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On the Weighted Generalization of Hermite-Hadamard Type Inclusions for Interval-Valued Convex Functions

Hüseyin Budak[a], **Hasan Kara**[a]

[a]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye*

Abstract. In this paper, firstly we give weighted Jensen inequality for interval valued functions.Then, by using Jensen inequality, we establish weighted Hermite-Hadamard type inclusions for interval-valued functions. Moreover, we obtain some inclusions of weighted Hermite-Hadamard type for co-ordinated convex interval-valued functions. These inclusions are generalizations of some results given in earlier works.

1. Introduction

Over the last century, integral inequalities have attracted interest of a good many researchers because of the importance in applied and pure mathematics. For example, Hermite-Hadamard inequalities, based on convex functions, have an important place in many areas of mathematics, specifically optimization theory. These inequalities, introduced by C. Hermite and J. Hadamard, express that if $\phi : I \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$, then

$$\phi\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \phi(\kappa_1) d\kappa_1 \leq \frac{\phi(\tau_1) + \phi(\tau_2)}{2}. \tag{1}$$

If ϖ is concave, both of the inequalities hold in the opposite direction. The best known results associated with these inequalities are Midpoint and Trapezoid inequalities which are frequently used in Special means and estimation errors (see [10, 14]). Afterwards, many authors derived new results related to these inequalities under various conditions of the mappings. Also, some researchers examined refinements, counterparts and generalizations of the inequalities (1).

The general structure of this paper consists of four main sections including introduction. In this section, we give weighted Hermite-Hadamard inequality for real valued functions and Hermite-Hadamard inclusion for interval valued functions. We also mention some related works in the literature. In Section 2, we present some basic informations about one and two variables interval-valued functions, respectively. In Section 3, we first provide weighted Jensen inclusion for interval valued-functions. Then we also prove some weighted Hermite-Hadamard type inclusions for interval-valued convex functions. Finally, by

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Email addresses: hsyn.budak@gmail.com (Hüseyin Budak), hasan64kara@gmail.com (Hasan Kara)

applying the inclusions given in Section 3, we establish weighted Hermite-Hadamard type inclusions for interval-valued co-ordinated convex functions in Section 4. We note that the opinion and technique of this work may inspire new research in this area.

The weighted version of the inequality (1), which is also named Hermite-Hadamard-Fejér inequality, was established by Fejér in [11] as follows:

Theorem 1.1. *Suppose that $\phi : [\tau_1, \tau_2] \to \mathbb{R}$ is a convex function, and let $\varpi : [\tau_1, \tau_2] \to \mathbb{R}$ be non-negative, integrable, and symmetric about $\kappa_1 = \frac{\tau_1+\tau_2}{2}$ (i.e. $\varpi(\kappa_1) = \varpi(\tau_1 + \tau_2 - \kappa_1)$). Then, we have the inequality*

$$\phi\left(\frac{\tau_1 + \tau_2}{2}\right)\int_{\tau_1}^{\tau_2} \varpi(\kappa_1)d\kappa_1 \leq \int_{\tau_1}^{\tau_2} \phi(\kappa_1)\varpi(\kappa_1)d\kappa_1 \leq \frac{\phi(\tau_1) + \phi(\tau_2)}{2}\int_{\tau_1}^{\tau_2} \varpi(\kappa_1)d\kappa_1. \tag{2}$$

Many mathematicians derived some generalizations and new results involving fractional integrals regarding to the inequality (2) to obtain new bounds for the left and right sides of the inequality (2) (for example, [1, 27–29]).

On the other side, interval analysis handled as one of the methods of solving interval uncertainty is an important material which is used in mathematical and computer models. Although this theory has a long history which may be dated back to Archimedes' calculation of the circumference of a circle, a considerable study was not published in this field until 1950s. The first book [17] about interval analysis was published by Ramon E. Moore known as the pioneer of interval calculus in 1966. Thereafter, a great many researchers started to investigate theories and applications of interval analysis. Recently, many authors have focused on integral inequalities obtained by using interval-valued functions. For example, Sadowska [26] established Hermite-Hadamard inequality for set-valued functions that is more general version of interval-valued mappings as follows:

Theorem 1.2. ([26]) *Suppose that $\Phi : [\tau_1, \tau_2] \to \mathbb{R}_I^+$ is interval–valued convex function such that $\Phi(t) = \left[\underline{\Phi}(t), \overline{\Phi}(t)\right]$. Then, we have*

$$\Phi\left(\frac{\tau_1 + \tau_2}{2}\right) \supseteq \frac{1}{\tau_2 - \tau_1}(IR)\int_{\tau_1}^{\tau_2} \Phi(\kappa_1)d\kappa_1 \supseteq \frac{\Phi(\tau_1) + \Phi(\tau_2)}{2}. \tag{3}$$

Furthermore, well-known inequalities such as Ostrowski, Minkowski and Beckenbach and their some applications were provided by considering interval-valued functions in [5, 6, 12, 24]. In addition, some inequalities involving interval-valued Riemann-Liouville fractional integrals were derived by Budak et al. in [4]. In [15], Liu et al. gave the definition of interval-valued harmonically convex functions, and so they obtain some Hermite-Hadamard type inequalities including interval-valued fractional integrals. On the other hand Budak et al. prove some weighted Fejer type inclusions in [3]. For more details about this topic, one can refer to [2, 7, 8, 13, 16, 18–21, 25, 30, 31].

