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# Existence Results for Nonlocal Hilfer-Type Integral-Multipoint Boundary Value Problems with Mixed Nonlinearities

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**Abstract.** In this paper, we investigate the existence of solutions for Hilfer-type fractional differential equations and inclusions involving mixed nonlinearities complemented with nonlocal fractional integral-multipoint boundary conditions. Our study is based on the tools of fixed point theory for both single-valued and multi-valued maps. Examples are constructed for illustrating the obtained results. We also discuss special cases concerning Langevin equation and inclusions in the given setting.

## 1. Introduction

Fractional differential equations are found to be of great value and interest in view of their applications in diverse disciplines of science and technology. Unlike the concept of classical derivative, there do exist several definitions of fractional derivative operators [1]-[3]. One important definition of fractional derivative, which represents both Riemann-Liouville and Caputo derivatives under suitable choice of parameters, was proposed by Hilfer in [4].

Initial value problems involving Hilfer fractional derivatives were studied by several authors, for example, see [5]-[9]. Some interesting results on boundary value problems involving Hilfer fractional differential equation can be found in the article [10]. For some recent results on Hilfer-type fractional differential equations, see [11]-[21].

In the present article, we introduce and investigate a new class of nonlocal multi-point integral boundary value problems involving Hilfer-type fractional differential operators of different orders. As a first problem, we discuss the existence and uniqueness of solutions for a nonlocal multi-point integral boundary value problem involving Hilfer fractional differential equation with mixed nonlinearities given by

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}\Big({}^{H}D^{\alpha_2,\beta_2}x(t) + g(t,x(t))\Big) = f(t,x(t)), \quad t\in[a,b], \\ x(a)=0,\ x(b)=\displaystyle\sum_{i=1}^{m}\mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k}x(\xi_k),\ a<\eta_i,\xi_k<b, \end{cases} \tag{1}$$

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where ${}^HD^{\alpha_j,\beta_j}$, $j = 1,2$ denotes the Hilfer fractional derivative operator of order $\alpha_j$, $0 < \alpha_j \le 1$ and parameter $\beta_j$, $0 \le \beta_j \le 1$, $j = 1,2$, $\mu_i$ $(i = 1,2,\dots,m)$, $q_k$ $(k = 1,2,\dots,n)$ are real constants, $a \ge 0$, $f,g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Notice that the boundary conditions in the problem (1) contain multi-point as well as multi-strip contributions. As a second problem, we study the multivalued analogue of the problem (1) in Section 4.

Here we emphasize that nonlocal conditions are important as these conditions help to model the phenomena occurring at different positions inside the given domain. Potential application of nonlocal conditions appear in diffusion processes [22], blood flow problems [23], bacteria self-organization models [24], etc.

We organize the rest of the paper as follows. In Section 2, we recall some preliminary concepts related to our study and investigate existence and uniqueness of solutions for the problem (1) in Section 3. Existence results for the multivalued analogue of problem (1) are proved in Section 4. A special case of the given problems in terms of Langevin equation and inclusions is discussed in Section 5. Some interesting observations are presented in the last section.

## 2. Preliminaries

Let us begin this section with some basic concepts.

**Definition 2.1.** *([4]) The Hilfer fractional derivative of order $\alpha$ and parameter $\beta$ for a continuous function $\sigma$ is defined by*

$$ {}^HD^{\alpha,\beta}\sigma(t) = I^{\beta(n-\alpha)}D^n I^{(1-\beta)(n-\alpha)}\sigma(t),\ n-1 < \alpha \le n,\ 0 \le \beta \le 1,\ t > a,\ D = \frac{d}{dt} $$

*and*

$$ I^{\omega}\sigma(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1}\sigma(s)ds, $$

*where $\omega \in \{\beta(n-\alpha), (1-\beta)(n-\alpha)\}$.*

Here we recall that the Hilfer fractional derivative corresponds to the Riemann-Liouville fractional derivative when $\beta = 0$, while the choice $\beta = 1$ changes the Hilfer fractional derivative into the Caputo fractional derivative [15].

**Lemma 2.2.** *([25]) Let $f \in L(a,b)$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, $0 \le \beta \le 1$ and $I^{(n-\alpha)(1-\beta)}f \in AC^k[a,b]$. Then*

$$ I^{\alpha}({}^HD^{\alpha,\beta}f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t \longrightarrow a^+} \frac{d^k}{dt^k}(I^{(1-\beta)(n-\alpha)}f)(t). $$

**Lemma 2.3.** *For any $u, y \in C[a,b]$, the unique solution of the linear fractional boundary value problem:*

$$ \begin{cases} {}^HD^{\alpha_1,\beta_1}({}^HD^{\alpha_2,\beta_2}x(t) + u(t)) = y(t),\ \ t \in [a,b] \\ x(a) = 0,\ x(b) = \sum_{i=1}^{m} \mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k}x(\xi_k), \end{cases} \tag{2} $$

*is*

$$ \begin{aligned} x(t) &= I^{\alpha_1+\alpha_2}y(t) - I^{\alpha_2}u(t) \\ &+ \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}u(b) - I^{\alpha_1+\alpha_2}y(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}y(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}u(\eta_i) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}y(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}u(\xi_k)\Big\}, \end{aligned} \tag{3} $$

*where $\epsilon_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1$ and it is assumed that*

$$\Delta = \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)} - \sum_{i=1}^{m}\mu_i\frac{(\eta_i-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)} - \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_2+\epsilon_1-1}}{\Gamma(q_k+\alpha_2+\epsilon_1)} \neq 0. \tag{4}$$

**Proof.** Applying the operator $I^{\alpha_1}$ on both sides of Hilfer fractional differential equation in (2) and then operating $I^{\alpha_2}$ on the resulting equation, it follows by Lemma 2.2 that

$$x(t) - \frac{c_1(t-a)^{\epsilon_2-1}}{\Gamma(\epsilon_2)} + I^{\alpha_2}u(t) - \frac{c_0}{\Gamma(\epsilon_1)}I^{\alpha_2}(t-a)^{\epsilon_1-1} = I^{\alpha_1+\alpha_2}y(t), \tag{5}$$

where $c_0, c_1 \in \mathbb{R}$ are unknown constants and $\epsilon_i = \alpha_i + \beta_i - \alpha_i\beta_i, i = 1, 2$. Computing the fourth term in (5), we get

$$x(t) - \frac{c_1(t-a)^{\epsilon_2-1}}{\Gamma(\epsilon_2)} + I^{\alpha_2}u(t) - \frac{c_0}{\Gamma(\alpha_2+\epsilon_1)}(t-a)^{\alpha_2+\epsilon_1-1} = I^{\alpha_1+\alpha_2}y(t). \tag{6}$$

Using the condition $x(a) = 0$ in (6), we find that $c_1 = 0$. Thus we have

$$x(t) + I^{\alpha_2}u(t) - \frac{c_0}{\Gamma(\alpha_2+\epsilon_1)}(t-a)^{\alpha_2+\epsilon_1-1} = I^{\alpha_1+\alpha_2}y(t), \tag{7}$$

which, on combining with the condition $x(b) = \sum_{i=1}^{m}\mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k}x(\xi_k)$, yields

$$\begin{aligned}
&c_0\Big(\frac{(b-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)} - \sum_{i=1}^{m}\mu_i\frac{(\eta_i-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)} - \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_2+\epsilon_1-1}}{\Gamma(q_k+\alpha_2+\epsilon_1)}\Big)\\
&= I^{\alpha_2}u(b) - I^{\alpha_1+\alpha_2}y(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}y(\eta_i) - \sum_{i=1}^{m}\mu_i I^{\alpha_2}u(\eta_i)\\
&+ \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}y(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}u(\xi_k).
\end{aligned}$$

