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On the Regularity of the Class of Generalized Drazin-Riesz Invertible Operators

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Abstract. In this paper, we prove that the class of all generalized Drazin-Riesz invertible operators forms a regularity.

1. Preliminary.

For a complex Banach space X, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X and $\mathcal{K}(X)$ be the ideal of compact operators in $\mathcal{L}(X)$. Let $\pi : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)/\mathcal{K}(X)$ be the canonical projection. For $T \in \mathcal{L}(X)$, let $\sigma(T)$, $\rho(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the spectrum, the resolvent set, the null space and the range of T, respectively. An operator T is said to be a *Fredholm* operator if $\dim \mathcal{N}(T) < \infty$ and $codim\, \mathcal{R}(T) < \infty$, which is equivalent to say that $\pi(T)$ is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. The *index* of a Fredholm operator T is defined by

$$ind(T) = \dim\, \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T).$$

The *essential spectrum* $\sigma_e(T)$ of T is defined as the set of all λ in $\mathbb{C}$ for which $T - \lambda I$ is not a Fredholm operator. T is said to be *Riesz* if $T - \lambda I$ is Fredholm for every $\lambda \in \mathbb{C} \setminus \{0\}$, i.e., $\sigma_e(T) \subseteq \{0\}$ (see [2]).

We say that T admits a complete reduction by the couple (M, N) and we write $(M, N) \in Red(T)$, if M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$. In this case, T is represented as a direct sum of the restrictions T_M and T_N, $T = T_M \oplus T_N$ [13].

Recall that the *ascent* of an operator T is the smallest non-negative integer p such that $\mathcal{N}(T^p) = \mathcal{N}(T^{p+1})$. If no such p exists we say that T is of infinite ascent. Analogously, the *descent* of an operator T is the smallest non-negative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if no such q exists we say that T is of infinite descent. $T \in \mathcal{L}(X)$ is said to be a *Browder* operator if T is Fredholm with finite ascent and descent. The *Browder spectrum* $\sigma_b(T)$ of T is defined as the set of all λ in $\mathbb{C}$ for which $T - \lambda I$ is not a Browder operator (see [2, 11]).

For a subset Λ of $\mathbb{C}$, we denote by *acc* Λ, *iso* Λ and *int* Λ respectively, the accumulation points, the isolated points and the interior of Λ. We denote by $D(\lambda, r)$ the open disc centered at $\lambda \in \mathbb{C}$ and with radius $r > 0$ and the corresponding closed disc is denoted by $\overline{D}(\lambda, r)$.

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Let $T \in \mathcal{L}(X)$. A subset σ of $\sigma(T)$ is called a *spectral set* of T if it is a clopen set in the topology of $\sigma(T)$ induced by the usual topology of $\mathbb{C}$. The *spectral projection* P_σ corresponding to σ is

$$P_\sigma = \frac{1}{2\pi i}\int_\Gamma (\lambda I - T)^{-1} d\lambda,$$

where Γ is a closed contour surrounding σ and its exterior contains $\sigma(T)\setminus\sigma$. It is easy to see that P_σ commutes with each element which commutes with T.

Following Drazin [4], a bounded linear operator $T \in \mathcal{L}(X)$ is said to be *Drazin invertible* if there exists an $S \in \mathcal{L}(X)$ such that

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is nilpotent.} \tag{1}$$

If such S exists then it is unique and it will be denoted by $S = T^D$ and it is called the *Drazin inverse* of T.

In [7] Koliha generalized the concept of the Drazin inverse: $T \in \mathcal{L}(X)$ is said to be *generalized Drazin invertible* (or *Koliha-Drazin invertible*) if there exists an $S \in \mathcal{L}(X)$ such that

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is quasinilpotent.} \tag{2}$$

If such S exists then it is unique and it will be denoted by $S = T^{gD}$ and it is called the *generalized Drazin inverse* of T.