2. Preliminaries

In this section we summarize some properties of one and two variables interval-valued functions.

2.1. Integral of Interval-Valued Functions

In this subsection, the notion of integral of the interval-valued mappings is mentioned. Before we can understand the definition of integrals of interval-valued functions, we need to give some concepts in the following.

A function φ is said to be an interval-valued function of t on $[\tau_1, \tau_2]$ if it assigns a non-empty interval to each $t \in [\tau_1, \tau_2]$

$$\varphi(t) = \left[\underline{\varphi}(t), \overline{\varphi}(t)\right].$$

A partition of $[\tau_1, \tau_2]$ is any finite ordered subset D having the form

$$D : \tau_1 = t_0 < t_1 < ... < t_n = \tau_2.$$

The mesh of a partition D is indicated by

$$mesh(D) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denote by $D([\tau_1, \tau_2])$ the set of all partition of $[\tau_1, \tau_2]$. Suppose that $D(\delta, [\tau_1, \tau_2])$ is the set of all $D \in D([\tau_1, \tau_2])$ such that $mesh(D) < \delta$. We take an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $i = 1, 2, ..., n$, and we define the sum

$$S(\varphi, D, \delta) = \sum_{i=1}^{n} \varphi(\xi_i)[t_i - t_{i-1}]$$

where $\varphi : [\tau_1, \tau_2] \to \mathbb{R}_{\mathcal{I}}$. The sum $S(\varphi, D, \delta)$ is said to be a Riemann sum of φ corresponding to $D \in D(\delta, [\tau_1, \tau_2])$.

Definition 2.1. ([22],[23]) $\varphi : [\tau_1, \tau_2] \to \mathbb{R}_{\mathcal{I}}$ is said to be an interval Riemann integrable function (*IR*-integrable) on $[\tau_1, \tau_2]$ if there exist $A \in \mathcal{P}$ and $\delta > 0$, for each $\varepsilon > 0$, such that

$$d(S(\varphi, D, \delta), A) < \varepsilon$$

for every Riemann sum S of φ corresponding to each $D \in D(\delta, [\tau_1, \tau_2])$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$, $1 \le i \le n$. In this case, A is called as the *IR*-integral of φ on $[\tau_1, \tau_2]$ and is denoted by

$$A = (IR)\int_{\tau_1}^{\tau_2} \varphi(t)dt.$$

The collection of all functions that are *IR*-integrable on $[\tau_1, \tau_2]$ will be denote by $\mathcal{IR}_{([\tau_1,\tau_2])}$.

The next theorem explains connection between *IR*-integrable and Riemann integrable (*R*-integrable):

Theorem 2.2. *Assume that $\varphi : [\tau_1, \tau_2] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued function such that $\varphi(t) = \left[\underline{\varphi}(t), \overline{\varphi}(t)\right]$. $\varphi \in \mathcal{IR}_{([\tau_1,\tau_2])}$ if and only if $\underline{\varphi}(t), \overline{\varphi}(t) \in \mathcal{R}_{([\tau_1,\tau_2])}$ and*

$$(IR)\int_{\tau_1}^{\tau_2} \varphi(t)dt = \left[(R)\int_{\tau_1}^{\tau_2} \underline{\varphi}(t)dt, (R)\int_{\tau_1}^{\tau_2} \overline{\varphi}(t)dt\right]$$

where $\mathcal{R}_{([\tau_1,\tau_2])}$ denotes the all R-integrable function.

It is easy to see that if $\varphi(t) \subseteq \psi(t)$ for all $t \in [\tau_1, \tau_2]$, then $(IR)\int_{\tau_1}^{\tau_2} \varphi(t)dt \subseteq (IR)\int_{\tau_1}^{\tau_2} \psi(t)dt$

2.2. Interval-Valued Double Integral and Co-ordinated Convexity

A set of numbers $\{t_{i-1}, \xi_i, t_i\}_{i=1}^{m}$ is called tagged partition P_1 of $[\tau_1, \tau_2]$ if

$$P_1 : \tau_1 = t_0 < t_1 < \ldots < t_n = \tau_2$$

and if $t_{i-1} \le \xi_i \le t_i$ for all $i = 1, 2, 3, \ldots, m$. Moreover if we have $\Delta t_i = t_i - t_{i-1}$, then P_1 is said to be $\delta-$fine if $\Delta t_i < \delta$ for all i. Let $\mathcal{P}(\delta, [\tau_1, \tau_2])$ denote the set of all $\delta-$fine partitions of $[\tau_1, \tau_2]$. If $\{t_{i-1}, \xi_i, t_i\}_{i=1}^{m}$ is a $\delta-$fine P_1 of $[\tau_1, \tau_2]$ and if $\{s_{j-1}, \eta_j, s_j\}_{j=1}^{n}$ is $\delta-$fine P_2 of $[\tau_3, \tau_4]$, then rectangles