Solving the above equation for $c_0$ and using the notation (4), we get

$$\begin{aligned}
c_0 &= \frac{1}{\Delta}\Big\{I^{\alpha_2}u(b) - I^{\alpha_1+\alpha_2}y(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}y(\eta_i)\\
&- \sum_{i=1}^{m}\mu_i I^{\alpha_2}u(\eta_i) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}y(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}u(\xi_k)\Big\}.
\end{aligned}$$

Substituting the value of $c_0$ in (7) leads to the solution (3). On the other hand, the converse of the lemma can be obtained by direct computation. This completes the proof. □

## 3. Main results for the problem (1)

We investigate the existence and uniqueness of solutions for the problem (1) by converting it into a fixed point problem $x = \mathcal{T}x$ with the aid of of Lemma 2.3, where the operator $\mathcal{T} : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ is defined by

$$\begin{aligned}
(\mathcal{T}x)(t) &= I^{\alpha_1+\alpha_2}f(t, x(t)) - I^{\alpha_2}g(t, x(t))\\
&+ \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b, x(b)) - I^{\alpha_1+\alpha_2}f(b, x(b)) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}f(\eta_i, x(\eta_i))\\
&- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i, x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}f(\xi_k, x(\xi_k)) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k, x(\xi_k))\Big\},
\end{aligned} \tag{8}$$

where $\Delta$ is given by (4) and $C([a,b],\mathbb{R})$ is the Banach space of all continuous real valued functions defined on $[a,b]$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$.

For the sake of computational convenience, we introduce the notations:

$$\begin{aligned}\omega_1 &= \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big(\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \\ &+ \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\sum_{i=1}^{m}|\mu_i|(\eta_i-a)^{\alpha_1+\alpha_2} + \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_1+\alpha_2}}{\Gamma(q_k+\alpha_1+\alpha_2+1)}\Big), \end{aligned} \tag{9}$$

$$\begin{aligned}\omega_2 &= \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big(\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \\ &+ \frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_2}}{\Gamma(q_k+\alpha_2+1)}\Big). \end{aligned} \tag{10}$$

Now we present our first main result which is concerned with the uniqueness of solutions for the problem (1) and relies on Banach contraction mapping principle [27].

**Theorem 3.1.** *Assume that*

*$(H_1)$ there exist positive constants $L_1$ and $L_2$ such that, for each $t \in [a,b]$ and $x, y \in \mathbb{R}$,*

$$|f(t,x(t)) - f(t,y(t))| \le L_1|x-y|, \;\; |g(t,x(t)) - g(t,y(t))| \le L_2|x-y|;$$

*$(H_2)$ $L_1\omega_1 + L_2\omega_2 < 1$, where $\omega_1$ and $\omega_2$ are respectively given by (9) and (10).*

*Then the problem (1) has a unique solution on $[a,b]$.*

**Proof.** Letting $\sup_{t\in[a,b]}|f(t,0)| = S_1$ and $\sup_{t\in[a,b]}|g(t,0)| = S_2$, we consider a closed and bounded ball: $B_\rho = \{x \in C([a,b],\mathbb{R}) : \|x\| \le \rho\}$ with

$$\rho \ge \frac{S_1\omega_1 + S_2\omega_2}{1-(L_1\omega_1+L_2\omega_2)}.$$

In the first step, it will be shown that $\mathcal{T}B_\rho \subset B_\rho$. For an arbitrary $x \in B_\rho$, we have

$$\begin{aligned}\|\mathcal{T}x\| &\le \sup_{t\in[a,b]}\Big\{I^{\alpha_1+\alpha_2}|f(t,x(t)) - f(t,0) + f(t,0)| + I^{\alpha_2}|g(t,x(t)) - g(t,0) + g(t,0)| \\ &\quad + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1+1)}\Big(I^{\alpha_2}|g(b,x(b)) - g(b,0) + g(b,0)| + I^{\alpha_1+\alpha_2}|f(b,x(b)) - f(b,0) + f(b,0)| \\ &\quad + \sum_{i=1}^{m}|\mu_i|I^{\alpha_1+\alpha_2}|f(\eta_i,x(\eta_i)) - f(\eta_i,0) + f(\eta_i,0)| + \sum_{i=1}^{m}|\mu_i|I^{\alpha_2}|g(\eta_i,x(\eta_i)) - g(\eta_i,0) + g(\eta_i,0)| \\ &\quad + \sum_{k=1}^{n}I^{q_k+\alpha_1+\alpha_2}|f(\xi_k,x(\xi_k)) - f(\xi_k,0) + f(\xi_k,0)| + \sum_{k=1}^{n}I^{q_k+\alpha_2}|g(\xi_k,x(\xi_k)) - g(\xi_k,0) + g(\xi_k,0)|\Big) \\ &\le (L_1\rho + S_1)\omega_1 + (L_2\rho + S_2)\omega_2 \le \rho,\end{aligned}$$

which implies that $\mathcal{T}B_\rho \subset B_\rho$.
Next, we show that $\mathcal{T}$ is a contraction. For $x, y \in C([a,b],\mathbb{R})$ and for each $t \in [a,b]$, we obtain

$$\begin{aligned}
& |\mathcal{T}x(t) - \mathcal{T}y(t)| \\
\leq\ & I^{\alpha_1+\alpha_2}|f(t,x(t)) - f(t,y(t))| + I^{\alpha_2}|g(t,x(t)) - g(t,y(t))| \\
& + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1+1)}\Big(I^{\alpha_2}|g(b,x(b)) - g(b,y(b))| + I^{\alpha_1+\alpha_2}|f(b,x(b)) - f(b,y(b))| \\
& + \sum_{i=1}^{m}|\mu_i|I^{\alpha_1+\alpha_2}|f(\eta_i,x(\eta_i)) - f(\eta_i,y(\eta_i))| + \sum_{i=1}^{m}|\mu_i|I^{\alpha_2}|g(\eta_i,x(\eta_i)) - g(\eta_i,y(\eta_i))| \\
& + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}|f(\xi_k,x(\xi_k)) - f(\xi_k,y(\xi_k))| + \sum_{k=1}^{n} I^{q_k+\alpha_2}|g(\xi_k,x(\xi_k)) - g(\xi_k,y(\xi_k))|,\Big) \\
\leq\ & (L_1\omega_1 + L_2\omega_2)\|x-y\|.
\end{aligned}$$

Taking the norm of the above inequality for $t \in [a,b]$, we get

$$\|\mathcal{T}x - \mathcal{T}y\| \leq (L_1\omega_1 + L_2\omega_2)\|x-y\|,$$

which, by $(H_2)$, implies that $\mathcal{T}$ is a contraction. In consequence, it follows by Banach fixed point theorem that the problem (1) has a unique solution on $[a,b]$. This completes the proof. □

In the following result, we establish the existence criteria for solutions of the problem (1) with the aid of Krasnosel'skiĭ's fixed point theorem [26].

