Filomat 36:14 (2022), 4963–4981
https://doi.org/10.2298/FIL2214963A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Estimation and Prediction for a Generalized Half-Normal Distribution Based on Left-Truncated and Right-Censored Data

Kambiz Ahmadi[a], Masoumeh Akbari[b], Liang Wang[c]

[a]*Department of Computer Science, Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran*
[b]*Department of Statistics, University of Mazandaran, Babolsar, Mazandaran, Iran*
[c]*School of Mathematics, Yunnan Normal University, Kunming, P.R. China*

Abstract. In this article the reliability estimation of the generalized half-normal distribution (GHN) is considered when data are subject to both left truncation and right censoring (LTRC). Since the EM-algorithm for the generalized gamma distribution (that includes GHN as a special case) based on LTRC data was developed in Balakrishnan and Mitra [Em-based likelihood inference for some lifetime distributions based on left truncated and right censored data and associated model discrimination; 2014, *South African Statistical Journal*, **48(2)**, 125–171], the maximum likelihood estimates, as well as asymptotic confidence intervals (CIs) and bootstrap CIs for the unknown parameters of GHN, are briefly discussed. For further study, we utilized a hierarchical Bayesian approach and proposed two sampling techniques, the Metropolis-Hastings algorithm and the slice sampler technique to carry out the Bayesian estimation procedure under squared error loss function, which can be easily extended to other loss function situations. In addition, the Bayesian prediction problem concerning the lifetime of a censored unit and the Bayesian estimates of the expected number of failures in a prefixed interval are investigated. Finally, some simulation studies are carried out to compare the performance of the proposed procedure with its competitor and data analysis of the electric power-transformers data is conducted to illustrate the purposes.

1. Introduction

The generalized half-normal (GHN) distribution is a flexible lifetime distribution with decreasing, increasing, and bathtub shapes of the hazard function proposed by Cooray and Ananda [1] for static fatigue data. This distribution was largely applied as model lifetimes in various fields of reliability analysis and lifetime studies. The probability density function (PDF) and the cumulative distribution function (CDF) of the GHN density function are given by

$$f(t;\alpha,\theta) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{t}\right)\left(\frac{t}{\theta}\right)^{\alpha}\exp\left\{-\frac{1}{2}\left(\frac{t}{\theta}\right)^{2\alpha}\right\}, \quad t \geq 0.$$

2020 *Mathematics Subject Classification*. 62N05 ; 62F10, 62F15

Keywords. Bayesian estimation, Confidence interval, Credible interval, EM algorithm, Generalized half-normal distribution, Hierarchical Bayesian model, Left truncation and right censoring, Maximum likelihood estimates

Received: 18 July 2022; Accepted: 22 September 2022

Communicated by Biljana Č. Popović

The work of Liang Wang is supported by the National Natural Science Foundation of China (No. 12061091) and the Yunnan Fundamental Research Projects(No. 202101AT070103).

Email addresses: k.ahmadi@sku.ac.ir (Kambiz Ahmadi), m.akbari@umz.ac.ir (Masoumeh Akbari), liang610112@163.com (Liang Wang)

and

$$F(t;\alpha,\theta) = 1 - 2\Phi\left(-\left(\frac{t}{\theta}\right)^{\alpha}\right), \quad t \geq 0,$$

where $\alpha > 0$ and $\theta > 0$ are the shape and scale parameters respectively, and $\Phi(\cdot)$ is the CDF of the standard normal distribution. Hereafter, we denote the GHN model with parameters α and θ by GHN(α,θ). It is noted that the shape parameter α affects the geometric shape of the failure density distribution curve of the GHN model, and the scale parameter θ not only determines the steepness of the failure density distribution curve, but also specifically exhibits the length of units lifetime. Its negatively and positively skewed density shapes make the GHN model a proper alternative to conventional exponential, Weibull and gamma distributions, among others. Therefore, given its flexible versatility and goodness-of-fit property, the GHN distribution could be used as a potential model in many reliability and lifetime studies as well as other application fields.

This distribution has been studied by many authors. For example, inferential issues for GHN distribution based on various censoring schemes where the focus on estimating the unknown parameters can be found in [2–6]. In completed data, Wang [7] derived different estimates of the unknown GHN parameters. Other works on the statistical inference of GHN can be found in [8–12].