Recently, Živković-Zlatanović and Cvetković [13] generalized the concept of generalized Drazin invertible operators:

Definition 1.1. [13] *An operator $T \in \mathcal{L}(X)$ is generalized Drazin-Riesz invertible if there exists an $S \in \mathcal{L}(X)$ such that*

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is Riesz;}$$

and such S is called a generalized Drazin-Riesz inverse of T.

We point out that $I - TS$ is a projection called the projection associated with the generalized Drazin-Riesz inverse S of T.

We denote by $\mathcal{L}(X)^{DR}$ the class of all bounded generalized Drazin-Riesz invertible operators, and by $\sigma_{DR}(T)$ the generalized Drazin-Riesz spectrum of T defined by

$$\sigma_{DR}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) \notin \mathcal{L}(X)^{DR}\}.$$

Let $T \in \mathcal{L}(X)^{DR}$ such that $0 \in acc\,\sigma(T)$ (if $0 \notin acc\,\sigma(T)$ then T is generalized Drazin invertible). By Proposition 2.7 and Corollary 2.4 [13], there exists $(M, N) \in Red(T)$ such that $T = T_M \oplus T_N$, T_M is invertible and T_N is Riesz with infinite spectrum. Hence, $\sigma(T_N) = \{0, \lambda_1, \lambda_2, \lambda_3, ...\}$ where $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ...$, and $\lambda_n \to 0$, as $n \to \infty$.
Since $0 \in \rho(T_M)$, there exists $\varepsilon > 0$ such that $D(0, \varepsilon) \subset \rho(T_M)$. As (λ_n) is a sequence which converges to zero, there exists $n_0 \in \mathbb{N}$ such that $(\lambda_n)_{n>n_0} \subset D(0, \varepsilon)$. Consequently, $(\lambda_n)_{n\geq n_0}$ is a sequence of non-zero Riesz points of T (see the proof of [13, Proposition 2.7]), and

$$\sigma(T_M) \cap \sigma(T_N) \subseteq (\mathbb{C} \setminus D(0, \varepsilon)) \cap \sigma(T_N) \subseteq \{\lambda_1, \lambda_2, ..., \lambda_{n_0}\}.$$

Now for each $n \geq n_0$, let σ_n and σ'_n be the closed sets defined by

$$\sigma_n = \{0, \lambda_{n+1}, \lambda_{n+2}, ...\}$$

and

$$\sigma'_n = (\sigma(T_M) \setminus (\sigma(T_M) \cap \sigma(T_N))) \cup \{\lambda_1, \lambda_2, ..., \lambda_n\} = \sigma(T) \setminus \sigma_n.$$

Notice that for all $n \geq n_0$, σ_n and σ'_n are spectral sets and $\sigma_n \cap \sigma(T_M) = \emptyset$. Through all this work, we take σ_n and σ'_n to be of index greater than or equal n_0.

Definition 1.2. *[12, Definition 2.2] Let $T \in \mathcal{L}(X)$ be a non-invertible operator with a bounded spectral set σ containing 0 and let P_σ be the corresponding spectral projection. The Drazin inverse of T relative to σ is defined by*

$$T^{D,\sigma} := (T - \xi P_\sigma)^{-1}(I - P_\sigma),$$

for some $\xi \in \mathbb{C}$ such that $|\xi| > r$ where $r = \sup_{\lambda\in\sigma} |\lambda|$.

The condition $|\xi| > r$ is required to ensure that $T - \xi P_\sigma$ is invertible. It is easy to see that $P_\sigma = I - TT^{D,\sigma}$, also $T^{D,\sigma}$ is independent of the value of ξ (see [12]).

Theorem 1.3. *[1, Theorem 2.2] Let $T \in \mathcal{L}(X)$ be generalized Drazin-Riesz invertible with $0 \in acc\,\sigma(T)$. Let n be large enough such that $r_n = \sup_{\lambda\in\sigma_n} |\lambda| < \frac{1}{2}$. Then*

$$T^{D,\sigma_n} = (T - P_{\sigma_n})^{-1}(I - P_{\sigma_n})$$

is a generalized Drazin-Riesz inverse for T.

