Filomat 36:14 (2022), 4893–4902
https://doi.org/10.2298/FIL2214893E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi-Fredholm Spectrum for Operator Matrices

I. El Ouali[a], M. Ech-Cherif El Kettani[a], M. Karmouni[b]

[a]*Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, LaSMA Laboratory, Fez, Morocco*
[b]*Cadi Ayyad University, Multidisciplinary Faculty, Safi, Morocco*

Dedicated to our Professor Mohamed AKKAR on the occasion of his 80th birthday.

Abstract. For $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$ we denote by M_C the operator matrix defined on $X \oplus Y$ by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, we prove that

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \bigcup_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \sigma_p(B) \cup \sigma_p(A^*),$$

where $\sigma_{qF}(.)$ (resp. $\sigma_p(.)$) denotes the quasi-Fredholm spectrum (resp. the point spectrum). Furthermore, we consider some sufficient conditions for M_C to be quasi-Fredholm and sufficient conditions to have

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) = \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$

.

1. Introduction

Let X and Y denote infinite dimensional complex Banach spaces and $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y. If $X = Y$ we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. For $T \in \mathcal{L}(X)$, we denote by $N(T)$ the kernel of T, by $R(T)$ the range of T and by $\sigma_p(T)$ the point spectrum of T.

An operator $T \in \mathcal{L}(X)$ is called quasi-nilpotent if and only if, for all $x \in X$, $\limsup_n \|T^n x\|^{\frac{1}{n}} = 0$, so $\sigma(T) = \{0\}$.

Recall that the degree of stable iteration is the quantity $dis(T) = inf \Delta(T)$ (with $dis(T) = +\infty$, if $\Delta(T) = \emptyset$) such that

$$\Delta(T) = \{n \in \mathbb{N} \,;\, \forall m \in \mathbb{N}, \quad m \geq n \implies R(T^m) \cap N(T) = R(T^n) \cap N(T)\}.$$

In 1980, Labrousse was the first to introduce the class of quasi-Fredholm operators [3] in the case of Hilbert spaces. In 1996, M. Mbekhta and V. Müller [6] generalized this class to Banach spaces. An operator $T \in \mathcal{L}(X)$ is called quasi-Fredholm if $dis(T) = d \in \mathbb{N}$, and for all $n \geq d$, $R(T) + N(T^n)$ and $R(T^n)$ are closed in X. We

2020 *Mathematics Subject Classification*. Primary 47A53; Secondary 47A10
Keywords. Operator matrices, Quasi-Fredholm operator, Quasi-Fredholm spectrum.
Received: 15 August 2021; Accepted: 11 June 2022
Communicated by Dragan S. Djordjević
Email addresses: issame.elouali@usmba.ac.ma (I. El Ouali), mustelkettani@yahoo.fr (M. Ech-Cherif El Kettani), mohamed.karmouni@uca.ac.ma (M. Karmouni)

denote by $qF(d)$ the class of quasi-Fredholm operators of degree d. An operator is quasi-Fredholm if it is quasi-Fredholm of some degree d. We denote by $qF(X)$ the set of all quasi-Fredholm operators on X.

Note that if $dis(T) = d$, we have: $R(T) + N(T^n)$ and $R(T^n)$ are closed in X, for all $n \geq d$, if and only if for all $n \geq d$, $R(T^n)$ is closed in X if and only if $R(T^{d+1})$ is closed in X (see [7, proposition 3]). For every bounded operator $T \in \mathcal{L}(X)$, let us define the quasi-Fredholm spectrum as follows :

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not quasi-Fredholm}\}.$$

Recall that $T \in \mathcal{L}(X)$ is called semi-regular if $R(T)$ is closed and $dis(T) = 0$.

An operator $T \in \mathcal{L}(X)$ is called of Kato type if there exists a pair of T-invariant closed subpaces (M, N) such that $X = M \oplus N$, $T_{|M}$ is semi-regular and $T_{|N}$ is nilpotent. A classic result from Labrousse [5] states that, in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators.

Let's consider the upper triangular operator matrix M_C defined on $X \oplus Y$ by :

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

with $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$. The spectra and related problems of M_C are extensively studied. In general, the upper triangular operator matrix does not conserve the properties of their diagonal elements. For example, in [3] the authors gave an example which show that this equality

$$\sigma(M_C) = \sigma(A) \cup \sigma(B), \text{ for arbitrary } C \in \mathcal{L}(Y, X),$$

is not always true. So, an obvious question arises: Under what conditions, on A and B, we will have equality? Serval articles have given answers to this question. For example, in [4] an answer is given by: $\sigma(A) \cap \sigma(B)$ has no interior points. This has prompted many authors to carry out similar studies concerning the perturbations of others spectra of upper triangular operator matrices, see for instance [2],[8], [9] and [10]. In their article [2], M. Barraa and M. Boumazgour established some results, on Hilbert spaces, concerning the spectrum associate to Kato-essential operators. Recall that any Kato-essential operator is a quasi-Fredholm operator of degree 0.

In this paper, we obtain some results concerning the perturbation of the quasi-Fredholm spectrum of M_C in the case of Banach spaces. This leads us to study the behavior of upper triangular operator matrices with powers.