$$\Delta_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$$

are the partition of the rectangle $\Delta = [\tau_1, \tau_2] \times [\tau_3, \tau_4]$ and the points (ξ_i, η_j) are inside the rectangles $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. Further, by $\mathcal{P}(\delta, \Delta)$ we denote the set of all $\delta-$fine partitions P of Δ with $P_1 \times P_2$, where $P_1 \in \mathcal{P}(\delta, [\tau_1, \tau_2])$ and $P_2 \in \mathcal{P}(\delta, [\tau_3, \tau_4])$. Let $\Delta A_{i,j}$ be the area of rectangle $\Delta_{i,j}$. In each rectangle $\Delta_{i,j}$, where $1 \leq i \leq m,\ 1 \leq j \leq n$, choose arbitrary (ξ_i, η_j) and get

$$S(F, P, \delta, \Delta) = \sum_{i=1}^{m} \sum_{j=1}^{n} F(\xi_i, \eta_j) \Delta A_{i,\ j}.$$

We call $S(F, P, \delta, \Delta)$ is integral sum of F associated with $P \in \mathcal{P}(\delta, \Delta)$.
Now we recall the concept of interval-valued double integral given by Zhao et al. in [31].

Theorem 2.3. ([31]) *Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then F is called $ID-$integrable on Δ with $ID-$integral $U = (ID) \iint\limits_{\Delta} F(t, s)\tau_4 A$, if for any $\epsilon > 0$ there exist $\delta > 0$ such that*

$$d(S(F, P, \delta, \Delta), U) < \epsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all $ID-$integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$.

Theorem 2.4. ([31]) *Let $\Delta = [\tau_1, \tau_2] \times [\tau_3, \tau_4]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is $ID-$integrable on Δ, then we have*

$$(ID) \iint\limits_{\Delta} F(s, t)\tau_4 A = (IR) \int_{\tau_1}^{\tau_2} (IR) \int_{\tau_3}^{\tau_4} F(s, t) ds dt.$$

Definition 2.5. ([32]) A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is said to be interval-valued co-ordinated convex function, if the following inequality holds:

$$\begin{aligned} & F(t\kappa_1 + (1-t)\kappa_2, su + (1-s)w) \\ \supseteq\ & tsF(\kappa_1, u) + t(1-s)F(\kappa_1, w) + s(1-t)F(\kappa_2, u) + (1-s)(1-t)F(\kappa_2, w), \end{aligned}$$

for all $(\kappa_1, \kappa_2), (u, w) \in \Delta$ and $s, t \in [0, 1]$.

3. Weighted Hermite-Hadamard Type Inclusions for Interval-Valued Convex Functions

In this section we prove some weighted Hermite-Hadamard type inclusions for interval valued convex functions.

First we need to following weighted Jensen inclusion:

Theorem 3.1 (Weighted Jensen Inclusion). *Let $g : [\tau_1, \tau_2] \to [\tau_1, \tau_2]$ be a function from $L^{\infty}[\tau_1, \tau_2]$ and $w : [\tau_1, \tau_2] \to \mathbb{R}$ be non-negative functions from $L^1[\tau_1, \tau_2]$ such that $\int_{\tau_1}^{\tau_2} w(t)dt \neq 0$. If $F : [\tau_1, \tau_2] \to \mathbb{R}_I$ is an interval–valued convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have*

$$F\left(\frac{1}{\int_{\tau_1}^{\tau_2} w(t)\, dt} \int_{\tau_1}^{\tau_2} w(t) g(t)\, dt\right) \supseteq \frac{1}{\int_{\tau_1}^{\tau_2} w(t)\, dt} (IR) \int_{\tau_1}^{\tau_2} F(g(t))\, w(t)\, dt.$$

Proof. The proof can be easily seen by applying the classical Jensen inequality to convex function $\underline{F}$ and concave function $\overline{F}$. □

Theorem 3.2. *Suppose that $F:[\tau_1,\tau_2]\to\mathbb{R}_I^+$ is interval–valued convex function such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. Let $g:[0,1]\to\mathbb{R}$ be a non-negative and Riemann integrable function and let $\lambda=\left(\int_0^1 tg(t)\,dt\right)\Big/\left(\int_0^1 g(t)\,dt\right)$. Then, we have*

$$F(\lambda\tau_1+(1-\lambda)\tau_2)\supseteq\frac{1}{G}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1)\,g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1\supseteq\lambda F(\tau_1)+(1-\lambda)F(\tau_2) \tag{4}$$

where

$$G=\int_{\tau_1}^{\tau_2}g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1.$$

Proof. By changing variable to $\kappa_1=t\tau_1+(1-t)\tau_2$, we get

$$\frac{1}{\int_{\tau_1}^{\tau_2}g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1)\,g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1=\frac{1}{\int_0^1 g(t)\,dt}(IR)\int_0^1F(t\tau_1+(1-t)\tau_2)\,g(t)\,dt. \tag{5}$$

Since F is interval-valued convex function on $[\tau_1,\tau_2]$, we have

$$F(t\tau_1+(1-t)\tau_2)\supseteq tF(\tau_1)+(1-t)F(\tau_2)$$

As g is non-negative and integrable on $[0,1]$, we can write

$$\begin{aligned}(IR)\int_0^1F(t\tau_1+(1-t)\tau_2)\,g(t)dt&\supseteq(IR)\int_0^1(tF(\tau_1)+(1-t)F(\tau_2))\,g(t)dt\\&=F(\tau_1)\int_0^1tg(t)dt+F(\tau_2)\int_0^1(1-t)g(t)dt.\end{aligned} \tag{6}$$