**Theorem 3.2.** *Assume that*

$(\widehat{H}_1)$ *there exist a positive constant $L_2$ such that, for each $t \in [a,b]$ and $x, y \in \mathbb{R}$,*

$$|g(t,x(t)) - g(t,y(t))| \leq L_2|x-y|, \sup_{t\in[a,b]}|g(t,0)| = S_2;$$

$(H_3)$ *there exists a function $\psi_1 \in C([a,b],\mathbb{R}^+)$ such that $|f(t,x)| \leqslant \psi_1(t), \forall (t,x) \in [a,b]\times\mathbb{R}$;*

$(H_4)$ *$\omega_2 L_2 < 1$, where $\omega_2$ is given by (10).*

*Then there exists at least one solution for the problem (1) on $[a,b]$.*

**Proof.** In order to verify the hypothesis of Krasnosel'skiĭ's fixed point theorem, we decompose the operator $\mathcal{T} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ into sum of two operators $\mathcal{T}_1$ and $\mathcal{T}_2$ from a bounded closed ball $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \leq r\}$ to $C([a,b],\mathbb{R})$, where $r \geq [\|\psi_1\|\omega_1 + S_2\omega_2]/(1 - L_2\omega_2)$, where $\omega_1$ and $\omega_2$ are respectively given by (9) and (10) and

$$\begin{aligned}
\mathcal{T}_1x(t) &= I^{\alpha_1+\alpha_2}f(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{-I^{\alpha_1+\alpha_2}f(b,x(b)) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}f(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}f(\xi_k,x(\xi_k))\Big\}, \\
\mathcal{T}_2x(t) &= -I^{\alpha_2}g(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x(b)) - \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x(\eta_i)) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x(\xi_k))\Big\}.
\end{aligned}$$

For arbitrary elements $x, y \in B_r$, we get

$$\begin{aligned} \|\mathcal{T}_1 x + \mathcal{T}_2 y\| &\leq I^{\alpha_1+\alpha_2}|f(t,x(t))| + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_1+\alpha_2}|f(b,x(b))| \\ &+ \sum_{i=1}^{m}|\mu_i|I^{\alpha_1+\alpha_2}|f(\eta_i,x(\eta_i))| + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}|f(\xi_k,x(\xi_k))|\Big\} \\ &+ I^{\alpha_2}|g(t,y(t))| + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}|g(b,y(b))| \\ &+ \sum_{i=1}^{m}|\mu_i|I^{\alpha_2}|g(\eta_i,y(\eta_i))| + \sum_{k=1}^{n} I^{q_k+\alpha_2}|g(\xi_k,y(\xi_k))|\Big\} \\ &\leq \|\psi_1\|\omega_1 + (L_2 r + S_2)\omega_2 \leq r, \end{aligned}$$

which shows that $\mathcal{T}_1 x(t) + \mathcal{T}_2 y(t) \in B_r$.
In order to show that $\mathcal{T}_2$ is a contraction mapping, we take $x, y \in B_r$. Then, $\forall t \in [a,b]$, we have

$$\begin{aligned} &\|\mathcal{T}_2 x - \mathcal{T}_2 y\| \\ \leq\ & \sup_{t\in[a,b]}\Big\{I^{\alpha_2}|g(t,x(t)) - g(t,y(t))| + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}|g(b,x(b)) - g(b,y(b))| \\ +\ & \sum_{i=1}^{m}|\mu_i|I^{\alpha_2}|g(\eta_i,x(\eta_i)) - g(\eta_i,y(\eta_i))| + \sum_{k=1}^{n} I^{q_k+\alpha_2}|g(\xi_k,x(\xi_k)) - g(\xi_k,y(\xi_k))|\Big\} \\ \leq\ & \omega_2 L_2\|x-y\|, \end{aligned}$$

which, in view of $(H_4)$, implies that $\mathcal{T}_2$ is a contraction.

Now we establish the compactness of the operator $\mathcal{T}_1$. It is easy to check that continuity of $\mathcal{T}_1$ follows from that of $f$. Moreover, the operator $\mathcal{T}_1$ is uniformly bounded as

$$\|\mathcal{T}_1 x\| \leq \|\psi_1\|\omega_1,$$

where $\omega_1$ is given by (9).
Let us set $\sup_{(t,x)\in[a,b]\times B_r}|f(t,x)| = \widehat{f} < \infty$ and show that $\mathcal{T}_1$ is equicontinuous. For $a \leq t_1 \leq t_2 \leq b$, we have

$$\begin{aligned} |\mathcal{T}_1 x(t_2) - \mathcal{T}_1 x(t_1)| &= \Big|\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}f(s,x(s))ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_1}(t_1-s)^{\alpha_1+\alpha_2-1}f(s,x(s))ds \\ +\ & \Big[\frac{(t_2-a)^{\alpha_2+\epsilon_1-1} - (t_1-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big]\Big\{-I^{\alpha_1+\alpha_2}f(b,x(b)) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}f(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}f(\xi_k,x(\xi_k))\Big\}\Big| \\ \leq\ & \widehat{f}\Big\{\frac{2(t_2-t_1)^{\alpha_1+\alpha_2} + |(t_2-a)^{\alpha_1+\alpha_2} - (t_1-a)^{\alpha_1+\alpha_2}|}{\Gamma(\alpha_1+\alpha_2+1)}\Big\} + \widehat{f}\,\frac{|(t_2-a)^{\alpha_2+\epsilon_1-1} - (t_1-a)^{\alpha_2+\epsilon_1-1}|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)} \\ \times\ & \Big\{\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_1+\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_1+\alpha_2}}{\Gamma(q_k+\alpha_1+\alpha_2+1)}\Big\}. \end{aligned}$$

Clearly the right-hand side of the above inequality tends to zero independently of $x$ as $t_2 \to t_1$. In consequence, it follows by Arzelá-Ascoli theorem that $\mathcal{T}_1$ is compact. Thus all the assumptions of Krasnosel'skiĭ's fixed point theorem are satisfied and hence its conclusion implies that there exists at least one solution for the problem (1). This completes the proof. □

Our next existence result is based on Leray-Schauder nonlinear alternative for single valued maps, which is stated below.