Theorem 1.4. *[1, Theorem 2.9] Let $T \in \mathcal{L}(X)$. Then T is generalized Drazin-Riesz invertible if and only if $0 \notin acc\,\sigma_b(T)$.*

Definition 1.5. *[11] Let R be a non-empty subset of $\mathcal{L}(X)$. R is called a regularity if it satisfies the following two conditions:*

1. *if $n \in \mathbb{N}$, then $A \in R$ if and only if $A^n \in R$.*
2. *if A, B, C and D are mutually commuting operators in $\mathcal{L}(X)$ such that $AC + BD = I$, then $AB \in R$ if and only if $A \in R$ and $B \in R$.*

A regularity $R \subset \mathcal{L}(X)$ assigns to each $T \in \mathcal{L}(X)$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin R\}$$

and called the spectrum of T corresponding to the regularity R. We notice that every regularity R contains all invertible operators, so that $\sigma_R(T) \subseteq \sigma(T)$. In general, $\sigma_R(T)$ is neither compact nor non-empty. However,

$$f(\sigma_R(T)) = \sigma_R(f(T))$$

for every analytic function f on a neighborhood of $\sigma(T)$ which is non-constant on each component of its domain of definition (see [8, 10, 11]).

Let R be a regularity in $\mathcal{L}(X)$. We say that R (or σ_R) satisfies the property:

- (P1): if $TS \in R \Leftrightarrow T \in R$ and $S \in R$ whenever $T, S \in \mathcal{L}(X)$.
- (P2): if T_n, $T \in \mathcal{L}(X)$, $T_n \to T$, $\mu_n \in \sigma_R(T_n)$ and $\mu_n \to \mu$, then $\mu \in \sigma_R(T)$ (upper semicontinuity of $\sigma_R(\cdot)$).
- (P3): if T_n, $T \in \mathcal{L}(X)$, $T_n \to T$, $TT_n = T_nT$ for every n, $\mu_n \in \sigma_R(T_n)$ and $\mu_n \to \mu$, then $\mu \in \sigma_R(T)$ (upper semicontinuity on commuting elements in $\mathcal{L}(X)$).
- (P4): T_n, $T \in \mathcal{L}(X)$, $T_n \to T$, $TT_n = T_nT$ for every n, then $\mu \in \sigma_R(T)$ if and only if there exits a sequence $\mu_n \in \sigma_R(T_n)$ such that $\mu_n \to \mu$ (continuity on commuting elements in $\mathcal{L}(X)$).

It is easy to see that (P2) or (P4) implies (P3). If R satisfies (P3) then $\sigma_R(T)$ is closed for every $T \in \mathcal{L}(X)$ (see [11]).

Berkani and Sarih [3, Theorem 2.3] proved that the class of all Drazin invertible operators forms a regularity. The result was extended to the class of all generalized Drazin invertible operators by Lubansky [9, Theorem 1.2].

The aim of this paper is to prove that the class $\mathcal{L}(X)^{DR}$ of all bounded generalized Drazin-Riesz invertible operators forms a regularity. Then the spectral mapping theorem for $\sigma_{DR}(T)$ holds. Finally, we investigate $\mathcal{L}(X)^{DR}$ under properties (Pi), $i \in \{1, 2, 3, 4\}$.

2. Main Results.

We begin by stating the main result of the paper.

Theorem 2.1. *The set $\mathcal{L}(X)^{DR}$ of all bounded generalized Drazin-Riesz invertible operators forms a regularity.*

To establish the proof of the theorem, we need the following Lemma.

Lemma 2.2. *Let A, B, C and D be mutually commuting operators in $\mathcal{L}(X)$ such that $AC + BD = I$. If $AB \in \mathcal{L}(X)^{DR}$, then $A \in \mathcal{L}(X)^{DR}$.*

Proof. Suppose that AB is generalized Drazin-Riesz invertible. Then by virtue of Theorem 1.3 there exists $n_0 \in \mathbb{N}$ such that $(AB)^{D,\sigma_{n_0}}$ is a generalized Drazin-Riesz inverse for AB. We have $P_{\sigma_{n_0}} = I - (AB)(AB)^{D,\sigma_{n_0}}$ and $ABP_{\sigma_{n_0}}$ is Riesz.