Combining inclusions (5) and (6) yields

$$\begin{aligned}&\frac{1}{\int_{\tau_1}^{\tau_2}g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1)\,g\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1\\&\supseteq\left(\frac{1}{\int_0^1g(t)dt}\int_0^1tg(t)dt\right)F(\tau_1)+\left(1-\frac{1}{\int_0^1g(t)dt}\int_0^1tg(t)dt\right)F(\tau_2)\\&=\lambda F(\tau_1)+(1-\lambda)F(\tau_2)\end{aligned}$$

which gives the proof right-hand side of (4).

In view of the assumption that F is interval-valued convex function, g is nonnegative and integrable on $[0,1]$, we deduce from the weighted Jensen inequality for interval-valued function that

$$\begin{aligned} &\frac{1}{\int_0^1 g(t)\,dt}(IR)\int_0^1 F(t\tau_1+(1-t)\tau_2)\,g(t)\,dt \\ \subseteq\ & F\left(\frac{1}{\int_0^1 g(t)\,dt}\int_0^1 (t\tau_1+(1-t)\tau_2)\,g(t)\,dt\right) \\ =\ & F\left[\left(\frac{1}{\int_0^1 g(t)\,dt}\int_0^1 tg(t)\,dt\right)\tau_1+\left(1-\frac{1}{\int_0^1 g(t)\,dt}\int_0^1 tg(t)\,dt\right)\tau_2\right] \\ =\ & F(\tau_1\lambda+(1-\lambda)\tau_2). \end{aligned}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. If we choose $g(t)=1$ for all $t\in[0,1]$, the Theorem 3.2 reduces to Theorem 1.2.

Theorem 3.4. *Suppose that $F:[\tau_1,\tau_2]\to\mathbb{R}_I^+$ is interval–valued convex function such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. Let g be a nonnegative and integrable function on $[\tau_1,\tau_2]$, and let*

$$\lambda=\left(\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g(\tau_1+\tau_2-\kappa_1)\,d\kappa_1\right)\Bigg/\left(\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g(\kappa_1)\,d\kappa_1\right).$$

Then, we have

$$F\left(\frac{\tau_1+\lambda\tau_2}{1+\lambda}\right)\supseteq\frac{1}{\int_{\tau_1}^{\tau_2}g(\kappa_1)\,d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1)\,g(\kappa_1)\,d\kappa_1\supseteq\frac{F(\tau_1)+\lambda F(\tau_2)}{1+\lambda}.$$

Proof. Since g is a nonnegative and integrable function on $[\tau_1,\tau_2]$, then it can be easily seen that $\varphi(\kappa_1)=g(\tau_2-(\tau_2-\tau_1)\kappa_1)$ is nonnegative and integrable function on $[0,1]$. Therefore, by applying Theorem 3.2 , we obtain

$$F(\mu\tau_1+(1-\mu)\tau_2)\supseteq\frac{1}{\int_{\tau_1}^{\tau_2}\varphi\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1)\,\varphi\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1\supseteq\mu F(\tau_1)+(1-\mu)F(\tau_2),$$

where

$$\mu=\frac{1}{\int_0^1\varphi(t)\,dt}\int_0^1 t\varphi(t)\,dt=\frac{1}{\int_0^1 g(\tau_2-(\tau_2-\tau_1)t)\,dt}\int_0^1 tg(\tau_2-(\tau_2-\tau_1)t)\,dt$$

$$= \frac{1}{\int_{\tau_1}^{\tau_2} (\tau_2 - \tau_1) g(\kappa_1) d\kappa_1} \int_{\tau_1}^{\tau_2} (\tau_2 - \kappa_1) g(\kappa_1) d\kappa_1$$

$$= \frac{1}{\int_{\tau_1}^{\tau_2} (\tau_2 - \kappa_1) g(\kappa_1) d\kappa_1 + \int_{\tau_1}^{\tau_2} (\tau_2 - \kappa_1) g(\tau_1 + \tau_2 - \kappa_1) d\kappa_1} \int_{\tau_1}^{\tau_2} (\tau_2 - \kappa_1) g(\kappa_1) d\kappa_1$$

$$= \frac{1}{1+\lambda}.$$

This completes the proof. □

Remark 3.5. Under assumption of Theorem 3.4, if we supose that g is symmetric about $\frac{\tau_1+\tau_2}{2}$, (i.e. $g(\kappa_1) = g(\tau_1 + \tau_2 - \kappa_1)$ for all $\kappa_1 \in [\tau_1, \tau_2]$), then Theorem 3.4 reduces to [15, Theorem 3.4 for $\alpha = 1$].