2. Main results

We start by the following lemmas which will be needed in the sequel.

Lemma 2.1. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$. Let $n \in \mathbb{N}$ such that $n \geq 2$. We have*

$$M_C^n = \begin{pmatrix} A^n & \sum_{k=0}^{n-1} A^{n-1-k} C B^k \\ 0 & B^n \end{pmatrix}.$$

Proof. By induction. □

Lemma 2.2. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$. Let $n \in \mathbb{N}^*$. We Have*

1. *$x \in N(A^n) \iff x \oplus 0 \in N(M_C^n)$, for all $x \in X$.*
2. *If B is injective, then for all $x \in X$ we have*

$$x \in R(A^n) \iff x \oplus 0 \in R(M_C^n).$$

3. *If B is injective, then for all $x \oplus y \in X \oplus Y$ we have*

$$x \oplus y \in R(M_C^n) \cap N(M_C) \iff x \in R(A^n) \cap N(A) \text{ and } y = 0.$$

Proof. 1. Let $x \in N(A^n)$, then $A^n x = 0$. Then

$$\begin{cases} A^n x + \sum_{k=0}^{n-1} A^{n-1-k} C B^k 0 = 0 \\ B^n 0 = 0 \end{cases}$$

Hence $x \oplus 0 \in N(M_C^n)$. The other implication is obvious.

2. Let $x \in R(A^n)$, then there exists $z \in X$ such that $A^n z = x$. Then

$$\begin{cases} A^n z + \sum_{k=0}^{n-1} A^{n-1-k} C B^k 0 = x \\ B^n 0 = 0 \end{cases}$$

Hence $x \oplus 0 \in R(M_C^n)$.
If $x \oplus 0 \in R(M_C^n)$, then there exists $z \in X$ and $t \in Y$ such that

$$\begin{cases} A^n z + \sum_{k=0}^{n-1} A^{n-1-k} C B^k t = x \\ B^n t = 0 \end{cases}$$

Since B is injective, $t = 0$. Indeed $B^n t = B(B^{n-1}) = 0 \implies B^{n-1} t = B(B^{n-2})t = 0 \implies \dots \implies B^2 t = B(Bt) = 0 \implies Bt = 0 \implies t = 0$. Hence $A^n z + \sum_{k=0}^{n-1} A^{n-1-k} C B^k 0 = A^n z = x$, it follows that $x \in R(A^n)$.

3. Let $x \oplus y \in R(M_C^n) \cap N(M_C)$. Then there exists $z \oplus t \in X \oplus Y$, such that

$$\begin{cases} A^n z + \sum_{k=0}^{n-1} A^{n-1-k} C B^k t = x \\ B^n t = y \end{cases} \quad \text{and} \quad \begin{cases} Ax + Cy = 0 \\ By = 0 \end{cases}$$

Since B is injective, $y = t = 0$. Hence:

$$\begin{cases} A^n z = x \\ y = 0 \end{cases}$$

Hence $x \in R(A^n) \cap N(A)$ *and* $y = 0$.
From 1 and 2, it is easy to see the reciprocal implication.
□

Theorem 2.3. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, $C \in \mathcal{L}(Y, X)$ and $d \in \mathbb{N}$. If B is injective and $M_C \in qF(d)$, then $A \in qF(d)$.*

Proof. Let $n \geq d$ and $x \in R(A^d) \cap N(A)$.
From lemma 2.2, we have $x \oplus 0 \in R(M_C^d) \cap N(M_C)$.
Since $dis(M_C) = d$, we have $x \oplus 0 \in R(M_C^n) \cap N(M_C)$. Hence, by lemma 2.2, $x \in R(A^n) \cap N(A)$. Hence $dis(A) \leq d$.
Suppose that $dis(A) = p < d$. Let $x \oplus y \in R(M_C^p) \cap N(M_C)$. From lemma 2.2, we have $x \in R(A^p) \cap N(A)$ and $y = 0$. Then $x \in R(A^d) \cap N(A)$ and $y = 0$. Then, by lemma 2.2, $x \oplus y \in R(M_C^d) \cap N(M_C)$, which is absurd because $dis(M_C) = d > p$. Hence $dis(A) = d$.
Let $n \geq d$ and $(x_k)_{k \in \mathbb{N}} \subseteq R(A^n)$ such that $x_k \to x$ when $k \to +\infty$.
By lemma 2.2, we have $(x_k \oplus 0)_k \subseteq R(M_c^n)$ and $x_k \oplus 0 \to x \oplus 0$ when $k \to +\infty$. Since $R(M_c^n)$ is closed, $(x \oplus 0) \in R(M_c^n)$. Then, by lemma 2.2, $x \in R(A^n)$. Hence $R(A^n)$ is closed in X.
Let $n \geq d$ and $(x_k)_{k \in \mathbb{N}} \subseteq R(A) + N(A^n)$ such that $x_k \to x$ when $k \to +\infty$. Then, for all $k \in \mathbb{N}$, we have $x_k = x_{k,1} + x_{k,2}$ such that $x_{k,1} \in R(A)$ and $x_{k,2} \in N(A^n)$. Hence, by lemma 2.2, we have $x_{k,1} \oplus 0 \in R(M_C)$ and $x_{k,2} \oplus 0 \in N(M_C^n)$, for all $k \in \mathbb{N}$. Then $(x_{k,1} \oplus 0) + (x_{k,2} \oplus 0) = x_k \oplus 0 \in R(M_C) + N(M_C^n)$, for all $k \in \mathbb{N}$. Since $R(M_C) + N(M_C^n)$ is closed, $x \oplus 0 \in R(M_C) + N(M_C^n)$. Then $x \oplus 0 = (x_1 \oplus 0) + (x_2 \oplus 0)$ such that $x_1 \oplus 0 \in R(M_C)$ and $x_2 \oplus 0 \in N(M_C^d)$. Since B is injective, the lemma 2.2 assures that $x_1 \in R(A)$ and $x_2 \in N(A^n)$. Hence $x = x_1 + x_2 \in R(A) + N(A^n)$. Thus $R(A) + N(A^n)$ is closed.
Therefore $A \in qF(d)$.
□

