > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\gamma}^t x_i(s)\mathrm{d}s = 0 \;\; a.s. \;\; (i = 1, 2, 3). \tag{37}$$

By Itô's formula and (37), we deduce

$$\begin{pmatrix} \ln x_1(t) \\ \ln x_2(t) \\ \ln x_3(t) \end{pmatrix} = \begin{pmatrix} \int_0^t B_1(\rho(s))\mathrm{d}s \\ -\int_0^t B_2(\rho(s))\mathrm{d}s \\ -\int_0^t B_3(\rho(s))\mathrm{d}s \end{pmatrix} - A \begin{pmatrix} \int_0^t x_1(s)\mathrm{d}s \\ \int_0^t x_2(s)\mathrm{d}s \\ \int_0^t x_3(s)\mathrm{d}s \end{pmatrix} + o(t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{38}$$

Case (i) : $|A_3| > 0$. According to system (38), we compute

$$A_{21}A_{32}\ln x_1(t) + A_{11}A_{32}\ln x_2(t) + M_{33}^{|A|}\ln x_3(t) = |A_3|t - |A|\int_0^t x_3(s)\mathrm{d}s + o(t). \tag{39}$$

Combining Lemma 3.3 with (39) yields that for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$M_{33}^{|A|}\ln x_3(t) \ge (|A_3| - \epsilon)\, t - |A|\int_0^t x_3(s)\mathrm{d}s. \tag{40}$$

In view of Lemma 3.1, (40) and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \frac{|A_3|}{|A|} \;\; a.s. \tag{41}$$

On the basis of system (38), we compute

$$A_{22}\ln x_1(t)-A_{12}\ln x_2(t)=M_{33}^{|A_1|}t-M_{33}^{|A|}\int_0^t x_1(s)\mathrm{d}s+A_{12}A_{23}\int_0^t x_3(s)\mathrm{d}s+o(t). \tag{42}$$

By Lemma 3.3 and (42), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$A_{22}\ln x_1(t)\leq\left(M_{33}^{|A_1|}+A_{12}A_{23}\limsup_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s+\epsilon\right)t-M_{33}^{|A|}\int_0^t x_1(s)\mathrm{d}s. \tag{43}$$

Based on Lemma 3.1, (43) and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty}t^{-1}\int_0^t x_1(s)\mathrm{d}s\leq\left(M_{33}^{|A|}\right)^{-1}\left(M_{33}^{|A_1|}+A_{12}A_{23}\limsup_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s\right)\triangleq\Gamma_{x_1}^{sup}\quad a.s. \tag{44}$$

According to (41), (44) and system (38), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_2(t)\leq\left(\Sigma_2-\frac{A_{21}}{A_{11}}\Sigma_1+A_{21}\Gamma_{x_1}^{sup}-A_{23}\frac{|A_3|}{|A|}+\epsilon\right)t-A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{45}$$

In view of (41), Lemma 3.3, Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty}t^{-1}\int_0^t x_2(s)\mathrm{d}s\leq A_{22}^{-1}\left(\Sigma_2-\frac{A_{21}}{A_{11}}\Sigma_1+A_{21}\Gamma_{x_1}^{sup}-A_{23}\frac{|A_3|}{|A|}\right)\quad a.s. \tag{46}$$

Combining (46) with system (38) yields that for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_3(t)\leq\left[\Sigma_3-\frac{A_{21}A_{32}}{A_{11}A_{22}}\Sigma_1+\frac{A_{32}}{A_{22}}\left(A_{21}\Gamma_{x_1}^{sup}-A_{23}\frac{|A_3|}{|A|}\right)+\epsilon\right]t-A_{33}\int_0^t x_3(s)\mathrm{d}s. \tag{47}$$

In view of (41), (47), Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s\leq A_{33}^{-1}\left[\Sigma_3-\frac{A_{21}A_{32}}{A_{11}A_{22}}\Sigma_1+\frac{A_{32}}{A_{22}}\left(A_{21}\Gamma_{x_1}^{sup}-A_{23}\frac{|A_3|}{|A|}\right)\right]\quad a.s. \tag{48}$$