Step 1. We show that $A(I - P_{\sigma_{n_0}}) \in \mathcal{L}(X)^{DR}$.

Set $U = B(AB)^{D,\sigma_{n_0}}$. Since $(AB)^{D,\sigma_{n_0}} P_{\sigma_{n_0}} = 0$ then

$$U(I - P_{\sigma_{n_0}}) = U.$$

We have U commutes with $A(I - P_{\sigma_{n_0}})$. Indeed, as $P_{\sigma_{n_0}}$ is a spectral projection of AB among σ_{n_0} and A commutes with B, then A and B commute with $P_{\sigma_{n_0}}$. Hence U commutes with $P_{\sigma_{n_0}}$. Therefore U commutes with $A(I - P_{\sigma_{n_0}})$. We have

$$UA = AU = AB(AB)^{D,\sigma_{n_0}} = I - P_{\sigma_{n_0}}.$$

Then

$$UA(I - P_{\sigma_{n_0}})U = (I - P_{\sigma_{n_0}})^2 U = U.$$

Also,

$$\begin{aligned} A(I - P_{\sigma_{n_0}}) - (A(I - P_{\sigma_{n_0}}))^2 U &= A(I - P_{\sigma_{n_0}}) - A^2(I - P_{\sigma_{n_0}})^2 U \\ &= A(I - P_{\sigma_{n_0}}) - A(I - P_{\sigma_{n_0}})^2 AU \\ &= A(I - P_{\sigma_{n_0}}) - A(I - P_{\sigma_{n_0}})^3 = 0 \text{ is Riesz.} \end{aligned}$$

Therefore, $A(I - P_{\sigma_{n_0}})$ is generalized Drazin-Riesz invertible and U is a generalized Drazin-Riesz inverse for $A(I - P_{\sigma_{n_0}})$. We notice also that

$$I - UA(I - P_{\sigma_{n_0}}) = P_{\sigma_{n_0}},$$

and then the projection associated with the generalized Drazin-Riesz inverse U of $A(I - P_{\sigma_{n_0}})$ coincides with $P_{\sigma_{n_0}}$.

Step 2. We show that $ACP_{\sigma_{n_0}} \in \mathcal{L}(X)^{DR}$.
Since

$$\begin{aligned} ACP_{\sigma_{n_0}} - (ACP_{\sigma_{n_0}})^2 &= AC(I - AC)P_{\sigma_{n_0}} \\ &= ACBDP_{\sigma_{n_0}} \\ &= (ABP_{\sigma_{n_0}})CD \end{aligned}$$

and $ABP_{\sigma_{n_0}}$ is a Riesz operator, then by virtue of [2, Theorem 3.112], $ACP_{\sigma_{n_0}} - (ACP_{\sigma_{n_0}})^2$ is a Riesz operator.

Also, by considering the analytic function $f(\lambda) = \lambda - \lambda^2$, $\forall \lambda \in \mathbb{C}$, we have

$$\begin{aligned} f(\sigma(\pi(ACP_{\sigma_{n_0}}))) &= \sigma(f(\pi(ACP_{\sigma_{n_0}}))) \\ &= \sigma(\pi(ACP_{\sigma_{n_0}}) - (\pi(ACP_{\sigma_{n_0}}))^2) \\ &= \sigma(\pi(ACP_{\sigma_{n_0}} - (ACP_{\sigma_{n_0}})^2)) = \{0\}. \end{aligned}$$

Hence $\sigma(\pi(ACP_{\sigma_{n_0}})) \subset \{0, 1\}$. Thus

$$0 \notin acc\, \sigma_e(ACP_{\sigma_{n_0}}) \text{ and } 0 \notin acc\, \sigma_e(I - ACP_{\sigma_{n_0}}).$$