Corollary 2.4. *Let H and K be two Hilbert spaces. Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, $C \in \mathcal{L}(H, K)$ and $d \in \mathbb{N}$. If A^* is injective and $M_C \in qF(d)$, then $B \in qF(d)$.*

Proof. We have

$$M_C^* = \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix}.$$

Since $M_C \in qF(d)$, $M_C^* \in qF(d)$ (See [1]). Hence, by following the same procedure as in the proof of theorem 2.3, we will have $B^* \in qF(d)$. Then $B \in qF(d)$. □

Corollary 2.5. *Let H and K be two Hilbert spaces. Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$. We have*

$$\sigma_{qF}(B) \cup \sigma_{qF}(A) \subseteq \bigcup_{C \in \mathcal{L}(K,H)} \sigma_{qF}(M_C) \cup \sigma_p(B) \cup \sigma_p(A^*).$$

Proof. Firstly, let $\lambda \in \rho_p(B) \cap \rho_{qF}(M_C)$. The theorem 2.3 entails $\lambda \in \rho_{qF}(A)$. Hence

$$\rho_p(B) \cap \rho_{qF}(M_C) \subseteq \rho_{qF}(A).$$

Thus $\sigma_{qF}(A) \subseteq \sigma_p(B) \cup \sigma_{qF}(M_C)$.
Secondly, let $\lambda \in \rho_p(A^*) \cap \rho_{qF}(M_C)$. The corollary 2.4 (*ii*) entails $\lambda \in \rho_{qF}(B)$. Hence

$$\rho_p(A^*) \cap \rho_{qF}(M_C) \subseteq \rho_{qF}(B).$$

Hence $\sigma_{qF}(B) \subseteq \sigma_p(A^*) \cup \sigma_{qF}(M_C)$. Therefore

$$\sigma_{qF}(B) \cup \sigma_{qF}(A) \subseteq \bigcup_{C \in \mathcal{L}(K,H)} \sigma_{qF}(M_C) \cup \sigma_p(B) \cup \sigma_p(A^*).$$

□

Example 2.6. *Let $X = Y = l^2(\mathbb{N})$. Let $A \in \mathcal{L}(l^2(\mathbb{N}))$ defined by $Ax = A(x_1, x_2, ...) = (x_1, 0, 0, ...)$. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ such that $Tx = T(x_1, x_2, ...) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$. Let $B = C = T$.*
It is easy to see that T is injective.

We have $dis(A) = 1$ (Indeed, we have $R(A^0) \cap N(A) = R(I) \cap N(A) = X \cap N(A) = N(A) = vect(e_2, e_3, ...)$ and, for all $n \in \mathbb{N}^*$, $A^n(x) = (x_1, 0, 0, 0, ...)$, then $R(A^n) \cap N(A) = 0$, thus $dis(A) = 1$) and $dim(R(A^n)) < \infty$ for all $n \in \mathbb{N}^*$, then $R(A^n)$ is closed for all $n \in \mathbb{N}^*$. Hence $A \in qF(1)$.

By lemma 2.2, it is easy to see that $dis(M_C) = 1$.

Let's show that $R(M_C)$ is not closed.
Let $(x_n)_n \subseteq l^2(N)$ defined by

$$x_n^k = \begin{cases} 1 & ; \quad k \leq n \\ 0 & ; \quad k > n \end{cases}$$

Thus

$$T(x_n^k) = \begin{cases} \frac{1}{k} & ; \quad k \leq n \\ 0 & ; \quad k > n \end{cases}$$

and $T(x_n) \to y = (1, \frac{1}{2}, \frac{1}{3}, ...)$ when $n \to +\infty$. Thus $y \notin R(B)$ (Indeed, we have $R(T) = \{(x_k) \in l^2(\mathbb{N}) \,/\, (kx_k) \in l^2(\mathbb{N})\}$, but $(ky_k)_k = (1, 1, 1, ...) \notin l^2(\mathbb{N})$. Hence $y = (y_k) \notin R(T)$). Then $(x_n \oplus x_n)_n \subseteq R(M_C)$ such that $x_n \oplus x_n \to y \oplus y$. Since $y \notin R(B)$, $y \oplus y \notin R(M_C)$. Hence $R(M_C)$ is not closed. It follows that M_C is not quasi-Fredholm.
This example shows that (This result is also true in case of Banach spaces)

$$\sigma_{qF}(A) \subsetneq \bigcup_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \sigma_p(B).$$