In other words, we have

$$\frac{A_{22}A_{33}M_{33}^{|A|}-A_{12}A_{21}A_{23}A_{32}}{A_{22}M_{33}^{|A|}}\limsup_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s\leq\Sigma_3-\frac{A_{21}A_{32}}{A_{11}A_{22}}\Sigma_1+\frac{A_{32}}{A_{22}}\left(A_{21}\frac{M_{33}^{|A_1|}}{M_{33}^{|A|}}-A_{23}\frac{|A_3|}{|A|}\right)\quad a.s. \tag{49}$$

In view of (49) and assumption ($\mathbf{H_2}$), we deduce

$$\limsup_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s\leq\frac{|A_3|}{|A|}\quad a.s. \tag{50}$$

Based on (41) and (50), we obtain

$$\lim_{t\to+\infty}t^{-1}\int_0^t x_3(s)\mathrm{d}s=\frac{|A_3|}{|A|}\quad a.s. \tag{51}$$

Substituting (51) into (44) yields

$$\limsup_{t\to+\infty}t^{-1}\int_0^t x_1(s)\mathrm{d}s\leq\left(M_{33}^{|A|}\right)^{-1}\left(M_{33}^{|A_1|}+A_{12}A_{23}\frac{|A_3|}{|A|}\right)=\frac{|A_1|}{|A|}\quad a.s. \tag{52}$$

Substituting (52) into (46) leads to

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le A_{22}^{-1}\left(\Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 + A_{21}\frac{|A_1|}{|A|} - A_{23}\frac{|A_3|}{|A|}\right) = \frac{|A_2|}{|A|} \quad a.s. \tag{53}$$

In view of system (38), we compute

$$A_{21}\ln x_1(t) + A_{11}\ln x_2(t) = M_{33}^{|A_2|}t - M_{33}^{|A|}\int_0^t x_2(s)\mathrm{d}s - A_{11}A_{23}\int_0^t x_3(s)\mathrm{d}s + o(t). \tag{54}$$

Based on Lemma 3.3 and (54), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$A_{11}\ln x_2(t) \ge \left(M_{33}^{|A_2|} - A_{11}A_{23}\frac{|A_3|}{|A|} - \epsilon\right)t - M_{33}^{|A|}\int_0^t x_2(s)\mathrm{d}s. \tag{55}$$

Thanks to (55), Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \left(M_{33}^{|A|}\right)^{-1}\left(M_{33}^{|A_2|} - A_{11}A_{23}\frac{|A_3|}{|A|}\right) = \frac{|A_2|}{|A|} \quad a.s. \tag{56}$$

Combining (53) with (56) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|A_2|}{|A|} \quad a.s. \tag{57}$$

Combining (57) with system (38) yields that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge \left(\Sigma_1 - A_{12}\frac{|A_2|}{|A|} - \epsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \le \left(\Sigma_1 - A_{12}\frac{|A_2|}{|A|} + \epsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{58}$$

In view of (58), Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = A_{11}^{-1}\left(\Sigma_1 - A_{12}\frac{|A_2|}{|A|}\right) = \frac{|A_1|}{|A|} \quad a.s. \tag{59}$$

$\mathfrak{Case}$ (ii) : $M_{33}^{|A_2|} > 0 > |A_3|$. Thanks to (39), we obtain

$$\limsup_{t\to+\infty} t^{-1}\ln\left[x_1^{A_{21}A_{32}}(t)x_2^{A_{11}A_{32}}(t)x_3^{M_{33}^{|A|}}(t)\right] \le |A_3| < 0 \quad a.s. \tag{60}$$

In view of (60) and Lemma 3.3, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = 0 \quad a.s. \tag{61}$$

From (54) and (61), we derive that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$A_{11}\ln x_2(t) \ge \left(M_{33}^{|A_2|} - \epsilon\right)t - M_{33}^{|A|}\int_0^t x_2(s)\mathrm{d}s. \tag{62}$$