We set $W = ACP_{\sigma_{n_0}}$. Since always $\sigma_b(W) = \sigma_e(W) \cup acc\, \sigma(W)$, and in order to find that $0 \notin acc\, \sigma_b(W)$, for the sake of contradiction we suppose that $0 \in acc\,(acc\, \sigma(W))$. Consequently, there exists $(\lambda_p)_p \subset acc\, \sigma(W)$ such that $\lambda_p \to 0$, as $p \longrightarrow \infty$. Thus for all $p \in \mathbb{N}$ there exists a sequence $(s_{p_k})_k \subset \sigma(W)$ such that $s_{p_k} \longrightarrow \lambda_p$, as $k \longrightarrow \infty$. Hence $f(s_{p_k}) \longrightarrow f(\lambda_p)$, as $k \longrightarrow \infty$, and using spectral mapping theorem, we get $(f(\lambda_p))_p \subset acc\, \sigma(f(W))$. Thus $f(\lambda_p) \longrightarrow 0$, as $p \longrightarrow \infty$ and so $0 \in acc\,(acc\; \sigma(f(W))) \subset acc\, \sigma_b(W - W^2)$, which is a contradiction. Therefore, we obtain that

$$0 \notin acc\, \sigma_b(ACP_{\sigma_{n_0}}).$$

Consequently, $ACP_{\sigma_{n_0}}$ is generalized Drazin-Riesz invertible by Theorem 1.4.

Step 3. We show that $AP_{\sigma_{n_0}} \in \mathcal{L}(X)^{DR}$.

Let Λ_{k_0} be the spectral set associated to the generalized Drazin-Riesz invertible operator $ACP_{\sigma_{n_0}}$ (as defined after Definition 1.1). Let us show that $V = CP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}}$ is a generalized Drazin-Riesz inverse of $AP_{\sigma_{n_0}}$.

We have V commutes with $AP_{\sigma_{n_0}}$, and

$$\begin{aligned} V(AP_{\sigma_{n_0}})V &= CP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}} AP_{\sigma_{n_0}} CP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}} \\ &= CP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}} = V. \end{aligned}$$

Also

$$\begin{aligned} AP_{\sigma_{n_0}} - (AP_{\sigma_{n_0}})^2 V &= AP_{\sigma_{n_0}}\left(I - AP_{\sigma_{n_0}} CP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}}\right) \\ &= AP_{\sigma_{n_0}}\left(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}}\right) \\ &= A(AC + BD)P_{\sigma_{n_0}}\left(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}}\right) \\ &= A(ACP_{\sigma_{n_0}})\left(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right) \\ &\quad + ABP_{\sigma_{n_0}}\left(D(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right). \end{aligned}$$

Since $ABP_{\sigma_{n_0}}$ is Riesz then $ABP_{\sigma_{n_0}}\left(D(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right)$ is Riesz by [2, Theorem 3.112]. Also, $ACP_{\sigma_{n_0}}\left(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right)$ is Riesz since $I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}}$ is a spectral projection of $ACP_{\sigma_{n_0}}$ among Λ_{k_0}. Now again by [2, Theorem 3.112], we get

$$A(ACP_{\sigma_{n_0}})\left(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right) + ABP_{\sigma_{n_0}}\left(D(I - ACP_{\sigma_{n_0}}(ACP_{\sigma_{n_0}})^{D,\Lambda_{k_0}})\right)$$

is Riesz. Hence, $AP_{\sigma_{n_0}} - AP_{\sigma_{n_0}} V AP_{\sigma_{n_0}}$ is Riesz. Therefore $AP_{\sigma_{n_0}}$ is generalized Drazin-Riesz invertible and V is a generalized Drazin-Riesz inverse for $AP_{\sigma_{n_0}}$.

Step 4. We show that $A \in \mathcal{L}(X)^{DR}$.