In many applied fields such as engineering, economics, medicine, biology, epidemiology, and demography, we often encounter failure observations with various data characteristics. One common appeared phenomenon for observations is truncation including left, right, and double truncation. For left truncation, it means lifetimes of units exceed the truncation time which, in general, has an unknown distribution function. Here, we consider lifetime data exceed a threshold. For example, some products are tested for a predetermined period of time to improve the performance of the product before shipping. The units that survive are considered to be appropriate for shipping, while failed units are discarded. So, the lifetime of the survived units has already exceeded the threshold value that the manufacturer decided to be the testing period. As another example, when the observations are measured by different instruments of unequal but known accuracy we encounter data that draws from a distribution with left truncation at a constant. Meanwhile, as another common feature of observations, censoring also appears frequently in many lifetime experiments due to different reasons such as time and cost limitations, etc. Censoring indicates that there are portion failure times of the tested units observed in experiments and possible censoring types are right, progressive and interval censoring. By right censoring, it is meant that the failure time of interest is only known to exceed the censoring time. Different from the sole data failure feature like truncation and censoring, left truncated and right censored (LTRC) data, as a widespread phenomenon for failure times, are more general in practical situations. For example, Hong et al. [13] considered the lifetime data of electric power-transformers in an electrical industry in the US, over a particular interval of time. The failure of a machine is observed only if it fails after 1980, as detailed record keeping on the lifetime of machines was started in that year. Complete information on the lifetime of machines installed after 1980 is available, while for machines installed before 1980, the installation dates are available but no information is available on machines installed and failed before 1980. For such dataset, it is observed that the associated lifetime data were left-truncated at the starting date of record keeping and right-censored at the ending date of the study, and should be appropriately adjusted as LTRC data.

Motivated by such previous reasons and due to the practical applications of GHN distribution, this paper considers reliability estimation for the GHN model when available observations are LTRC data. Of late, a lot of attention has been paid to LTRC data and various studies have been discussed by many authors. For example, Balakrishnan and Mitra [14–16] developed the steps of Expectation-Maximization (EM) algorithm to estimate the unknown parameters of the lognormal, Weibull, and gamma distributions based on LTRC data. In Balakrishnan and Mitra [17], the EM-algorithm for generalized gamma (GG) distribution based on LTRC data was developed. In their study, the GG distribution is a model that includes lognormal, Weibull, gamma, and the GHN discussed in this paper, as special cases. Kundu and Mitra [18] provided the Bayesian inference of the unknown parameters of the Weibull distribution based on LTRC data. They considered fairly flexible priors on the scale and shape parameters and computed the Bayes estimates of the unknown parameters and the associated credible intervals using Gibbs sampling technique. Kundu et al. [19] considered both the classical and Bayesian estimates of the Weibull parameters for the LTRC competing risks data.

In this paper, different analyzing inferential methods for GHN under LTRC data are compared using classical and Bayesian perspectives. As mentioned above, GHN is a special case of GG distribution. In general, the fitting ability of statistical models increases with the number of parameters, thus in order to select the best model among a certain number of candidates it is necessary to use criteria that allow balancing fitting ability against model complexity. The GG distribution's ability to behave like other more commonly-used life distributions and its mathematical complexity cause this distribution is not often used to model life data by itself. Due to the reason that the GHN distribution has its own lifetime characteristic showing flexible and various data fitting ability but less complexity than GG in model structure, which provides some trade-off between simple and complexity lifetime models in data analysis. Furthermore, to the best of our knowledge, no theoretical result has been reported for Bayesian inference for the GHN model in the setting of LTRC data so far.

The outline of the paper is as follows. Section 2 is dedicated to a short description of LTRC data and some notations. Section 3 gives a brief background of the classical approach to find the point and interval estimate of the unknown parameters. These are related to MLE, EM algorithm, asymptotic confidence intervals (ACIs), and bootstrap confidence intervals (BCIs). Bayesian estimates and the associated credible intervals for the unknown parameters and some Bayesian prediction issues facing LTRC data are discussed in Section 4. Section 5 presents a simulation study to illustrate the performances of the proposed methods. Finally, we present a real-life example for illustrating the applications of our results in Section 6 and conclude this paper in Section 7.

2. Data description and notation

Consider a lifetime experiment with $n \in \mathbb{N}$ identical units. Its lifetimes are described by independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_n$. It is assumed that the lifetimes of the units being tested have a GHN(α, θ), where parameters α and θ are unknown. Corresponding to i-th unit we assume that there is a prefixed left truncation point, τ_i, and a prefixed right censoring point, c_i. Each unit can be placed on the test before or after the corresponding left truncation point. If the i-th unit has been put on a test before τ_i and it is failed before τ_i, i.e. $X_i < \tau_i$, no information is available about X_i. The information regarding the i-th unit is available only if it is failed after τ_i, or it is being censored after c_i.