According to (62), Lemma 3.1 and the arbitrariness of ϵ, we have

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge \frac{M_{33}^{|A_2|}}{M_{33}^{|A|}} \quad a.s. \tag{63}$$

In view of (42) and (61), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$A_{22}\ln x_1(t) \le \left(M_{33}^{|A_1|}+\epsilon\right)t - M_{33}^{|A|}\int_0^t x_1(s)\mathrm{d}s. \tag{64}$$

Thanks to (63), Lemma 3.3, Lemma 3.1 and the arbitrariness of ϵ, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{M_{33}^{|A_1|}}{M_{33}^{|A|}} \quad a.s. \tag{65}$$

Substituting (61) and (65) into system (38) yields that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \le \left(\Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 + A_{21}\frac{M_{33}^{|A_1|}}{M_{33}^{|A|}}+\epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{66}$$

Based on (66), Lemma 3.1 and the arbitrariness of ϵ, we deduce

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le A_{22}^{-1}\left(\Sigma_2 - \frac{A_{21}}{A_{11}}\Sigma_1 + A_{21}\frac{M_{33}^{|A_1|}}{M_{33}^{|A|}}\right) = \frac{M_{33}^{|A_2|}}{M_{33}^{|A|}} \quad a.s. \tag{67}$$

Combining (63) with (67) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{M_{33}^{|A_2|}}{M_{33}^{|A|}} \quad a.s. \tag{68}$$

Combining (68) with system (38) yields that for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge \left(\Sigma_1 - A_{12}\dfrac{M_{33}^{|A_2|}}{M_{33}^{|A|}} - \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \le \left(\Sigma_1 - A_{12}\dfrac{M_{33}^{|A_2|}}{M_{33}^{|A|}} + \epsilon\right)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{69}$$

Based on (69), Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{M_{33}^{|A_1|}}{M_{33}^{|A|}} \quad a.s. \tag{70}$$

𝔆𝔞𝔰𝔢 (iii) : $\Sigma_1 > 0 > M_{33}^{|A_2|}$. Then, $\lim_{t\to+\infty} x_3(t) = 0$ a.s. In view of (54),

$$\limsup_{t\to+\infty} t^{-1}\ln\left(x_1^{A_{21}}(t)x_2^{A_{11}}(t)\right) \le M_{33}^{|A_2|} < 0 \quad a.s. \tag{71}$$

On the basis of (71) and Lemma 3.3, we derive that $\lim_{t\to+\infty} x_2(t) = 0$ a.s. Thus, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\begin{cases} \ln x_1(t) \ge (\Sigma_1 - \epsilon)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s, \\ \ln x_1(t) \le (\Sigma_1 + \epsilon)t - A_{11}\displaystyle\int_0^t x_1(s)\mathrm{d}s. \end{cases} \tag{72}$$

In the light of (72), Lemma 3.1 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{\Sigma_1}{A_{11}} \quad a.s. \tag{73}$$

𝔆𝔞𝔰𝔢 (iv) : $0 > \Sigma_1$. Then, $\lim_{t\to+\infty} x_1(t) = 0$ a.s. Consequently, based on Lemma 3.3, we obtain that $\lim_{t\to+\infty} x_2(t) = \lim_{t\to+\infty} x_3(t) = 0$ a.s. □

Remark 3.5. *If* $\mathbb{S} = \{1\}$ *and* $\mu_{ij}(\theta) = C_{ij}$ $(\theta \in [-\tau, 0])$, *then system (4) becomes the system discussed in [31].*