Example 2.7. *Let $X = Y = l^2(\mathbb{N})$. Let $S \in \mathcal{L}(l^2(\mathbb{N}))$ defined by $Sx = S(x_1, x_2, ...) = (x_1, 0, 0, ...)$. Let $T : l^2(N) \to l^2(N)$ such that $Tx = T(x_1, x_2, ...) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$. Let $A = C = T^*$ and $B = S^*$. We have*

$$M_C^* = \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix} = \begin{pmatrix} T & 0 \\ T & S \end{pmatrix}.$$

By the same procedure of example 2.6, it is easy to see that $A^* = T$ is injective and $B^* \in qF(1)$, but M_C^* is not quasi-Fredholm. Hence M_C is not quasi-Fredholm.
This example shows that

$$\sigma_{qF}(B) \subsetneq \bigcup_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \sigma_p(A^*).$$

Corollary 2.8. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. We have:*

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \bigcup_{C\in\mathcal{L}(K,H)} \sigma_{qF}(M_C) \cup \sigma_p(B) \cup \sigma_p(A^*)$$

Proof. It is obvious from theorem 2.3, example 2.6 and example 2.7 □

Lemma 2.9. *Let $x \oplus y \in X \oplus Y$. For all positive integer n, we have:*

$$x \oplus y \in R(M_0) + N(M_0^n) \iff x \in R(A) + N(A^n) \quad \text{and} \quad y \in R(B) + N(B^n)$$

Proof. Let $n \in \mathbb{N}$. If $x \oplus y \in R(M_0) + N(M_0^n)$, then:

$$x \oplus y = (x_1 \oplus y_1) + (x_2 \oplus y_2) \text{ such that } x_1 \oplus y_1 \in R(M_0) \text{ and } x_2 \oplus y_2 \in N(M_0^n).$$

Hence

$$\begin{cases} x_1 \in R(A) \\ y_1 \in R(B) \end{cases} \quad \text{and} \quad \begin{cases} x_2 \in N(A^n) \\ y_2 \in N(B^n) \end{cases}$$

Hence

$$\begin{cases} x = x_1 + x_2 \in R(A) + N(A^n) \\ y = y_1 + y_2 \in R(B) + N(B^n) \end{cases}$$

Conversely, if $x \in R(A) + N(A^n)$ and $y \in R(B) + N(B^n)$, then

$$\begin{cases} x = x_1 + x_2 \\ y = y_1 + y_2 \end{cases} \text{ such that } \begin{cases} x_1 \in R(A) \text{ and } x_2 \in N(A^n) \\ y_1 \in R(B) \text{ and } y_2 \in N(B^n) \end{cases}$$

It follows that

$$\begin{cases} x_1 \oplus y_1 \in R(M_0) \\ x_2 \oplus y_2 \in N(M_0^n) \end{cases}$$

Therfore $x \oplus y = (x_1 \oplus y_1) + (x_2 \oplus y_2) \in R(M_0) + N(M_0^n)$. □

Theorem 2.10. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. If $A \in qF(d)$ and $B \in qF(d')$ such that $d' \leq d$, then $M_0 \in qF(d)$.*

Proof. Suppose that $A \in qF(d)$ and $B \in qF(d')$ such that $d' \leq d$.
Firstly, let's show that $dis(M_0) = d$.
Let $n \geq d$. We have:

$$\begin{aligned} x \oplus y \in R(M_0^d) \cap N(M_0) &\iff \begin{cases} x \in R(A^d) \cap N(A) \\ y \in R(B^d) \cap N(B) \end{cases} \\ &\iff \begin{cases} x \in R(A^n) \cap N(A) \\ y \in R(B^n) \cap N(B) \end{cases} \\ &\iff x \oplus y \in R(M_0^n) \cap N(M_0) \end{aligned}$$

Thus $R(M_0^n) \cap N(M_0) = R(M_0^d) \cap N(M_0)$. Hence $d \in \Delta(M_0)$, which implies that $dis(M_0) \leq d$.
If $dis(M_0) = d'' < d$, then $R(A^n) \cap N(A) = R(A^{d''}) \cap N(A)$, for all $n \geq d''$. Since $dis(A) = d > d''$, that is absurd.
Hence $dis(M_0) = d$.