We have $A(I-P_{\sigma_{n_0}})$ and $AP_{\sigma_{n_0}}$ are generalized Drazin-Riesz invertible, and

$$(A(I-P_{\sigma_{n_0}}))(AP_{\sigma_{n_0}}) = (AP_{\sigma_{n_0}})(A(I-P_{\sigma_{n_0}})) = 0,$$

by [6, Proposition 2.8], we obtain that $A(I-P_{\sigma_{n_0}}) + AP_{\sigma_{n_0}} = A$ is generalized Drazin-Riesz invertible. □

We are now in a position to prove our main result.

Proof. of Theorem 2.1. Let $A \in \mathcal{L}(X)$. If A is generalized Drazin-Riesz invertible, then A^n is also generalized Drazin-Riesz invertible. Conversely, assume that A^n is generalized Drazin-Riesz invertible. Then $0 \notin acc\sigma_b(A^n)$. As $\sigma_e(A^n) = \{\lambda^n : \lambda \in \sigma_e(A)\}$, and $\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\}$, we conclude immediately that $0 \notin acc\ \sigma_b(A)$. Finally A is generalized Drazin-Riesz invertible by Theorem 1.4.

Now let A, B, C and D be mutually commuting operators in $\mathcal{L}(X)$ such that $AC + BD = I$. If A and B are generalized Drazin-Riesz invertible. Then AB is also generalized Drazin-Riesz invertible.
Indeed, for

$$Q = I - (I - P_{\sigma_{n_0,B}})(I - P_{\sigma_{n_0,A}})$$

let us show that Q is a projection satisfying

$$QAB = ABQ,\ AB + Q \text{ is Browder and } ABQ \text{ is Riesz.}$$

By Theorem 2.1 [12], $A + P_{\sigma_{n_0,A}}$ and $B + P_{\sigma_{n_0,B}}$ are invertible and so Browder operators. Also $AP_{\sigma_{n_0,A}}$ and $BP_{\sigma_{n_0,B}}$ are Riesz operators. Since $A + P_{\sigma_{n_0,A}}$ and $B + P_{\sigma_{n_0,B}}$ commute, we conclude by [5, Theorem 7.9.2] that $(A + P_{\sigma_{n_0,A}})(B + P_{n_0,B})$ is a Browder operator.
Now we show that ABQ is Riesz. Since $Q = P_{\sigma_{n_0,A}} + P_{\sigma_{n_0,B}} - P_{\sigma_{n_0,A}}P_{\sigma_{n_0,B}}$, we have

$$ABQ = AP_{\sigma_{n_0,A}}B + ABP_{\sigma_{n_0,B}} - AP_{\sigma_{n_0,A}}BP_{\sigma_{n_0,B}}.$$

$AP_{\sigma_{n_0,A}}$ and $BP_{\sigma_{n_0,B}}$ are Riesz operators and commute respectively with B and A, then by [2, Theorem 3.112] $AP_{\sigma_{n_0,A}}B$, $ABP_{\sigma_{n_0,B}}$ and $AP_{\sigma_{n_0,A}}BP_{\sigma_{n_0,B}}$ are Riesz operators. Thus, ABQ is a Riesz operator, again by [2, Theorem 3.112]. Therefore, $I + ABQ$ is a Browder operator by [2, Theorem 3.111].
Since $I + ABQ$ is Browder and $(\mathcal{R}(Q), \mathcal{N}(Q)) \in Red(I + AB)$, we deduce by [13, Lemma 2.1] that $(I + ABQ)_{\mathcal{R}(Q)}$ is a Browder operator. We have

$$(I + ABQ)Q = 0_{\mathcal{N}(Q)} \oplus (I + AB)_{\mathcal{R}(Q)}.$$

Also

$$\begin{aligned} AB(I-Q) &= (A + P_{\sigma_{n_0,A}})(B + P_{\sigma_{n_0,B}})(I-Q) \\ &= ((A + P_{\sigma_{n_0,A}})(B + P_{\sigma_{n_0,B}}))_{\mathcal{N}(Q)} \oplus 0_{\mathcal{R}(Q)}, \end{aligned}$$