**Lemma 3.3.** *(Leray-Schauder nonlinear alternative [27]) Let $M$ be a closed and convex subset of a Banach space $\mathcal{E}$, $\mathcal{V}$ be an open subset of $M$ with $0 \in \mathcal{V}$. Suppose that $G : \overline{\mathcal{V}} \to M$ is continuous, compact map (that is, $G(\overline{\mathcal{V}})$ is a relatively compact subset of $M$). Then either (i) $G$ has a fixed point in $\overline{\mathcal{V}}$, or (ii) there is $v \in \partial\mathcal{V}$ (the boundary of $\mathcal{V}$ in $M$) and $\lambda \in (0,1)$ with $v = \lambda G v$.*

**Theorem 3.4.** *Let $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that*

*($H_5$) there exist functions $p_1, p_2 \in C([a,b], \mathbb{R}^+)$ and nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$, such that $|f(t,x)| \leq p_1(t)\psi_1(\|x\|)$ and $|g(t,x)| \leq p_2(t)\psi_2(\|x\|)$ for all $(t,x) \in [a,b] \times \mathbb{R}$;*

*($H_6$) there exists a positive real number $M$ such that*

$$\frac{M}{\|p_1\|\psi_1(M)\omega_1 + \|p_2\|\psi_2(M)\omega_2} > 1, \tag{11}$$

*where $\omega_1$ and $\omega_2$ are respectively given by (9) and (10).*

*Then there exists at least one solution for the problem (1) on $[a,b]$.*

**Proof.** The proof will be completed in three steps.

**Step 1.** *$\mathcal{T} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by (8) maps bounded sets into bounded sets in $C([a,b],\mathbb{R})$.*
Let $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \leq r\}$ be a closed and bounded set in $C([a,b],\mathbb{R})$. Then, for $x \in B_r$, we have

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq I^{\alpha_1+\alpha_2}|f(t,x(t))| + I^{\alpha_2}|g(t,x(t))| \\
&+ \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}|g(b,x(b))| + I^{\alpha_1+\alpha_2}|f(b,x(b))| + \sum_{i=1}^{m}|\mu_i| I^{\alpha_1+\alpha_2}|f(\eta_i,x(\eta_i))| \\
&+ \sum_{i=1}^{m}|\mu_i| I^{\alpha_2}|g(\eta_i,x(\eta_i))| + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}|f(\xi_k,x(\xi_k))| + \sum_{k=1}^{n} I^{q_k+\alpha_2}|g(\xi_k,x(\xi_k))|\Big\} \\
&\leq \|p_1\|\psi_1(r)\omega_1 + \|p_2\|\psi_2(r)\omega_2.
\end{aligned}$$

Consequently, $\|\mathcal{T}x\| \leq \|p_1\|\psi_1(r)\omega_1 + \|p_2\|\psi_2(r)\omega_2$.

**Step 2.** *$\mathcal{T}$ maps bounded sets into equicontinuous sets of $C([a,b],\mathbb{R})$.*
For $a \leq t_1 \leq t_2 \leq b$ and $x \in B_r$, we have

$$\begin{aligned}
&|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \\
\leq\ & \|p_1\|\psi_1(r)\Big(\frac{2(t_2-t_1)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{|(t_2-a)^{\alpha_1+\alpha_2} - (t_1-a)^{\alpha_1+\alpha_2}|}{\Gamma(\alpha_1+\alpha_2+1)}\Big) \\
+\ & \|p_2\|\psi_2(r)\Big(\frac{2(t_2-t_1)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{|(t_2-a)^{\alpha_2} - (t_1-a)^{\alpha_2}|}{\Gamma(\alpha_2+1)}\Big) \\
+\ & \frac{\Big|(t_2-a)^{\alpha_2+\epsilon_1-1} - (t_1-a)^{\alpha_2+\epsilon_1-1}\Big|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)} \\
\times\ & \Big\{\|p_1\|\psi_1(r)\Big(\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_1+\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_1+\alpha_2}}{\Gamma(q_k+\alpha_1+\alpha_2+1)}\Big) \\
+\ & \|p_2\|\psi_2(r)\Big(\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_k-a)^{q_k+\alpha_2}}{\Gamma(q_k+\alpha_2+1)}\Big)\Big\},
\end{aligned}$$

which tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}$ is completely continuous.

**Step 3.** *Here we establish the boundedness of the set of all solutions to $x = \delta\mathcal{T}x$, with $\delta \in (0,1)$.*
As in Step 1, one can obtain

$$\|x\| \leq \|p_1\|\psi_1(\|x\|)\omega_1 + \|p_2\|\psi_2(\|x\|)\omega_2,$$

which can alternatively be written as

$$\frac{\|x\|}{\|p_1\|\psi_1(\|x\|)\omega_1 + \|p_2\|\psi_2(\|x\|)\omega_2} \leq 1.$$

According to assumption $(H_6)$, there exists a positive constant $M$ such that $\|x\| \neq M$. Let us define a set $\mathcal{U} = \{x \in C([a,b],\mathbb{R}) : \|x\| < M\}$. Note that the operator $\mathcal{T} : \overline{\mathcal{U}} \to C([a,b],\mathbb{R})$ is completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x = \delta\mathcal{T}x$ for $\delta \in (0,1)$. Then, by Lemma 3.3, we deduce that $\mathcal{T}$ has at least one fixed point in $\overline{\mathcal{U}}$, which is a solution of the problem (1) on $[a,b]$. This completes the proof. □

**Example 3.5.** *Consider the following nonlinear boundary value problem*

$$\begin{cases} {}^HD^{1/2,1/4}({}^HD^{2/3,4/5}x(t) + g(t,x(t)) = f(t,x(t)), \quad t \in [0,1] \\ x(0) = 0,\ x(1) = \frac{1}{20}x\Big(\frac{1}{6}\Big) + \frac{1}{13}x\Big(\frac{1}{3}\Big) + \frac{1}{12}x\Big(\frac{1}{2}\Big) + I^{7/2}x\Big(\frac{2}{3}\Big) + I^{5/2}x\Big(\frac{5}{6}\Big), \end{cases} \tag{12}$$

*where $\alpha_1 = 1/2, \beta_1 = 1/4, \alpha_2 = 2/3, \beta_2 = 4/5, a = 0, b = 1, \mu_1 = 1/20, \mu_2 = 1/13, \mu_3 = 1/12, q_1 = 7/2, q_2 = 5/2, \eta_1 = 1/6, \eta_2 = 1/3, \eta_3 = 1/2, \xi_1 = 2/3, \xi_2 = 5/6, m = 3, n = 2$. Using the given data, it is found that $\epsilon_1 = 5/8, \alpha_2 + \epsilon_1 = 31/24 > 1, |\Delta| \approx 1.3158, \omega_1 \approx 1.7869$ and $\omega_2 \approx 2.2113$.*

**(a)** *For illustrating Theorem 3.1, we consider*

$$f(t,x(t)) = \frac{e^{-2t} + \tan^{-1}x}{10\sqrt{81+\sin t}},\ g(t,x(t)) = \frac{1+t}{350}\Big(\frac{|x|}{|x|+1} + 3x + 60\Big). \tag{13}$$

*It is easy to verify that $f$ and $g$ satisfy Lipschitz condition with $L_1 = 1/90, L_2 = 6/350$, and $L_1\omega_1 + L_2\omega_2 \approx 0.0578 < 1$. As the assumptions of Theorem 3.1 hold true, therefore the problem (12) with $f$ and $g$ given by (13) has a unique solution on $[0,1]$.*

**(b)** *In order to illustrate Theorem 3.2, we take*

$$f(t,x(t)) = \frac{2\tan^{-1}x + \pi}{2\pi(1+t)},\ g(t,x(t)) = \frac{e^{-t}}{\pi}\tan^{-1}x + \frac{1}{t^2+1}. \tag{14}$$

*Note that the function $f$ is continuous with $|f(t,x)| \leq \frac{1}{1+t} = \psi_1(t)$, $g$ is continuous and satisfies the Lipschitz condition with $L_2 = 1/\pi$, $S_2 = 1$. Moreover, $L_2\omega_2 \approx 0.7039$. Clearly all the hypotheses of Theorem 3.2 are satisfied and hence the problem (12) with the values of $f$ and $g$ given by (14) has at least one solution on $[0,1]$.*