4. Weighted Hermite-Hadamard Type Inclusions for Interval-Valued Co-ordinated Convex Functions

In this section, we present some Hermite-Hadamard type inclusions for interval-valued inclusions for co-ordinated convex function:

Theorem 4.1. *Let $F : \Delta \to \mathbb{R}_I^+$ be a inteval-valued co-ordinated convex functions on Δ. Let $g_1 : [0,1] \to \mathbb{R}$ and $g_2 : [0,1] \to \mathbb{R}$ be two a nonnegative and Riemann integrable functions and let*

$$\lambda = \frac{1}{\int_0^1 g_1(t)\,dt} \int_0^1 t g_1(t)\,dt \text{ and } \beta = \frac{1}{\int_0^1 g_2(s)\,ds} \int_0^1 s g_2(s)\,ds.$$

Then, one has the following inclusions

$$\begin{aligned}
& F(\lambda\tau_1 + (1-\lambda)\tau_2, \beta\tau_3 + (1-\beta)\tau_4) \qquad (7)\\
\supseteq\ & \frac{1}{2}\Bigg[\frac{1}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1, \beta\tau_3 + (1-\beta)\tau_4)\, g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right) d\kappa_1 \\
& + \frac{1}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\lambda\tau_1 + (1-\lambda)\tau_2, \kappa_2)\, g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right) d\kappa_2\Bigg] \\
\supseteq\ & \frac{1}{G_1G_2}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2) g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right) g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right) d\kappa_2 d\kappa_1 \\
\supseteq\ & \frac{1}{2}\Bigg[\frac{\beta}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_3) g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right) d\kappa_1 + \frac{(1-\beta)}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_4) g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right) d\kappa_1 \\
& + \frac{\lambda}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_1,\kappa_2) g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right) d\kappa_2 + \frac{(1-\lambda)}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_2,\kappa_2) g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right) d\kappa_2\Bigg] \\
\supseteq\ & \beta\lambda f(\tau_1,\tau_3) + \lambda(1-\beta) F(\tau_1,\tau_4) + (1-\lambda)\beta f(\tau_2,\tau_3) + (1-\beta)(1-\lambda) F(\tau_2,\tau_4)
\end{aligned}$$

where

$$G_1 = \int_{\tau_1}^{\tau_2} g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1, \quad G_2 = \int_{\tau_3}^{\tau_4} g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2.$$

Proof. Since F is an interval-valued co-ordinated convex functions on Δ, if we define the mappings $F_{\kappa_1} : [\tau_3, \tau_4] \to \mathbb{R}_I^+$, $F_{\kappa_1}(\kappa_2) = F(\kappa_1, \kappa_2)$, then $F_{\kappa_1}(\kappa_2)$ is interval-valued convex on $[\tau_3, \tau_4]$ for all $\kappa_1 \in [\tau_1, \tau_2]$. If we apply the inclusion (4) for the interval-valued convex function $F_{\kappa_1}(\kappa_2)$, then we have

$$F_{\kappa_1}(\beta\tau_3 + (1-\beta)\tau_4) \supseteq \frac{1}{\int_{\tau_3}^{\tau_4} g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2}(IR)\int_{\tau_3}^{\tau_4} F_{\kappa_1}(\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 \supseteq \beta f_{\kappa_1}(\tau_3) + (1-\beta)F_{\kappa_1}(\tau_4). \tag{8}$$

That is,

$$F(\kappa_1, \beta\tau_3 + (1-\beta)\tau_4) \supseteq \frac{1}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 \supseteq \beta f(\kappa_1,\tau_3) + (1-\beta)F(\kappa_1,\tau_4). \tag{9}$$

Multiplying with $\frac{g_1(\frac{\tau_2-\kappa_1}{\tau_2-\kappa_1})}{G_1}$ the inclusion (9) and integrating with respect to κ_1 from τ_1 to τ_2, we obtain

$$\begin{aligned} &\frac{1}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1, \beta\tau_3 + (1-\beta)\tau_4)\, g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 \\ \supseteq\ & \frac{1}{G_1G_2}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 d\kappa_1 \\ \supseteq\ & \frac{\beta}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_3)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 + \frac{(1-\beta)}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_4)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1. \end{aligned} \tag{10}$$

Similarly, as F is interval-valued co-ordinated convex functions on Δ, if we define the mappings $F_{\kappa_2} : [\tau_1, \tau_2] \to \mathbb{R}_I^+$, $F_{\kappa_2}(\kappa_1) = F(\kappa_1, \kappa_2)$, then $F_{\kappa_2}(\kappa_1)$ is inteval-valued convex on $[\tau_1, \tau_2]$ for all $\kappa_2 \in [\tau_3, \tau_4]$. Utilizing the inclusion (4) for the interval-valued convex function $F_{\kappa_2}(\kappa_1)$, then we obtain the inclusion

$$F_{\kappa_2}(\lambda\tau_1 + (1-\lambda)\tau_2) \supseteq \frac{1}{\int_{\tau_1}^{\tau_2} g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2} F_{\kappa_2}(\kappa_1)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 \supseteq \lambda f_{\kappa_2}(\tau_1) + (1-\lambda)F_{\kappa_2}(\tau_2), \tag{11}$$

i.e.

$$\begin{aligned} F(\lambda\tau_1 + (1-\lambda)\tau_2, \kappa_2) \ &\supseteq\ \frac{1}{\int_{\tau_1}^{\tau_2} g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\kappa_2)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 \\ &\supseteq\ \lambda f(\tau_1,\kappa_2) + (1-\lambda)F(\tau_2,\kappa_2). \end{aligned} \tag{12}$$

Multiplying with $\frac{g_2(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3})}{G_2}$ the inclusion (12) and integrating with respect to κ_2 on $[\tau_3, \tau_4]$, we get

$$\frac{1}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\lambda\tau_1 + (1-\lambda)\tau_2, \kappa_2)\, g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 \tag{13}$$

$$\supseteq \frac{1}{G_1G_2}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 d\kappa_1$$

$$\supseteq \frac{\lambda}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_1,\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 + \frac{(1-\lambda)}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_2,\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2.$$

Summing the inclusions (10) and (13), we obtain the second and third inclusions in (7).