Secondly, let's show that $R(M_0^n)$ is closed in $X \oplus Y$, $\forall n \geq d$.
Let $n \geq d$ and $(x_k \oplus y_k)_{k\geq 0} \in R(M_0^n)$ such that $x_k \oplus y_k \to x \oplus y$ when $k \to +\infty$. Hence, for all $k \in \mathbb{N}$, we have:

$$\begin{cases} x_k \in R(A^n) \\ y_k \in R(B^n) \end{cases}$$

Since $R(A^n)$ and $R(B^n)$ are closed, we have

$$\begin{cases} x \in R(A^n) \\ y \in R(B^n) \end{cases}$$

Therefore $x \oplus y \in R(M_0^n)$. Thus $R(M_0^n)$ is closed in $X \oplus Y$, $\forall n \geq d$.
So it remains to show that $R(M_0) + N(M_0^n)$ is closed in $X \oplus Y$, $\forall n \geq d$.
Let $n \geq d$ and let $(x_k \oplus y_k)_{k\geq O} \in R(M_0) + N(M_0^n)$ such that $x_k \oplus y_k \to x \oplus y$ when $k \to +\infty$. Hence, by lemma 2.9, for all $k \in \mathbb{N}$, we have:

$$x_k \oplus y_k \in R(M_0) + N(M_0^n) \iff x_k \in R(A) + N(A^n) \quad and \quad y_k \in R(B) + N(B^n).$$

Since $R(A) + N(A^n)$ and $R(B) + N(B^n)$ are closed, hence $x \in R(A) + N(A^n)$ and $y \in R(B) + N(B^n)$. Hence, by lemma 2.9, $x \oplus y \in R(M_0) + N(M_0^n)$. Thus $R(M_0) + N(M_0^n)$ is closed in $X \oplus Y$, $\forall n \geq d$. □

Remark 2.11. *A and B play symmetrical roles in the matrix M_0. That is: If $A \in qF(d)$ and $B \in qF(d')$ such that $d \leq d'$, then $M_0 \in qF(d')$. Indeed, the proof is analogous to that of the theorem 2.10.*

Corollary 2.12. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. We have*

$$\sigma_{qF}(M_0) \subseteq \sigma_{qF}(A) \cup \sigma_{qF}(B).$$

Proof. Let $\lambda \in \rho_{qF}(A) \cap \rho_{qF}(B)$, then, by theorem 2.10, $\lambda \in \rho_{qF}(M_0)$. Hence $\rho_{qF}(A) \cap \rho_{qF}(B) \subseteq \rho_{qF}(M_0)$. Therefore $\sigma_{qF}(M_0) \subseteq \sigma_{qF}(A) \cup \sigma_{qF}(B)$. □

Example 2.13. *Let Z and Y be two Banach spaces.*

Let $D \in \mathcal{L}(Z)$ such that D is injective and $R(D)$ is not closed. Then we have: $\Delta(T) = \mathbb{N}$, then $dis(D) = 0$. We consider the operator matrix $A = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$ acting on $X = Z \oplus Z$. It follows that $dis(A) = 0$ and $R(A)$ is not closed (Indeed. We have $x \oplus y \in R(A) \iff (\theta.z + \theta.t) \oplus (D.z + \theta.t) = x \oplus y$ for some $(z, t) \in Z^2 \iff x = 0$ and $y \in R(D)$. Let $(y_k)_k \subseteq R(D)$ such that $y_k \to z \notin R(D)$. Then $(0 \oplus y_k)_k \subseteq R(A)$ and $0 \oplus y_k \to 0 \oplus z$. If $0 \oplus z \in R(A)$, then $z \in R(D)$ which is absurde). Hence A is not quasi-Fredholm. Let $B \in \mathcal{L}(Y)$ a quasi-Fredholm operator of degree 2. By The proposition 3 in [2] (it suffices to show that $R(M_0^3)$ is closed), it is easy to see that $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{L}(X \oplus Y)$ is a quasi-Fredholm of degree 2.

Conclusion: $M_C \in qF(2)$ and $B \in qF(2)$ but A is not quasi-Fredholm operator.
By this example, there exists $\lambda \in \rho_{qF}(M_0)$ such that $\lambda \notin \rho_{qF}(A) \cap \rho_{qF}(B)$. Hence $\rho_{qF}(M_0) \not\subset \rho_{qF}(A) \cap \rho_{qF}(B)$. Thus $\sigma_{qF}(A) \cup \sigma_{qF}(B) \not\subset \sigma_{qF}(M_0)$. Hence

$$\sigma_{qF}(M_0) \subsetneq \sigma_{qF}(A) \cup \sigma_{qF}(B).$$

Corollary 2.14. *Let $n \in \mathbb{N}^*$. Let $X_1, X_2, ..., X_n$ be Banach spaces. Let $T_k \in \mathcal{L}(X)$, for all $1 \leq k \leq n$. If $T \in qF(d_k)$, for all $1 \leq k \leq n$. Then $\bigoplus_{k=1}^n T_k \in qF(d)$ such that $d = max\{d_k \,/\, 1 \leq k \leq n\}$.*

Proof. By induction. □

Corollary 2.15. *Let $n \in \mathbb{N}^*$. Let $X_1, X_2, ..., X_n$ be Banach spaces. Let $T_k \in \mathcal{L}(X_k)$, for all $1 \leq k \leq n$. We have*

$$\sigma_{qF}(\bigoplus_{k=1}^{n} T_k) \subsetneq \bigcup_{k=1}^{n} \sigma_{qF}(T_k).$$

Theorem 2.16. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, then the following statements hold:*

1. *If $M_0 \in qF(d)$, then $A \in qF(d)$ or $B \in qF(d)$.*
2. *If $M_0 \in qF(d)$ and $dis(A) = dis(B)$, then $A \in qF(d)$ and $B \in qF(d)$.*