**(c)** *We demonstrate the application of Theorem 3.4 by choosing*

$$f(t,x) = \frac{3}{29+3t}\Big(\frac{|x|}{|x|+1} + |x| + 1/2\Big), g(t,x) = \frac{\cos^2 t}{\sqrt{t^2+121}}|x|. \tag{15}$$

*Observe that the condition $(H_5)$ is satisfied as*

$$|f(t,x)| \leq \frac{3}{29+3t}(\|x\| + 3/2) = p_1(t)\psi_1(\|x\|), |g(t,x)| \leq \frac{\cos^2 t}{\sqrt{t^2+121}}\|x\| = p_2(t)\psi_2(\|x\|),$$

*and the condition $(H_6)$ holds for $M > 0.4515$. Thus the conclusion of Theorem 3.4 applies to the problem (12) with $f$ and $g$ given by (15).*

## 4. Multivalued analogue of problem ( 1)

In this section, we discuss the existence of solutions for the multivalued analogue of problem ( 1) given by

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}({}^{H}D^{\alpha_2,\beta_2}x(t) + g(t, x(t))) \in F(t, x(t)),\ t \in [a, b], \\ x(a) = 0,\ x(b) = \sum_{i=1}^{m} \mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k} x(\xi_k),\ a < \eta_i, \xi_k < b, \end{cases} \tag{16}$$

where $F : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Let us first fix our terminology and state the known fixed point theorems for multivalued maps that we need in the forthcoming analysis. We define some spaces related to our work as follows: $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$; $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$; $\mathcal{P}_{c,cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex and compact}\}$; $\mathcal{P}_{b,cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded and closed}\}$ and $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$.

**Definition 4.1.** *A multi-valued map $F : X \to \mathcal{P}(X)$ is said to be **upper semi-continuous** on $X$ if for each $x_o \in X$, the set $F(x_o)$ is a nonempty closed subset of $X$ and there exist an open neighborhood $\mathcal{N}_o$ of $x_o$ such that $F(\mathcal{N}_o) \subseteq \mathbb{B}$ for each open set $\mathbb{B}$ of $X$ containing $F(x_o)$.*

**Lemma 4.2 (Nonlinear alternative for Kakutani maps [27]).** *Let $C$ be a closed convex subset of a Banach space $E$ and $U$ be an open subset of $C$ with $0 \in U$. Suppose that $G : \overline{U} \to \mathcal{P}_{c,cp}(C)$ is an upper semicontinuous compact map. Then either $G$ has a fixed point in $\overline{U}$ or there is an element $u \in \partial U$ such that $u \in \mu G(u)$ with $\mu \in (0, 1)$.*

Let $(X, d)$ be a metric space induced from the normed space $(X; \|\cdot\|)$.Let $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [28]).

**Definition 4.3.** *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

(*a*) *$\theta$−Lipschitz if and only if there exists $\theta > 0$ such that*

$$H_d(N(x), N(y)) \leq \theta d(x, y) \text{ for each } x, y \in X;$$

(*b*) *a contraction if and only if it is $\theta$−Lipschitz with $\theta < 1$.*

**Lemma 4.4.** *(Lasota and Opial [29]) Let $X$ be a Banach space. Let $F : [a, b] \times \mathbb{R}^3 \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an $L^1-$ Carathéodory multivalued map and let $\Theta$ be a linear continuous mapping from $L^1([a, b], \mathbb{R})$ to $C([a, b], \mathbb{R})$. Then the operator*

$$\Theta \circ S_{F,u} : C([a, b], \mathbb{R}) \to \mathcal{P}_{cp,c}(C([a, b], \mathbb{R})),\ \ u \mapsto (\Theta \circ S_{F,u})(u) = \Theta(S_{F,u})$$

*is a closed graph operator in $C([a, b], \mathbb{R}) \times C([a, b], \mathbb{R}) \times C([a, b], \mathbb{R})$.*

**Lemma 4.5.** *[Covitz and Nadler [30]] Let $(X, d)$ be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$, where $FixN$ is the set of fixed points of the multivalued operator $N$.*

**Theorem 4.6.** *Suppose that the following conditions hold:*

($M_1$) *$F : [a, b] \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ has convex, compact values and is $L^1$-Caratheodory;*

($M_2$) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([a, b], \mathbb{R}^+)$ such that*

$$\|F(t, x)\|_p = \sup\{|x| : x \in F(t, x)\} \leq p(t)\psi(\|x\|), \forall (t, x) \in [a, b] \times \mathbb{R};$$

($M_3$) *$|g(t, x)| \leqslant A(t)$ for each $(t, x) \in [a, b] \times \mathbb{R}$ with $A \in C([a, b], \mathbb{R}^+)$;*

$(M_4)$ *there exists a real number $M > 0$ such that*

$$\frac{M}{\|p\|\psi(M)\omega_1 + \|A\|\omega_2} > 1,$$

*where $\omega_1$ and $\omega_2$ are respectively given by (9) and (10).*

*Then the boundary value problem (16) has at least one solution on $[a, b]$.*

**Proof.** Define an operator $\mathcal{G} : C([a, b], \mathbb{R}) \to \mathcal{P}(C([a, b], \mathbb{R}))$ as

$$\begin{aligned} \mathcal{G}(x)(t) &= \Big\{h \in C([a, b], \mathbb{R}) : h(t) = I^{\alpha_1+\alpha_2}v(t) - I^{\alpha_2}g(t, x(t)) \\ &+ \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b, x(b)) - I^{\alpha_1+\alpha_2}v(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i, x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k, x(\xi_k))\Big\},\ t \in [a, b], v \in S_{F,x}\Big\}, \end{aligned} \tag{17}$$

where $S_{F,x} = \{v \in L^1([a, b], \mathbb{R}) : v(t) \in F(t, x(t))\ for\ a.e.\ on\ [a, b]\}$ is the set of selections of $F$ for each $x \in C([a, b], \mathbb{R})$. We complete the proof in several steps.

**I.** By $(M_1)$, $F$ is convex and so is $S_{F,x}$. Therefore, $\mathcal{G}(x)$ is convex for each $x \in C([a, b], \mathbb{R})$.

**II.** In order to show that $\mathcal{G}$ maps bounded sets into bounded sets in $C([a, b], \mathbb{R})$, let $B_r = \{x \in C([a, b], \mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C([a, b], \mathbb{R})$. Then, $\forall\ h \in \mathcal{B}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= I^{\alpha_1+\alpha_2}v(t) - I^{\alpha_2}g(t, x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b, x(b)) - I^{\alpha_1+\alpha_2}v(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i, x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k, x(\xi_k))\Big\}, \end{aligned}$$

where $\Delta$ is given by (4). By the conditions $(M_2) - (M_3)$, it follows that $\|h\| \leq \|p\|\psi(r)\omega_1 + \|A\|\omega_2$, where $\omega_1$ and $\omega_2$ are respectively given by (9) and (10).