Since $F(\kappa_1, \beta\tau_3 + (1-\beta)\tau_4)$ is inteval-valued convex on $[\tau_1, \tau_2]$ and $g_1(\kappa_1)$ is positive and integrable, using the first inclusion in (4), we have

$$F(\lambda\tau_1 + (1-\lambda)\tau_2,\ \beta\tau_3 + (1-\beta)\tau_4) \supseteq \frac{1}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1, \beta\tau_3 + (1-\beta)\tau_4)\, g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1. \tag{14}$$

And similarly, since $F(\lambda\tau_1 + (1-\lambda)\tau_2, \kappa_2)$ is interval-valued convex on $[\tau_3, \tau_4]$ and $g_2(\kappa_2)$ is positive and integrable,by the first inclusion in (4), we get

$$F(\lambda\tau_1 + (1-\lambda)\tau_2, \beta\tau_3 + (1-\beta)\tau_4) \supseteq \frac{1}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\lambda\tau_1 + (1-\lambda)\tau_2, \kappa_2)\, g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2. \tag{15}$$

Summing the inclusions (14) and (15), we obtain the first inclusion in (7).

Since $F(\kappa_1, \tau_3)$ and $F(\kappa_1, \tau_4)$ are interval-valued convex on $[\tau_1, \tau_2]$ and $g_1(\kappa_1)$ is positive, integrable, by the second inclusion in (4), we get

$$\frac{\beta}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_3)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 + \frac{(1-\beta)}{G_1}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_4)g_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1 \tag{16}$$

$$\supseteq \beta\lambda f(\tau_1,\tau_3) + \beta(1-\lambda)F(\tau_2,\tau_3) + (1-\beta)\lambda f(\tau_1,\tau_4) + (1-\beta)(1-\lambda)F(\tau_2,\tau_4).$$

And similarly, since $F(\tau_1, \kappa_2)$ and $F(\tau_2, \kappa_2)$ are interval-valued convex on $[\tau_3, \tau_4]$ and $g_2(\kappa_2)$ is positive, integrable, by the second inclusion in (4), we have the folllowing inclusion

$$\frac{\lambda}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_1,\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 + \frac{(1-\lambda)}{G_2}(IR)\int_{\tau_3}^{\tau_4} F(\tau_2,\kappa_2)g_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 \tag{17}$$

$$\supseteq \beta\lambda f(\tau_1,\tau_3) + \lambda(1-\beta)F(\tau_1,\tau_4) + (1-\lambda)\beta f(\tau_2,\tau_3) + (1-\beta)(1-\lambda)F(\tau_2,\tau_4).$$

By summing the resulting inclusions (16) and (17), then we obtain the last inclusion in (7). This completes the proof. □

Remark 4.2. Under assumptions of Theorem 4.1 with $g_1(t) = 1$ and $g_2(s) = 1$ for all $t, s \in [0,1]$, then we have

$$F\left(\frac{\tau_1+\tau_2}{2}, \frac{\tau_3+\tau_4}{2}\right) \tag{18}$$

$$\supseteq \frac{1}{2}\left[\frac{1}{(\tau_2-\tau_1)}(IR)\int_{\tau_1}^{\tau_2} F\left(\kappa_1, \frac{\tau_3+\tau_4}{2}\right)d\kappa_1 + \frac{1}{(\tau_4-\tau_3)}(IR)\int_{\tau_3}^{\tau_4} F\left(\frac{\tau_1+\tau_2}{2}, \kappa_2\right)d\kappa_2\right]$$

$$\supseteq \frac{1}{(\tau_2-\tau_1)(\tau_4-\tau_3)}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1$$

$$\supseteq \frac{1}{4}\left[\frac{1}{(\tau_2-\tau_1)}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_3)d\kappa_1 + \frac{1}{(\tau_2-\tau_1)}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_4)d\kappa_1 + \frac{1}{(\tau_4-\tau_3)}(IR)\int_{\tau_3}^{\tau_4} F(\tau_1,\kappa_2)d\kappa_2 + \frac{1}{(\tau_4-\tau_3)}(IR)\int_{\tau_3}^{\tau_4} F(\tau_2,\kappa_2)d\kappa_2\right]$$

$$\supseteq \frac{f(\tau_1,\tau_3)+F(\tau_1,\tau_4)+f(\tau_2,\tau_3)+F(\tau_2,\tau_4)}{4}$$

which was proved by Zhao et al. in [32].