Proof. By a similar proof of than of theorem 2.10, it is easy to see that:

$$\begin{cases} \text{For all } n \in \mathbb{N}. \text{ If } R(M_0^n) \text{ is closed, then } R(A^n) \text{ and } R(B^n) \text{ are closed.} \\ \text{For all } n \in \mathbb{N}. \text{ If } R(M_0) + N(M_0^n) \text{ is closed, then } R(A) + N(A^n) \text{ and } R(B) + N(B^n) \text{ are closed.} \end{cases}$$

1. Suppose that $dis(M_0) = d$. It is easy to see that $dis(A) \leq d$ and $dis(B) \leq d$. Suppose that $dis(B) \leq dis(A) = d' < d$. Then $x \oplus y \in R(M_0^{d'}) \cap N(M_0) \iff x \oplus y \in R(M_0^d) \cap N(M_0)$. Hence $dis(M_0) < d$, which is absurd. Thus
$$\begin{cases} dis(A) = d \\ dis(B) \leq d \end{cases} \quad or \quad \begin{cases} dis(B) = d \\ dis(A) \leq d \end{cases}$$
Therefore, if $M_0 \in qF(d)$, then $A \in qF(d)$ or $B \in qF(d)$.
2. If $dis(M_0) = d$, then $dis(A) = d$ and $dis(B) \leq d$ (A and B play symmetrical role). Suppose that $dis(A) = dis(B)$, then $dis(A) = dis(B) = d$. Hence, if $M_0 \in qF(d)$, then $A \in qF(d)$ and $B \in qF(d)$. Therefore, if $M_0 \in qF(d)$ and $dis(A) = dis(B)$, then $A \in qF(d)$ or $B \in qF(d)$.

□

Proposition 2.17. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. We have*

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \sigma_{qF}(M_0) \cup \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) \neq dis(B - \lambda)\}.$$

Proof. Let $\lambda \in \rho_{qF}(M_0) \cap \{\lambda \in \mathbb{C} \ / \ dis(A-\lambda) = dis(B-\lambda)\}$. By theorem 2.16 (2), it follows that $\lambda \in \rho_{qF}(A) \cap \rho_{qF}(B)$. Hence $\rho_{qF}(M_0) \cap \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) = dis(B - \lambda)\} \subseteq \rho_{qF}(A) \cap \rho_{qF}(B)$. Hence $\sigma_{qF}(A) \cup \sigma_{qF}(B) \subseteq \sigma_{qF}(M_0) \cup \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) \neq dis(B - \lambda)\}$.
Furthermore, the theorem 2.10 ensures the existence of a $\lambda \in \rho_{qF}(A) \cap \rho_{qF}(B)$ such that $\lambda \notin \rho_{qF}(M_0) \cap \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) = dis(B - \lambda)\}$. Hence $\rho_{qF}(A) \cap \rho_{qF}(B) \not\subset \rho_{qF}(M_0) \cap \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) = dis(B - \lambda)\}$. Which implies that $\sigma_{qF}(M_0) \cup \{\lambda \in \mathbb{C} \ / \ dis(A - \lambda) \neq dis(B - \lambda)\} \not\subset \sigma_{qF}(A) \cup \sigma_{qF}(B)$. □

Lemma 2.18. *(See Theorem 1.110, p 73, [1]) If $T \in \mathcal{L}(X)$ is quasi-Fredholm and $K \in \mathcal{L}(X)$ is finite-dimensional, then $T + K$ is quasi-Fredholm.*

Proposition 2.19. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. We have:*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \subseteq \sigma_{qF}(A) \cup \sigma_{qF}(B).$$

Proof. Let $C \in \mathcal{L}(Y, X)$ a finite-dimensional operator. Hence $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X \oplus Y)$ is finite-dimentional. We have

$$M_C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.$$

Since A and B are quasi-Fredholm, M_0 is quasi-Fredholm. The lemma 2.18 ensures that M_C is quasi-Fredholm.

Let $\lambda \in \rho_{qF}(A) \cap \rho_{qF}(B)$. Then there exists $C_0 \in \mathcal{L}(Y, X)$ ($C_0 \neq 0$), such that $\lambda \in \rho_{qF}(M_{C_0})$. Hence $\rho_{qF}(A) \cap \rho_{qF}(B) \subseteq \rho_{qF}(M_{C_0})$. Thus $\rho_{qF}(A) \cap \rho_{qF}(B) \subseteq \bigcup_{C \in \mathcal{L}(Y,X)} \rho_{qF}(M_C)$. Hence $\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \subseteq \sigma_{qF}(A) \cup \sigma_{qF}(B)$. □

Example 2.20. *Let Z be a Banach space. Let $X = Z \oplus Z \oplus Z$. Let $D \in \mathcal{L}(Z)$ ($D \neq 0$) such that D is injective and with not closed range. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(X)$. Hence A is nilpotent of index 2 and $dis(A) = 0$. Hence A is not a quasi-Fredholm operator (Indeed. $R(D)$ is not closed $R(A)$ is not closed hence $A \notin qF(0)$). Let $T \in \mathcal{L}(Z)$ such that $T \in qF(2)$. Let $B = \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(X)$, then $AB = 0$ and $B \in qF(2)$. Let $M_B = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \in \mathcal{L}(X \oplus X)$. Hence $M_B^n = \begin{pmatrix} 0 & B^n \\ 0 & B^n \end{pmatrix}$, for all $n \geq 2$. Let's show that $M_B \in qF(2)$.*