Remark 3.6. *If* $\mathbb{S} = \{1\}$, $\gamma_j(\mu, \cdot) = 0$ $(\mu \in \mathbb{Z})$, $\mu_{ii}(\theta) = C_{ii}$ $(\theta \in [-\tau, 0])$, $a_{ij} = 0$ $(i \neq j)$ *and* $\mu_{ij}(\theta)$ *are defined as follows:*

$$\mu_{12}(\theta) = \begin{cases} \widetilde{a_{12}}, & -\tau_1 < \theta \le 0, \\ 0, & -\tau_{12} \le \theta \le -\tau_1, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} \widetilde{a_{21}}, & -\tau_2 < \theta \le 0, \\ 0, & -\tau_{21} \le \theta \le -\tau_2, \end{cases}$$

$$\mu_{23}(\theta) = \begin{cases} \widetilde{a_{23}}, & -\tau_3 < \theta \le 0, \\ 0, & -\tau_{23} \le \theta \le -\tau_3, \end{cases} \quad \mu_{32}(\theta) = \begin{cases} \widetilde{a_{32}}, & -\tau_4 < \theta \le 0, \\ 0, & -\tau_{32} \le \theta \le -\tau_4, \end{cases}$$

then system (4) becomes the following system discussed in [45]:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - \widetilde{a_{12}}x_2(t-\tau_1)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + \widetilde{a_{21}}x_1(t-\tau_2) - a_{22}x_2(t) - \widetilde{a_{23}}x_3(t-\tau_3)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + \widetilde{a_{32}}x_2(t-\tau_4) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t). \end{cases}$$

4. Numerical simulations

In this section we provide some numerical simulations to show the effectiveness of our main theoretical results by using the Milstein approach mentioned in [46]. For simplicity, we suppose that system (4) has only two regimes, namely $\mathbb{S} = \{1, 2\}$. Then system (4) is a hybrid system of the following two subsystems:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(1) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,1)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(1) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,1)\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(1) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,1)\right], \end{cases} \tag{74}$$

and

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[(r_1(2) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,2)\right], \\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(2) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,2)\right], \\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(2) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,2)\right]. \end{cases} \tag{75}$$

Let $\tau_{ji} = \ln 2$, $\mu_{ji}(\theta) = \mu_{ji}\mathrm{e}^{\theta}$, $\gamma_j(\mu, i) = \gamma_j(i)$, $\lambda(\mathbb{Z}) = 1$. Denote

$$\mathrm{Param}(i) = \begin{pmatrix} r_1(i) & a_{11} & a_{12} & 0 & \mu_{11} & \mu_{12} & 0 & \sigma_1(i) & \gamma_1(i) \\ r_2(i) & a_{21} & a_{22} & a_{23} & \mu_{21} & \mu_{22} & \mu_{23} & \sigma_2(i) & \gamma_2(i) \\ r_3(i) & 0 & a_{32} & a_{33} & 0 & \mu_{32} & \mu_{33} & \sigma_3(i) & \gamma_3(i) \end{pmatrix}.$$

Then system (4) may be regarded as the result of regime switching between subsystems (74) and (75) with the following parameters, respectively,

$$\mathrm{Param}(1) = \begin{pmatrix} 0.9 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.3 & 0.2 & 0.4 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.3 & 0.3 & 0 & 0.2 & 0.2 & 0.1 & 0.1 \end{pmatrix},$$

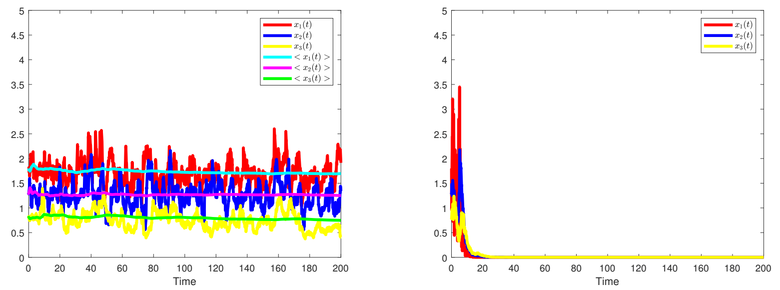
$$\mathrm{Param}(2) = \begin{pmatrix} 0.5 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0 & 1.2 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.2 & 0.4 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.3 & 0.3 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix},$$

subject to $x_1(\theta) = 1.8\mathrm{e}^{\theta}$, $x_2(\theta) = 1.3\mathrm{e}^{\theta}$, $x_3(\theta) = 0.8\mathrm{e}^{\theta}$, $\theta \in [-\ln 2, 0]$. For subsystem (74), we compute $B_1(1) = 0.795 + \ln 1.1$, $B_2(1) = 0.405 - \ln 1.1$, $B_3(1) = 0.305 - \ln 1.1$, $|A_1| = 0.24375 + 0.25\ln 1.1$, $|A_2| = 0.16965 + 0.27\ln 1.1$,