Theorem 4.3. *Let $F:\Delta\to\mathbb{R}_I^+$ be a interval-valued co-ordinated convex functions on Δ. Let $g_1:[\tau_1,\tau_2]\to\mathbb{R}$ and $g_2:[\tau_3,\tau_4]\to\mathbb{R}$ be two a nonnegative and Riemann integrable functions and let*

$$\lambda = \frac{1}{\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\kappa_1)\,d\kappa_1}\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\tau_1+\tau_2-\kappa_1)\,d\kappa_1$$

and

$$\beta = \frac{1}{\int_{\tau_3}^{\tau_4}(\tau_4-\kappa_2)\,g_2(\kappa_2)\,d\kappa_2}\int_{\tau_3}^{\tau_4}(\tau_4-\kappa_2)\,g_2(\tau_3+\tau_4-\kappa_2)\,d\kappa_2.$$

Then, one has the following inclusions

$$F\left(\frac{\tau_1+\lambda\tau_2}{1+\lambda},\frac{\tau_3+\beta\tau_4}{1+\beta}\right)$$

$$\supseteq \frac{1}{2}\left[\frac{1}{G_3}(IR)\int_{\tau_1}^{\tau_2} F\left(\kappa_1,\frac{\tau_3+\beta\tau_4}{1+\beta}\right)g_1(\kappa_1)\,d\kappa_1 + \frac{1}{G_4}(IR)\int_{\tau_3}^{\tau_4} F\left(\frac{\tau_1+\lambda\tau_2}{1+\lambda},\kappa_2\right)g_2(\kappa_2)\,d\kappa_2\right]$$

$$\supseteq \frac{1}{G_3G_4}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)g_1(\kappa_1)\,g_2(\kappa_2)\,d\kappa_2 d\kappa_1$$

$$\supseteq \frac{1}{2}\left[\frac{1}{(1+\beta)G_3}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_3)g_1(\kappa_1)d\kappa_1 + \frac{\beta}{(1+\beta)G_3}(IR)\int_{\tau_1}^{\tau_2} F(\kappa_1,\tau_4)g_1(\kappa_2)d\kappa_1 + \frac{1}{(1+\lambda)G_4}(IR)\int_{\tau_3}^{\tau_4} F(\tau_1,\kappa_2)g_2(\kappa_2)\,d\kappa_2 + \frac{\lambda}{(1+\lambda)G_4}(IR)\int_{\tau_3}^{\tau_4} F(\tau_2,\kappa_2)g_2(\kappa_2)\,d\kappa_2\right]$$

$$\supseteq \frac{F(\tau_1,\tau_3)+\beta f(\tau_1,\tau_4)+\lambda f(\tau_2,\tau_3)+\lambda\beta f(\tau_2,\tau_4)}{(1+\lambda)(1+\beta)}$$

where $G_3=\int_{\tau_1}^{\tau_2} g_1(\kappa_1)\,d\kappa_1$ and $G_4=\int_{\tau_3}^{\tau_4} g_2(\kappa_2)\,d\kappa_2$.

Proof. Based on the assumption that g_1 and g_2 are nonnegative, integrable functions on $[\tau_1,\tau_2]$ and $[\tau_3,\tau_4]$, respectively. Then, it can be shown that $\varphi_1(t)=g_1(\tau_2-(\tau_2-\tau_1)t)$ and $\varphi_2(s)=g_2(\tau_4-(\tau_4-\tau_3)s)$ are nonnegative, integrable functions on $[0,1]$. Thus, by using Theorem 4.1, we can write the following inclusions,

$$F(\gamma\tau_1+(1-\gamma)\tau_2,\delta\tau_3+(1-\delta)\tau_4)$$

$$\supseteq \frac{1}{2}\left[\frac{1}{\int_{\tau_1}^{\tau_2}\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F\left(\kappa_1,\delta\tau_3+(1-\delta)\tau_4\right)\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1\right.$$

$$\left.+\frac{1}{\int_{\tau_3}^{\tau_4}\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2}(IR)\int_{\tau_3}^{\tau_4}F\left(\gamma\tau_1+(1-\gamma)\tau_2,\kappa_2\right)\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2\right]$$

$$\supseteq \frac{1}{\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4}\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 d\kappa_1}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4}F(\kappa_1,\kappa_2)\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2 d\kappa_1$$

$$\supseteq \frac{1}{2}\left[\frac{\delta}{\int_{\tau_1}^{\tau_2}\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1,\tau_3)\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1\right.$$

$$+\frac{(1-\delta)}{\int_{\tau_1}^{\tau_2}\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1,\tau_4)\varphi_1\left(\frac{\tau_2-\kappa_1}{\tau_2-\tau_1}\right)d\kappa_1$$

$$+\frac{\gamma}{\int_{\tau_3}^{\tau_4}\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2}(IR)\int_{\tau_3}^{\tau_4}F(\tau_1,\kappa_2)\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2$$

$$\left.+\frac{(1-\gamma)}{\int_{\tau_3}^{\tau_4}\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2}(IR)\int_{\tau_3}^{\tau_4}F(\tau_2,\kappa_2\varphi_2\left(\frac{\tau_4-\kappa_2}{\tau_4-\tau_3}\right)d\kappa_2\right]$$

$$\supseteq \delta\gamma f\left(\tau_1,\tau_3\right)+\gamma\left(1-\delta\right)F\left(\tau_1,\tau_4\right)+(1-\gamma)\delta f\left(\tau_2,\tau_4\right)+\left(1-\delta\right)\left(1-\gamma\right)F\left(\tau_2,\tau_4\right).$$