Step 1: If $1 \in \Delta(M_B)$, then $1 \in \Delta(M_B^*)$. Hence $R(M_B^*) \cap N(M_B^*) = R((M_B^*)^2) \cap N(M_B^*)$. Since $dis(B) = dis(B^*) = 2$, there exists $y_0 \in R(B^*) \cap N(B^*)$ such that $y_0^* \notin R((B^*)^2) \cap N(B^*)$ and $y_0^* \neq 0$. Thus $0 \oplus y_0^* \in R(M_B^*) \cap N(M_B^*)$. Therefore $0 \oplus y_0^* \in R((M_B^*)^2) \cap N(M_B^*)$. Hence $y_0^* = 0$, which is absurd. Thus $dis(M_B) \geq 2$. Let $n \geq 2$. For all $x \oplus y \in X \oplus X$, we have $x \oplus y \in R(M_B^2) \cap N(M_B) \Leftrightarrow x = y \in R(B^2) \cap N(B) \Leftrightarrow x = y \in R(B^n) \cap N(B) \Leftrightarrow x \oplus y \in R(M_B^n) \cap N(M_B)$. Therefore $dis(M_B) = 2$.
Step 2: Let $n \geq 2$ and $(x_k \oplus y_k)_{k \in \mathbb{N}} \in R(M_B^n)$ such that $x_k \oplus y_k \to x \oplus y$ when $k \to +\infty$. Let $k \in \mathbb{N}$, then there exists $t_k \in X$ such that $B^n t_k = x_k = y_k$. Since $B \in qF(2)$, $R(B^n)$ is closed in X. Hence $y \in R(B^n)$. Thus $y \oplus y = x \oplus y \in R(M_B^n)$. Hence $R(M_B^n)$ is closed.
Therefore $M_B \in qF(2)$.
Conclusion: For all $A \in \mathcal{L}(X)$ and $A \in \mathcal{L}(Y)$, we have:

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \subsetneq \sigma_{qF}(A) \cup \sigma_{qF}(B).$$

Indeed, The inclusion is proved in proposition 2.19. By this example, we have $\rho_{qF}(M_B) \not\subset \rho_{qF}(A) \cap \rho_{qF}(B)$. Hence $\sigma_{qF}(A) \cup \sigma_{qF}(B) \not\subset \sigma_{qF}(M_B)$. It follows that $\sigma_{qF}(A) \cup \sigma_{qF}(B) \not\subset \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C)$.

Proposition 2.21. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Then there exists an operator $C \in \mathcal{L}(Y, X)$ ($C \neq 0$) such that:*

1. *If M_C is quasi-Fredholm, then $A \in qF$ or $B \in qF$.*
2. *If M_C is quasi-Fredholm and $dis(A) = dis(B)$, then A and B are quasi-Fredholm.*

M_C is quasi-Fredholm and $dis(A) = dis(B)$ which implies that A and B are quasi-Fredholm.

Proof. Let $C \in \mathcal{L}(Y, X)$ a finite-dimensional operator. Thus $M = \begin{pmatrix} 0 & -C \\ 0 & 0 \end{pmatrix}$ is finite-dimensional. Suppose that M_C is quasi-Fredholm. By lemma 2.18 $M_C + M = M_0$ is quasi-Fredholm. Hence, by a direct application of the theorem 2.16, we will have the requested result. □

Proposition 2.22. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. We have:*

1. $$\sigma_{qF}(A) \cap \sigma_{qF}(B) \subseteq \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$
2. $$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \{\lambda \in \mathbb{C}\ ;\ dis(A - \lambda) \neq dis(B - \lambda)\}.$$

Proof. 1. Let $C_0 \in \mathcal{L}(Y, X)$ ($C_0 \neq 0$) be a finite-dimensional operator. By proposition 2.21 we have $\lambda \in \rho_{qF}(M_{C_0}) \implies \lambda \in \rho_{qF}(A) \cup \rho_{qF}(B)$. Hence $\rho_{qF}(M_{C_0}) \subseteq \rho_{qF}(A) \cup \rho_{qF}(B)$. Thus $\sigma_{qF}(A) \cap \sigma_{qF}(B) \subseteq \sigma_{qF}(M_{C_0})$. It follows that $\sigma_{qF}(A) \cap \sigma_{qF}(B) \subseteq \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{qF}(M_C)$.