**III.** Here we establish that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C([a, b], \mathbb{R})$. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathcal{B}(x)$, we have

$$\begin{aligned} & |h(t_2) - h(t_1)| \\ \leq\ & I^{\alpha_1+\alpha_2}|v(t_2) - v(t_1)| + I^{\alpha_2}|g(t_2, x(t_2)) - g(t_1, x(t_1))| \\ +\ & \Big|\frac{(t_2-a)^{\alpha_2+\epsilon_1-1} - (t_1-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big| \times \Big\{I^{\alpha_2}|g(b, x(b))| + I^{\alpha_1+\alpha_2}|v(b)| \\ +\ & \sum_{i=1}^{m}|\mu_i| I^{\alpha_1+\alpha_2}|v(\eta_i)| + \sum_{i=1}^{m}|\mu_i| I^{\alpha_2}|g(\eta_i, x(\eta_i))| + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}|v(\xi_k)| + \sum_{k=1}^{n} I^{q_k+\alpha_2}|g(\xi_k, x(\xi_k))|\Big\} \\ \leq\ & \|p\|\psi(r)\Big(\frac{2(t_2-t_1)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{|(t_2-a)^{\alpha_1+\alpha_2} - (t_1-a)^{\alpha_1+\alpha_2}|}{\Gamma(\alpha_1+\alpha_2+1)}\Big) \\ +\ & \frac{\|A\|}{\Gamma(\alpha_2+1)}(2(t_2-t_1)^{\alpha_2} + |(t_2-a)^{\alpha_2} - (t_1-a)^{\alpha_2}|) + \Big|\frac{(t_2-a)^{\alpha_2+\epsilon_1-1} - (t_1-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big| \\ \times\ & \Big\{\|p\|\psi(r)\Big(\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_1+\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_i-a)^{q_k+\alpha_1+\alpha_2}}{\Gamma(q_k+\alpha_1+\alpha_2+1)}\Big) \\ +\ & \|A\|\Big(\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{m}|\mu_i(\eta_i-a)^{\alpha_2}| + \sum_{k=1}^{n}\frac{(\xi_i-a)^{q_k+\alpha_2}}{\Gamma(q_k+\alpha_2+1)}\Big)\Big\} \to 0, \end{aligned}$$

as $t_2 \to t_1$ independently of $x \in B_r$. Therefore, it follows by the Arzelá-Ascoli theorem that the operator $\mathcal{G}$ is completely continuous.

**IV.** Now we show that $\mathcal{G}$ is upper semicontinuous. It is equivalent to show that $\mathcal{G}$ has a closed graph by Lemma 4.4.
Let $x_n \to \widehat{x}, h_n \in \mathcal{G}(x_n)$ with $h_n \to \widehat{h}$, then we need to show that $\widehat{h} \in \mathcal{G}(\widehat{x})$.
For $h_n \in \mathcal{G}(x_n)$, we can find $v_n \in S_{F,x_n}$ such that, $\forall t \in [a,b]$,

$$\begin{aligned} h_n(t) &= I^{\alpha_1+\alpha_2}v_n(t) - I^{\alpha_2}g(t,x_n(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x_n(b)) - I^{\alpha_1+\alpha_2}v_n(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v_n(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x_n(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v_n(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x_n(\xi_k))\Big\} \end{aligned}$$

Next it will be shown that there exists $\widehat{v} \in S_{F,\widehat{x}}$ such that

$$\begin{aligned} \widehat{h}(t) &= I^{\alpha_1+\alpha_2}\widehat{v}(t) - I^{\alpha_2}g(t,\widehat{x}(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,\widehat{x}(b)) - I^{\alpha_1+\alpha_2}\widehat{v}(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}\widehat{v}(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,\widehat{x}(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}\widehat{v}(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,\widehat{x}(\xi_k))\Big\}, \quad \forall t \in [a,b]. \end{aligned}$$

Introduce a linear continuous operator $T : L^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ as

$$\begin{aligned} v &\longmapsto (Tv)(t) = I^{\alpha_1+\alpha_2}v(t) - I^{\alpha_2}g(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x(b)) - I^{\alpha_1+\alpha_2}v(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x(\xi_k))\Big\}. \end{aligned}$$

Notice that $\|h_n - \widehat{h}\| \to 0$ as $n \to \infty$. Then, by Lemma 4.4, $T \circ S_F$ is a closed graph operator. Moreover, we have that $h_n(t) \in T(S_{F,x_n})$. As $x_n \to \widehat{x}$, we get

$$\begin{aligned} \widehat{h}(t) &= I^{\alpha_1+\alpha_2}\widehat{v}(t) - I^{\alpha_2}g(t,\widehat{x}(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,\widehat{x}(b)) - I^{\alpha_1+\alpha_2}\widehat{v}(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}\widehat{v}(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,\widehat{x}(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}\widehat{v}(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,\widehat{x}(\xi_k))\Big\}, \quad \forall t \in [a,b], \end{aligned}$$

for some $\widehat{v} \in S_{F,\widehat{x}}$.

**V.** In this step, it will be shown that there exists an open set $U \subseteq C([a,b],\mathbb{R})$ with $x \notin \delta\mathcal{G}(x)$ for any $\delta \in (0,1)$ and all $x \in \partial U$.
Let $\delta \in (0,1)$ and $x \in \delta\mathcal{G}(x)$, then there exists $v \in L^1([a,b],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [a,b]$,

$$\begin{aligned} x(t) &= \delta I^{\alpha_1+\alpha_2}v(t) - \delta I^{\alpha_2}g(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\delta\Big\{I^{\alpha_2}g(b,x(b)) - I^{\alpha_1+\alpha_2}v(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x(\xi_k))\Big\} \end{aligned}$$

As in the step **II**, one can obtain $\|x\| \le \|p\|\psi(\|x\|)\omega_1 + \|A\|\omega_2$, which can alternatively be written as

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\omega_1 + \|A\|\omega_2} \le 1.$$

On the other hand, by $(M_4)$, we can find $M > 0$ with $\|x\| \neq M$. Define an open set $U = \{x \in C([a,b],\mathbb{R}) : \|x\| < M\}$ and note that the operator $\mathcal{G} : \overline{U} \to \mathcal{P}_{c,cp}(C([a,b],\mathbb{R}))$ is compact multivalued map and upper semicontinuous with convex closed values. From the choice of $U$, there is no $x \in \partial U$ such that $x \in \delta\mathcal{G}$. Therefore, by Lemma 4.2, $\mathcal{G}$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (16). This completes the proof. □

In the following result, we apply Lemma 4.5 to prove the existence of at least one solution for the problem (16) when the multivalued map $F : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonconvex values.

**Theorem 4.7.** *Let $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant L and the following hypotheses hold:*

$(M_5)$ $F : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(.,x) : [a,b] \times \mathbb{R} \to \mathcal{P}_{b,cl}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$;

$(M_6)$ $H_d(F(t,x), F(t,y)) \le m(t)|x-y|$ *for almost all* $t \in [a,b]$ *and* $x, y \in \mathbb{R}$ *with* $m \in C([a,b],\mathbb{R}^+)$ *and* $d(0, F(t,0)) \le m(t)$ *for almost all* $t \in [a,b]$;

$(M_7)$ $(\|m\|\omega_1 + L\omega_2) < 1$.

*Then there exists at least one solution for the problem (16) on* $[a,b]$.