This gives

$$F\left(\gamma\tau_1+(1-\gamma)\tau_2,\delta\tau_3+(1-\delta)\tau_4\right)$$

$$\supseteq \frac{1}{2}\left[\frac{1}{G_3}(IR)\int_{\tau_1}^{\tau_2}F\left(\kappa_1,\delta\tau_3+(1-\delta)\tau_4\right)g_1\left(\kappa_1\right)d\kappa_1+\frac{1}{G_4}(IR)\int_{\tau_3}^{\tau_4}F\left(\gamma\tau_1+(1-\gamma)\tau_2,\kappa_2\right)g_2\left(\kappa_2\right)d\kappa_2\right]$$

$$\supseteq \frac{1}{G_3G_4}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4}F(\kappa_1,\kappa_2)g_1\left(\kappa_1\right)g_2\left(\kappa_2\right)d\kappa_2 d\kappa_1$$

$$\supseteq \frac{1}{2}\left[\frac{\delta}{G_3}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1,\tau_3)g_1(\kappa_1)d\kappa_1+\frac{(1-\delta)}{G_3}(IR)\int_{\tau_1}^{\tau_2}F(\kappa_1,\tau_4)g_1(\kappa_1)d\kappa_1\right.$$

$$\left.+\frac{\gamma}{G_4}(IR)\int_{\tau_3}^{\tau_4}F(\tau_1,\kappa_2)g_2\left(\kappa_2\right)d\kappa_2+\frac{(1-\gamma)}{G_4}(IR)\int_{\tau_3}^{\tau_4}F(\tau_2,\kappa_2)g_2\left(\kappa_2\right)d\kappa_2\right]$$

$$\supseteq \quad \delta\gamma f(\tau_1,\tau_3) + \gamma(1-\delta) F(\tau_1,\tau_4) + (1-\gamma)\delta f(\tau_2,\tau_4) + (1-\delta)(1-\gamma) F(\tau_2,\tau_4),$$

where

$$\begin{aligned}
\gamma &= \frac{1}{\int_0^1 g_1(\tau_2-(\tau_2-\tau_1)t)\,dt}\int_0^1 t g_1(\tau_2-(\tau_2-\tau_1)t)\,dt \\
&= \frac{1}{\int_{\tau_1}^{\tau_2}(\tau_2-\tau_1)\,g_1(\kappa_1)\,d\kappa_1}\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\kappa_1)\,d\kappa_1 \\
&= \frac{\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\kappa_1)\,d\kappa_1}{\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\kappa_1)\,d\kappa_1+\int_{\tau_1}^{\tau_2}(\tau_2-\kappa_1)\,g_1(\tau_1+\tau_2-\kappa_1)\,d\kappa_1} = \frac{1}{1+\lambda}
\end{aligned}$$

and similarly

$$\delta = \frac{1}{\int_0^1 g_2(\tau_4-(\tau_4-\tau_3)s)\,ds}\int_0^1 s g_2(\tau_4-(\tau_4-\tau_3)s)\,ds = \frac{1}{1+\beta}.$$

This completes the proof. □

Corollary 4.4. *Under assumptions of Theorem 4.3, let $g_1(\kappa_1) = g_1(\tau_1+\tau_2-\kappa_1)$ for any $\kappa_1 \in [\tau_1,\tau_2]$ and $g_2(\kappa_2) = g_2(\tau_3+\tau_4-\kappa_2)$ for any $\kappa_2 \in [\tau_3,\tau_4]$, then we have the following inclusion*

$$\begin{aligned}
& F\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right) \\
\supseteq\; & \frac{1}{2}\left[\frac{1}{G_3}(IR)\int_{\tau_1}^{\tau_2} F\left(\kappa_1,\frac{\tau_3+\tau_4}{2}\right)g_1(\kappa_1)d\kappa_1 + \frac{1}{G_4}(IR)\int_{\tau_3}^{\tau_4} F\left(\frac{\tau_1+\tau_2}{2},\kappa_2\right)g_2(\kappa_2)d\kappa_2\right] \\
\supseteq\; & \frac{1}{G_3G_4}(ID)\int_{\tau_1}^{\tau_2}\int_{\tau_3}^{\tau_4} F(\kappa_1,\kappa_2)g_1(\kappa_1)g_2(\kappa_2)d\kappa_2 d\kappa_1 \\
\supseteq\; & \frac{1}{4}\left[\frac{1}{G_3}(IR)\int_{\tau_1}^{\tau_2}[F(\kappa_1,\tau_3)+F(\kappa_1,\tau_4)]\,g_1(\kappa_1)d\kappa_1 + \frac{1}{G_4}(IR)\int_{\tau_3}^{\tau_4}[F(\tau_1,\kappa_2)+F(\tau_2,\kappa_2)]\,g_2(\kappa_2)d\kappa_2\right] \\
\supseteq\; & \frac{F(\tau_1,\tau_3)+F(\tau_1,\tau_4)+F(\tau_2,\tau_3)+F(\tau_2,\tau_4)}{4}.
\end{aligned}$$

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