$|A_3| = 0.0507 + 0.66 \ln 1.1 = 0.1136 > 0$. Based on Theorem 3.4 (i), all species in subsystem (74) are persistent in mean and

$$\begin{cases} \lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)ds = \frac{|A_1|}{|A|} = \frac{0.2676}{0.156} = 1.7154 \ \ a.s. \\ \lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)ds = \frac{|A_2|}{|A|} = \frac{0.1954}{0.156} = 1.2526 \ \ a.s. \\ \lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)ds = \frac{|A_3|}{|A|} = \frac{0.1136}{0.156} = 0.7282 \ \ a.s. \end{cases} \tag{76}$$

For subsystem (75), we compute $B_1(2) = -0.42 + \ln 1.2$, $B_2(2) = 0.42 - \ln 1.2$, $B_3(2) = 0.42 - \ln 1.2$, $\Sigma_1 = -0.42 + \ln 1.2 = -0.2377 < 0$. In view of Theorem 3.4(iv), all species in subsystem (75) are extinctive. See Figure 1 (a) and Figure 1 (b), respectively.



(a) (b)

Figure 1: (a) shows the solution to subsystem (74) with Param (1). This subfigure represents that all species in regime 1 are persistent in mean; (b) shows three sample paths of subsystem (75) with Param (2). This subfigure represents that all species in regime 2 are extinctive.

Case 1. $\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 9 & -9 \end{pmatrix}$. Then $\pi = (\pi_1, \pi_2) = \left(\frac{9}{10}, \frac{1}{10}\right)$. Thus, we have $|A_1| = 0.208875 + 0.225 \ln 1.1 + 0.025 \ln 1.2$, $|A_2| = 0.141345 + 0.243 \ln 1.1 + 0.027 \ln 1.2$, $|A_3| = 0.01791 + 0.594 \ln 1.1 + 0.066 \ln 1.2 = 0.0866 > 0$. Based on Theorem 3.4 (i), all species in system (4) are persistent in mean (see Figure 2 (a)) and

$$\begin{cases} \lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)ds = \frac{|A_1|}{|A|} = \frac{0.2349}{0.156} = 1.5058 \ \ a.s. \\ \lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)ds = \frac{|A_2|}{|A|} = \frac{0.1694}{0.156} = 1.0859 \ \ a.s. \\ \lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)ds = \frac{|A_3|}{|A|} = \frac{0.0866}{0.156} = 0.5551 \ \ a.s. \end{cases} \tag{77}$$

Case 2. $\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -9 & 9 \\ 11 & -11 \end{pmatrix}$. Then $\pi = (\pi_1, \pi_2) = \left(\frac{11}{20}, \frac{9}{20}\right)$. Thus, we compute $|A_3| = -0.096855 + 0.363 \ln 1.1 + 0.297 \ln 1.2 = -0.0081 < 0$, $M_{33}^{|A_1|} = 0.222825 + 0.055 \ln 1.1 + 0.045 \ln 1.2$, $M_{33}^{|A_2|} = 0.025425 + 0.495 \ln 1.1 + 0.405 \ln 1.2 = 0.1464 > 0$. By Theorem 3.4 (ii), $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive (see Figure 2 (b)) and

$$\begin{cases} \lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)ds = \frac{M_{33}^{|A_1|}}{M_{33}^{|A|}} = \frac{0.2363}{0.3} = 0.7877 \ \ a.s. \\ \lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)ds = \frac{M_{33}^{|A_2|}}{M_{33}^{|A|}} = \frac{0.1464}{0.3} = 0.4880 \ \ a.s. \end{cases} \tag{78}$$