Now we get

$$\begin{aligned} AB + Q &= ABQ + Q + AB(I-Q) \\ &= (I + ABQ)Q + AB(I-Q) \\ &= (0_{\mathcal{N}(Q)} \oplus (I + ABQ)_{\mathcal{R}(Q)}) + \Big(((A + P_{\sigma_{n_0,A}})(B + P_{\sigma_{n_0,B}}))_{\mathcal{N}(Q)} \oplus 0_{\mathcal{R}(Q)}\Big) \\ &= ((A + P_{\sigma_{n_0,A}})(B + P_{\sigma_{n_0,B}}))_{\mathcal{N}(Q)} \oplus (I + ABQ)_{\mathcal{R}(Q)}. \end{aligned}$$

As $((A + P_{\sigma_{n_0,A}})(B + P_{\sigma_{n_0,B}}))_{\mathcal{N}(Q)}$ and $(I + ABQ)_{\mathcal{R}(Q)}$ are Browder, we conclude by [13, Lemma 2.1] that $AB + Q$ is a Browder operator. Finally, [13, Theorem 2.3] leads to conclude that AB is generalized Drazin-Riesz invertible.

Conversely, if AB is generalized Drazin-Riesz invertible, then by Lemma 2.2, we conclude that A and B are generalized Drazin-Riesz invertible. □

Combining Theorem 2.1 and [11, Theorem I.6.7], the spectral mapping theorem holds for $\sigma_{DR}(T)$

Theorem 2.3. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If f is any function holomorphic in an open neighborhood of $\sigma(T)$ and non-constant on any component of $\sigma(T)$, then*

$$f(\sigma_{DR}(T)) = \sigma_{DR}(f(T)).$$

We investigate some topological properties of $\sigma_{DR}(T)$.

Proposition 2.4. *Let X be a Banach space and $T \in \mathcal{L}(X)$. We have :*
1) $\sigma_{DR}(T) = acc\,\sigma_b(T)$ *and* $\sigma_{DR}(T)$ *is closed.*
2) $\sigma_{DR}(T) \subset \sigma_{KD}(T) \subset \sigma_D(T) \subset \sigma(T)$.
3) $\sigma_{DR}(T) = \emptyset$ *if and only if* $\sigma_b(T)$ *is a finite set.*

Proof. 1) By Theorem 1.4, $\sigma_{DR}(T) = acc\ \sigma_b(T)$. The closure of $\sigma_{DR}(T)$ is assured by $\sigma_{DR}(T) = acc\ \sigma_b(T)$, $\sigma_b(T)$ being closed, we have $acc\ \sigma_b(T) = \sigma_{DR}(T)$ is closed.
2) The inclusions are obvious.
3) If $\sigma_b(T)$ is finite, its every point is isolated in $\sigma_b(T)$, therefore not in $acc\ \sigma_b(T)$.

Conversely, suppose that $\sigma_{DR}(T) = \emptyset$. By way of contradiction, suppose that $\sigma_b(T)$ is infinite. As $\sigma_b(T)$ is a compact set (closed in the compact set $\sigma(T)$), it has an accumulation point μ, so $\mu \in\ acc\ \sigma_b(T) = \sigma_{DR}(T)$, which is a contradiction. □

In general the class of generalized Drazin-Riesz invertible operators does not satisfy the property (P1).

Example 2.5. Let S be the right shift operator defined on $\ell_2(\mathbb{N})$ by

$$S(x_0, x_1, ...) = (0, x_0, x_1, ...),\ \forall (x_0, x_1, ...).$$

Set $T_1 = S \oplus 0$ and $T_2 = 0 \oplus S$ on $X = \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$. Since

$$\sigma(S) = \sigma_b(S) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},$$

then

$$\sigma(T_i) = \sigma_b(T_i) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},\ \text{ for } i = 1, 2.$$

We have $T_1T_2 = 0 \in \mathcal{L}(X)^{DR}$. But $0 \in\ acc\ \sigma_b(T_i),\ i = 1, 2$. Thus T_1 and T_2 do not belong to $\mathcal{L}(X)^{DR}$.