**Proof.** In view of the assumption $(M_5)$, one can notice that the set $S_{F,x} \neq \emptyset$ for each $x \in C([a,b],\mathbb{R})$ and hence $F$ has a measurable selection (see Theorem III.6 in [32]). Now we show that $\mathcal{G}(x) \in \mathcal{P}_{cl}(C([a,b],\mathbb{R})$ *for each* $x \in C([a,b],\mathbb{R})$, where $\mathcal{G}$ is defined by (17). Let $\{u_n\}_{n\ge 0} \in \mathcal{G}(x)$ be such that $u_n \to u$ as $n \to \infty$ in $C([a,b],\mathbb{R})$. Then $u \in C([a,b],\mathbb{R})$ and assume that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$\begin{aligned} u_n(t) &= I^{\alpha_1+\alpha_2}v_n(t) - I^{\alpha_2}g(t,x_n(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x_n(b)) - I^{\alpha_1+\alpha_2}v_n(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v_n(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x_n(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v_n(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x_n(\xi_k))\Big\}. \end{aligned}$$

Since $F$ has compact values by $(M_5)$, we can find $v \in L^1([a,b],\mathbb{R})$ such that $v_n \to v$ as $n \to \infty$. In consequence, $v \in S_{F,x}$ and for each $t \in [a,b]$, we have $u_n(t) \to u(t)$, where

$$\begin{aligned} u(t) &= I^{\alpha_1+\alpha_2}v(t) - I^{\alpha_2}g(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x(b)) - I^{\alpha_1+\alpha_2}v(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x(\xi_k))\Big\}, \quad \forall t \in [a,b]. \end{aligned}$$

Hence $u \in \mathcal{G}(x)$, which implies that $\mathcal{G}(x) \in \mathcal{P}_{cl}(C([a,b],\mathbb{R})$.
Next, we show that $\mathcal{G}$ is a contraction, that is, $H_d(\mathcal{G}(x),\mathcal{G}(y)) \le \theta\|x-y\|, \forall x,y \in C([a,b],\mathbb{R})$ and $\theta \in (0,1)$. Let $x, y \in C([a,b],\mathbb{R})$ and $h_1 \in \mathcal{G}(x)$, then there exists $v_1(t) \in F(t,x(t))$ such that, $\forall t \in [a,b]$,

$$\begin{aligned} h_1(t) &= I^{\alpha_1+\alpha_2}v_1(t) - I^{\alpha_2}g(t,x(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{I^{\alpha_2}g(b,x(b)) - I^{\alpha_1+\alpha_2}v_1(b) + \sum_{i=1}^{m}\mu_i I^{\alpha_1+\alpha_2}v_1(\eta_i) \\ &- \sum_{i=1}^{m}\mu_i I^{\alpha_2}g(\eta_i,x(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2}v_1(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2}g(\xi_k,x(\xi_k))\Big\}. \end{aligned}$$

By $(M_6)$, we have

$$H_d(F(t,x), F(t,y)) \le m(t)|x(t)-y(t)|.$$

Therefore, we can find $w \in F(t,y(t))$ such that

$$|v_1(t) - w| \le m(t)|x-y|, \ \ t \in [a,b].$$

Now we define $\mathcal{V} : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - y(t)|\}.$$

Since $\mathcal{V} \cap F(t, y(t))$ is measurable (Proposition 3.4 in [32]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{V}$. Hence we can find $v_2(t) \in F(t, y(t))$ such that

$$|v_1(t) - v_2(t)| \leq m(t)|x(t) - y(t)|, \ \forall t \in [a,b].$$

For each $t \in [a,b]$, let

$$\begin{aligned} h_2(t) &= I^{\alpha_1+\alpha_2} v_2(t) - I^{\alpha_2} g(t, y(t)) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{ I^{\alpha_2} g(b, y(b)) - I^{\alpha_1+\alpha_2} v_2(b) + \sum_{i=1}^{m} \mu_i I^{\alpha_1+\alpha_2} v_2(\eta_i) \\ &- \sum_{i=1}^{m} \mu_i I^{\alpha_2} g(\eta_i, y(\eta_i)) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2} v_2(\xi_k) - \sum_{k=1}^{n} I^{q_k+\alpha_2} g(\xi_k, y(\xi_k))\Big\}. \end{aligned}$$

Then we have

$$|h_1(t) - h_2(t)| \leq (\|m\|\omega_1 + L\omega_2)\|x - y\|,$$

which implies that $\|h_1 - h_2\| \leq (\|m\|\omega_1 + L\omega_2)\|x - y\|$. Interchanging the roles of $x$ and $y$, we have

$$H_d(\mathcal{G}(x), \mathcal{G}(y)) \leq (\|m\|\omega_1 + L\omega_2)\|x - y\|,$$

which, by $(M_7)$, shows that $\mathcal{G}$ is a contraction. Thus it follows by Lemma 4.5 that $\mathcal{G}$ has a fixed point, which is indeed a solution of the problem (16) on $[a,b]$. □

## 5. Special case-Langevin equation and inclusions

Letting $g(t, x(t)) = \lambda x(t)$ in the problems (1) and (16), we obtain the nonlocal multi-point integral boundary value problems involving Hilfer type Langevin fractional differential equation and inclusions respectively given by

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}({}^{H}D^{\alpha_2,\beta_2} + \lambda)x(t) = f(t, x(t)), \ t \in [a,b], \\ x(a) = 0, \ x(b) = \displaystyle\sum_{i=1}^{m} \mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k} x(\xi_k), \ a < \eta_i, \xi_k < b, \end{cases} \tag{18}$$

and

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}({}^{H}D^{\alpha_2,\beta_2} + \lambda)x(t) \in F(t, x(t)), \ t \in [a,b], \\ x(a) = 0, \ x(b) = \displaystyle\sum_{i=1}^{m} \mu_i x(\eta_i) + \sum_{k=1}^{n} I^{q_k} x(\xi_k), \ a < \eta_i, \xi_k < b. \end{cases} \tag{19}$$

### 5.1. *Existence and uniqueness results for the problem (18)*

In relation to the problem (18), the fixed point operator $\widehat{\mathcal{T}} : C([a,b], \mathbb{R}) \to C([a,b], \mathbb{R})$ is

$$\begin{aligned} (\widehat{\mathcal{T}} x)(t) &= I^{\alpha_1+\alpha_2} f(t, x(t)) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{ \lambda I^{\alpha_2} x(b) - I^{\alpha_1+\alpha_2} f(b, x(b)) + \sum_{i=1}^{m} \mu_i I^{\alpha_1+\alpha_2} f(\eta_i, x(\eta_i)) \\ &- \lambda \sum_{i=1}^{m} \mu_i I^{\alpha_2} x(\eta_i) + \sum_{k=1}^{n} I^{q_k+\alpha_1+\alpha_2} f(\xi_k, x(\xi_k)) - \lambda \sum_{k=1}^{n} I^{q_k+\alpha_2} x(\xi_k)\Big\}. \end{aligned}$$

Now we formulate the existence and uniqueness results for the problem (18). One can prove these results with the aid of the above operator following the method of proof employed in Section 3. So we do not provide the proofs for these results.

**Theorem 5.1.** *Assume that*

($\overline{H}_1$) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $|f(t,x)| \leqslant \phi(t)$, $\forall (t,x) \in [a,b] \times \mathbb{R}$, *where* $\phi(t) \in C([a,b], \mathbb{R}^+)$;

($\overline{H}_2$) $|\lambda|\omega_2 < 1$, *where* $\omega_2$ *is given by (10).*

*Then the problem (18) has at least one solution on* $[a,b]$.