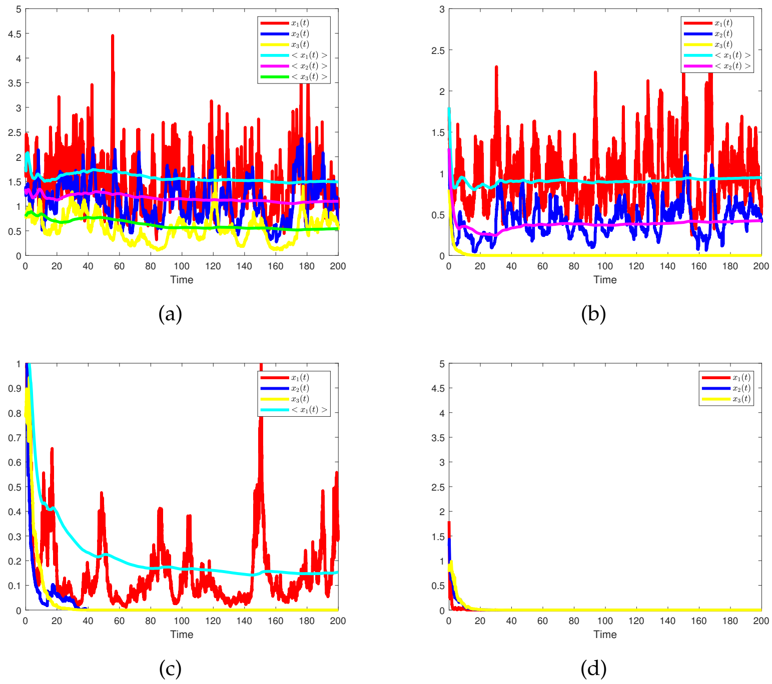


Figure 2: (a) shows the solution to hybrid system (4) with $\pi = \left(\frac{9}{10}, \frac{1}{10}\right)$. This subfigure represents that all species in **Case 1** are persistent in mean; (b) shows the solution to hybrid system (4) with $\pi = \left(\frac{11}{20}, \frac{9}{20}\right)$. This subfigure represents that in **Case 2**, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ is extinctive; (c) shows the solution to hybrid system (4) with $\pi = \left(\frac{1}{4}, \frac{3}{4}\right)$. This subfigure represents that in **Case 3**, $x_1(t)$ is persistent in mean, while $x_2(t)$ and $x_3(t)$ are extinctive; (d) shows the solution to hybrid system (4) with $\pi = \left(\frac{1}{6}, \frac{5}{6}\right)$. This subfigure represents that all species in **Case 4** are extinctive. Other parameters in Figure 2 are the same as those in Figure 1.

Case 3. $\Gamma = (\gamma_{ij})_{2\times2} = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$. Then $\pi = (\pi_1, \pi_2) = \left(\frac{1}{4}, \frac{3}{4}\right)$. Thus, we obtain $M_{33}^{|A_2|} = -0.194625 + 0.225 \ln 1.1 + 0.675 \ln 1.2 = -0.0501 < 0$, $\Sigma_1 = -0.11625 + 0.25 \ln 1.1 + 0.75 \ln 1.2 = 0.0443 > 0$. By Theorem 3.4 (iii), $x_1(t)$ is persistent in mean, while $x_2(t)$, $x_3(t)$ are extinctive (see Figure 2 (c)) and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{\Sigma_1}{A_{11}} = \frac{0.0443}{0.3} = 0.1477 \quad a.s. \tag{79}$$

Case 4. $\Gamma = (\gamma_{ij})_{2\times2} = \begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix}$. Then $\pi = (\pi_1, \pi_2) = \left(\frac{1}{6}, \frac{5}{6}\right)$. Hence, we have $\Sigma_1 = \frac{-1.305 + \ln 1.1 + 5 \ln 1.2}{6} = -0.0497 < 0$. Based on Theorem 3.4 (iv), all species in system (4) are extinctive (see Figure 2 (d)).

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