However, the class of generalized Drazin-Riesz invertible operators satisfies property (P1) in a very special case.

Theorem 2.6. *Let X be a Banach space. Then the following conditions are equivalent:*

i) $\mathcal{L}(X)^{DR}$ *has property* (P1);
ii) $\mathcal{L}(X)^{DR} = \mathcal{L}(X)$;
iii) $\sigma_{DR}(T) = \emptyset$ for all $T \in \mathcal{L}(X)$;
iv) Each bounded operator has a finite Browder spectrum.

Proof. i) ⇒ ii): Suppose that $\mathcal{L}(X)^{DR}$ satisfies (P_1). As 0 belongs to $\mathcal{L}(X)^{DR}$ and commutes with all elements, for any $A \in \mathcal{L}(X)$ the product $0 = 0A \in \mathcal{L}(X)^{DR}$; and so $A \in \mathcal{L}(X)^{DR}$ for all $A \in \mathcal{L}(X)$, hence $\mathcal{L}(X) = \mathcal{L}(X)^{DR}$.
ii) ⇒ iii): It is obvious.
iii) ⇒ iv): follows at once from Proposition 2.4.
iv) ⇒ i): If each element T of $\mathcal{L}(X)$ has a finite Browder spectrum, then all elements of $\sigma_b(T)$ are isolated for all $T \in \mathcal{L}(X)$. Hence by Theorem 1.4, T is generalized Drazin-Riesz invertible. Trivially we have $\mathcal{L}(X)^{DR}$ satisfies property (P1). □

The class $\mathcal{L}(X)^{DR}$ is not necessarily open in $\mathcal{L}(X)$ as shown by the following example.

Example 2.7. For a nonzero positive integer n, let S_n be the weighted right shift operator defined on $\ell^2(\mathbb{N})$ by

$$S_n x = (x_1, x_2, ...) = (0, \frac{1}{n}x_1, \frac{1}{n}x_2, ...).$$

Then $\|S_n\| = \frac{1}{n}$. Let $V(0,\eta)$ be the open disk in $\mathcal{L}(\ell^2(\mathbb{N}))$ centered at 0 and with radius $\eta > 0$. Then for n large enough we have $S_n \in D(0,\eta)$. Since

$$\sigma(S_n) = \sigma_b(S_n) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n}\},$$

then for each n, S_n is not generalized Drazin-Riesz invertible and the sequence (S_n) converges to 0 which is generalized Drazin-invertible operator with a generalized Drazin-Riesz inverse 0. Therefore, $\mathcal{L}(\ell^2(\mathbb{N}))^{DR}$ is not open in $\mathcal{L}(\ell^2(\mathbb{N}))$.

By virtue of [11, Proposition I.6.9], a regularity R is an open set of $\mathcal{L}(X)$ if and only if R satisfies (P2). The previous example shows that $\mathcal{L}(X)^{DR}$ does not satisfy (P2) in general. The special case when $\mathcal{L}(X)^{DR}$ satisfies (P2) is the following

Theorem 2.8. *$\mathcal{L}(X)^{DR}$ satisfies properties* (P2), (P3), and (P4) if and only if $\mathcal{L}(X)^{DR} = \mathcal{L}(X)$.

Proof. If $\mathcal{L}(X)^{DR} = \mathcal{L}(X)$, then properties (P2), (P3), and (P4) are obviously satisfied. Conversely, suppose that there exists $T \in \mathcal{L}(X) \setminus \mathcal{L}(X)^{DR}$. Then $0 \in \sigma_{DR}(T) = acc\,\sigma_b(T)$, set $T_n = \frac{1}{n}T$ for all $n \in \mathbb{N}$. Thus $0 \in acc\,\sigma_b(T_n) = \sigma_{DR}(T_n)$, and $T_n \longrightarrow 0$, as $n \longrightarrow \infty$. However, $0 \notin \sigma_{DR}(0)$, then property (P3) does not hold, and therefore neither (P2) nor (P4). □

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