**Theorem 5.2.** *Assume that the following conditions hold:*

($\overline{H}_3$) $|f(t,x) - f(t,y)| \leq L|x-y|, L > 0$ *for each* $t \in [a,b]$ *and* $x, y \in \mathbb{R}$.

($\overline{H}_4$) $L\omega_1 + |\lambda|\omega_2 < 1$, *where* $\omega_1, \omega_2$ *are given by (9) and (10) respectively.*

*Then the problem (18) has a unique solution on* $[a,b]$.

*5.1.1. Examples*

In this subsection, we illustrate the above-stated results.

**Example 5.3.** *Consider an integral-multipoint boundary value problem for Hilfer type Langevin equation given by*

$$\begin{cases} {}^{H}D^{1/2,1/4}({}^{H}D^{2/3,4/5} + 1/9)x(t) = \dfrac{\tan^{-1} x}{\pi(1+t)} + \dfrac{1}{2(1+t)}, \quad t \in [0,1], \\ x(0) = 0, \ x(1) = (1/20)x(1/6) + (1/13)x(1/3) + (1/12)x(1/2) + I^{7/2}x(2/3) + I^{5/2}x(5/6). \end{cases} \tag{20}$$

*Here* $\alpha_1 = 1/2, \beta_1 = 1/4, \alpha_2 = 2/3, \beta_2 = 4/5, \lambda = 1/9, a = 0, b = 1, \mu_1 = 1/20, \mu_2 = 1/13, \mu_3 = 1/12, q_1 = 7/2, q_2 = 5/2, \eta_1 = 1/6, \eta_2 = 1/3, \eta_3 = 1/2, \xi_1 = 2/3, \xi_2 = 5/6, m = 3, n = 2$. *Clearly* $\alpha_2 + \epsilon_1 = 31/24 > 1$, *and* $|f(t,x)| \leq 1/(1+t) = \phi(t)$. *With the given values, it is found that* $|\Delta| \approx 1.3158, \omega_1 \approx 1.7869, \omega_2 \approx 2.2113$, *and* $|\lambda|\omega_2 \approx 0.2457 < 1$. *As all the conditions of Theorem 5.1 are satisfied, therefore its conclusion implies that the problem (20) has at least one solution in* $[0,1]$.

**Example 5.4.** *Consider the problem*

$$\begin{cases} {}^{H}D^{1/2,1/2}({}^{H}D^{1/2,1/7} + 1/10)x(t) = (1/20)\tan^{-1} x + e^{-t}, \quad t \in [0,7/10] \\ x(0) = 0, x(7/10) = (1/11)x(1/5) + (1/10)x(3/10) + I^{7/2}x(2/5) + I^{9/2}x(1/2). \end{cases} \tag{21}$$

*It is easy to check that* $|f(t,x) - f(t,y)| \leq L|x-y|$ *with* $L = 1/20$. *Using the given data, we find that* $|\Delta| \approx 1.1601, \omega_1 \approx 1.3512, \omega_2 \approx 1.8603$ *and* $L\omega_1 + |\lambda|\omega_2 = 0.2536 < 1$. *Since the hypothesis of Theorem 5.2 holds true, so its conclusion applies to the problem (21).*

*5.2. Existence results for the problem (19)*

**Theorem 5.5.** *Assume that hypotheses* $(M_1)$ *and* $(M_2)$ *hold. In addition, there exists a positive real number* $\mathcal{M}$ *such that*

$$\frac{(1 - |\lambda|\omega_2)\mathcal{M}}{\|p\|\psi(\mathcal{M})\omega_1} > 1,$$

*where* $\omega_1, \omega_2$ *are given by (9) and (10) respectively. Then the boundary value problem (19) has at least one solution on* $[a,b]$.

**Theorem 5.6.** *Suppose that the conditions* $(M_5)$ *and* $(M_6)$ *are satisfied. Then the problem (19) has at least one solution on* $[a,b]$ *if* $\|m\|\omega_1 + |\lambda|\omega_2 < 1$, *where* $\omega_1, \omega_2$ *are respectively given by (9) and (10).*

**Example 5.7.** *Consider the inclusions problem*

$$\begin{cases} {}^{H}D^{1/2,1/4}({}^{H}D^{2/3,4/5} + 1/9)x(t) \in F(t, x(t)), \ t \in [0,1], \\ x(0) = 0, \ x(1) = (1/20)x(1/6) + (1/13)x(1/3) + (1/12)x(1/2) + I^{7/2}x(2/3) + I^{5/2}x(5/6). \end{cases} \tag{22}$$

*Here* $\alpha_1 = 1/2, \beta_1 = 1/4, \alpha_2 = 2/3, \beta_2 = 4/5, \lambda = 1/9, a = 0, b = 1, \mu_1 = 1/20, \mu_2 = 1/13, \mu_3 = 1/12, q_1 = 7/2, q_2 = 5/2, \xi_1 = 2/3, \xi_2 = 5/6, m = 3, n = 2$. *Using the given values, we find that* $\epsilon_1 = 5/8, \alpha_2 + \epsilon_1 = \dfrac{31}{24} > 1, |\Delta| \approx 1.3158, \omega_1 \approx 1.7869, \omega_2 \approx 2.2113$. *Let us take*

$$F(t,x(t)) = \Big[\frac{1}{17e^{-t}}(|x| + \frac{|x|}{|x|+2} + 1/7), \;\; \frac{(1+t)}{20}\sin|x|\Big]. \tag{23}$$

*Then* $p(t) = e/17$ *and* $\psi(\|x\|) = \|x\| + 8/7$. *By (5.5), we have* $\mathcal{M} > 0.6968$. *Therefore, the hypothesis of Theorem 5.5 holds true and hence it follows by its conclusion that the problem (22) with* $F(t,x(t))$ *given by (23) has at least one solution on* $[0,1]$.
*For illustrating Theorem 5.6, we consider*

$$F(t,x(t)) = \Big[\frac{t+1}{16}(\tan^{-1}|x| + 1/5), \;\; \frac{(1+\sin t)(|x|)}{32(1+|x|)} + \frac{1}{4}\Big]. \tag{24}$$

*Then F is continuous and measurable, and*

$$H_d(F(t,x), F(t,y)) \le \frac{1}{16}(t+1)|x-y|, \forall x, y \in \mathbb{R}.$$

*Moreover,* $d(0, F(t,0)) \le m(t)$, *where* $m(t) = (t+1)/16$, $\|m\| = 1/8$. *Furthermore,* $\|m\|\omega_1 + |\lambda|\omega_2 \simeq 0.4690 < 1$. *Since the assumptions of Theorem 5.6 are satisfied, therefore we deduce by its conclusion that the problem (22) with* $F(t,x(t))$ *given by (24) has at least one solution on* $[0,1]$.

## 6. Conclusions

We have presented the existence criteria for solutions of Hilfer-type fractional differential equation and inclusions involving mixed nonlinearities equipped with nonlocal integral-multipoint boundary conditions. The standard fixed point theorems are applied to achieve the desired results. As a special case, we deduce the new existence results for Hilfer-type Langevin equation and inclusions, which generalize the results obtained in [15]. In the nutshell, the work established in the article is new and enrich the related literature.

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