2. Let $C_0 \in \mathcal{L}(Y,X)$ ($C_0 \neq 0$) be a finite-dimensional operator. By proposition 2.21 we have

$$\lambda \in \rho_{qF}(M_{C_0}) \cap \{\lambda \in \mathbb{C}\ ;\ dis(A-\lambda) = dis(B-\lambda)\} \implies \lambda \in \rho_{qF}(A) \cap \rho_{qF}(B).$$

But the equivalence is not satisfied. In fact, if A and B are quasi-Fredholm, then we do not have necessary $dis(A) = dis(B)$. Hence $\rho_{qF}(M_{C_0}) \cap \{\lambda \in \mathbb{C}\ ;\ dis(A-\lambda) = dis(B-\lambda)\} \subsetneq \rho_{qF}(A) \cap \rho_{qF}(B)$. It follows that

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \{\lambda \in \mathbb{C},\quad dis(A-\lambda) \neq dis(B-\lambda)\}.$$

□

Corollary 2.23. *If* $\{\lambda \in \mathbb{C}\ ;\ dis(A-\lambda) \neq dis(B-\lambda)\} \subseteq \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C)$ *Then*

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) = \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$

Corollary 2.24. *Let* $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ *such that* A *and* B *are quasi-nilpotent and injective. We have:*

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) = \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$

Proof. We have

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) \subsetneq \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C) \cup \{\lambda \in \mathbb{C}\ ;\ dis(A-\lambda) \neq dis(B-\lambda)\}.$$

and

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C) \subsetneq \sigma_{qF}(A) \cup \sigma_{qF}(B).$$

Since A and B are quasi-nilpotent, $\sigma(A) = \sigma(B) = 0$. Then $A-\lambda$ and $B-\lambda$ are injectif for all $\lambda \in \mathbb{C}^*$. Thus $dis(A-\lambda) = dis(B-\lambda) = 0$.

Furthermore A and B are injectif, then $dis(A) = dis(B) = 0$. Therefore $dis(A-\lambda) = dis(B-\lambda) = 0$ for all $\lambda \in \mathbb{C}$. Hence $\{\lambda \in \mathbb{C}\ ;\ dis(A-\lambda) \neq dis(B-\lambda)\} = \emptyset$. Hence

$$\sigma_{qF}(A) \cup \sigma_{qF}(B) = \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$

□

Example 2.25. *Let* Z *be a Banach space. Let* $X = Z \oplus Z \oplus Z$. *Let* $T \in \mathcal{L}(Z)$ *bounded below. Hence* $T \in qF(0)$. *Let* $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(X)$. *Hence* $A \in qF(d)$. *Let* $S \in \mathcal{L}(Z)$ *such that* $dis(S) = +\infty$. *Let* $B = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(X)$, *then* $dis(B) = +\infty$ *and* $AB = 0$. *Let* $M_B = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \in \mathcal{L}(X \oplus X)$. *Hence* $M_B^n = \begin{pmatrix} 0 & B^n \\ 0 & B^n \end{pmatrix}$, *for all* $n \geq 2$. *Thus* $dis(M_B) = +\infty$. *It follows that* M_B *is not quasi-Fredholm.*

By this example, there exists $\lambda \in \rho_{qF}(A) \cup \rho_{qF}(B)$ *but* $\lambda \notin \rho_{qF}(M_B)$. *Hence* $\rho_{qF}(A) \cup \rho_{qF}(B) \not\subset \rho_{qF}(M_B)$. *Thus* $\sigma_{qF}(M_B) \not\subset \sigma_{qF}(A) \cap \sigma_{qF}(B)$. *It follows that*

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C) \not\subset \sigma_{qF}(A) \cap \sigma_{qF}(B).$$

By proposition 2.22, we have

$$\sigma_{qF}(A) \cap \sigma_{qF}(B) \subsetneq \bigcap_{C\in\mathcal{L}(Y,X)} \sigma_{qF}(M_C).$$

References

[1] P. Aiena, Fredholm and Local Spectral Theory II, Lecture Notes in Mathematics, vol. 2235. Springer, New York (2018).
[2] M. Barraa, M. Boumazgour, On the perturbations of spectra of upper triangular operator matrices, J. Math. Anal. Appl. 347 (2008) 315-322.
[3] H. K. Du, J. Pan, Perturbation of Spectrums of 2×2 Operator Matrices, Proceedings of the American Mathe- matical Society, 121(1994), 761-766.
[4] J. K. Han, H. Y. Lee, W. Y. Lee, Invertible completions of 2×2 upper triangular operator matrices, Proceedings of the American Mathematical Society 128 (1999) 119-123.
[5] J.P. Labrousse, Les opérateurs quasi Fredholm: une généralisation des opérateurs semi-Fredholm, Rend. Circ. Mat, Palermo 29 (1980), 161-258.
[6] M. Mbekhta, V. Müller, On the axiomatic theory of spectrum II, Studia Math., 119 (1996), 129-147.
[7] V. Müller, On the Kato decomposition of quasi-Fredholm and B-Fredholm operators, Preprint ESI, 2001.
[8] A. Tajmouati, M. Abkari, M. Karmouni, Generalized Drazin-type spectra of Operator matrices, Proyecciones Journal of Mathematics 37(2018), 119-131.
[9] A. Tajmouati, M. Karmouni, S. Alaoui Chrifi, Generalized Drazin-Riesz invertibility for operator matrices. Adv. Oper. Theory 5(2020), 347-358 .
[10] E H, Zerouali, H. Zguitti, Perturbation of spectra of operator matrices and local spectral theory, J. Math. Anal. Appl. 324 (2006) 992-1005.