Data from experiment involving left truncation and right censoring can be conveniently represented by $\{(Y_i, \nu_i, \delta_i); i = 1, 2, \ldots, n\}$, where the discrete random variable ν_i denotes truncation indicator; i.e. $\nu_i = 0$ if the i-th observation is truncated and 1 if it is not truncated. The discrete random variable δ_i denotes censoring indicator, i.e. $\delta_i = 0$ if the i-th observation is censored and 1 if it is not censored. Y_i is equal to X_i if the lifetime of the i-th unit is observed and to c_i if it is right censored, i.e. $Y_i = \min(X_i, c_i)$. Accordingly, we denote the observed data by $\{(y_1, \nu_1, \delta_1), (y_2, \nu_2, \delta_2), \ldots, (y_n, \nu_n, \delta_n)\}$. Now consider the index sets $S_1, S_2, S_{c_1}, S_{c_2}$ as follows:

$S_1 = \{i : \nu_i = 1\}$: $i \in S_1$ implies that the lifetime of i-th unit is not left truncated at τ_i.

$S_2 = \{i : \nu_i = 0\}$: $i \in S_2$ implies that the lifetime of i-th unit is left truncated at τ_i.

$S_{c_1} = \{i : \delta_i = 1\}$: $i \in S_{c_1}$ implies that the lifetime of i-th unit is not right censored at c_i.

$S_{c_2} = \{i : \delta_i = 0\}$: $i \in S_{c_2}$ implies that the lifetime of i-th unit is right censored at c_i.

Let m be the number of elements in S_{c_1} and $S = S_1 \cup S_2$.

3. Frequentist estimation

In this section, statistical inference is based on the maximum likelihood estimation and confidence intervals for the unknown parameters when the sample consists of LTRC data.

3.1. Maximum likelihood estimates

Let us denote the unknown parameter vector of the distribution by $\lambda = (\alpha, \theta)$. The likelihood function of the observed data $\{(y_1, \nu_1, \delta_1), (y_2, \nu_2, \delta_2), \ldots, (y_n, \nu_n, \delta_n)\}$ is given by

$$\begin{aligned} L(\lambda) &= \prod_{i \in S} \left\{ \{f(y_i;\lambda)\}^{\delta_i \nu_i} \{S(y_i;\lambda)\}^{(1-\delta_i)\nu_i} \Big\{ \frac{f(y_i;\lambda)}{S(\tau_i;\lambda)} \Big\}^{\delta_i(1-\nu_i)} \Big\{ \frac{S(y_i;\lambda)}{S(\tau_i;\lambda)} \Big\}^{(1-\delta_i)(1-\nu_i)} \right\} \\ &\propto \frac{\alpha^m}{\theta^{m\alpha}} \exp\Big\{ -\frac{1}{2} \sum_{i \in S_{c_1}} u_i^2 \Big\} \Big(\prod_{i \in S_{c_1}} y_i^{\alpha-1} \Big) \Big(\prod_{i \in S_{c_2}} \Phi(-u_i) \Big) \Big(\prod_{i \in S_2} \frac{1}{\Phi(-v_i)} \Big), \end{aligned} \tag{1}$$

where $u_i = (y_i/\theta)^\alpha$ and $v_i = (\tau_i/\theta)^\alpha$. Therefore, setting the first partial derivatives of the logarithm of likelihood with respect to α and θ to zero, the MLEs of α and θ, say $\hat{\alpha}$ and $\hat{\theta}$, can be obtained by solving the following nonlinear equations:

$$\frac{m}{\alpha} + \sum_{i \in S_{c_1}} (1 - u_i^2) \ln y_i - \sum_{i \in S_{c_2}} u_i h(u_i) \ln y_i + \sum_{i \in S_2} v_i h(v_i) \ln \tau_i = 0, \tag{2}$$

$$m - \sum_{i \in S_{c_1}} u_i^2 - \sum_{i \in S_{c_2}} u_i h(u_i) + \sum_{i \in S_2} v_i h(v_i) = 0, \tag{3}$$

where $h(\cdot)$ is the hazard function of the standard normal distribution. Clearly, the system of nonlinear equations (2) and (3) cannot be solved analytically and mathematical or statistical software should apply to get a numerical solution via iterative techniques. Here, R package *nleqslv* is used to find the roots of these two non-linear equations by the Broyden method.

3.2. EM algorithm based estimation

The EM algorithm, originally suggested by Dempster et al. [20], is a broadly applicable iterative algorithm used to find MLEs in the presence of incomplete data, missing data, truncated distributions, and censored observations; see [21]. Each iteration of the EM algorithm consists of two steps called the Expectation step (the E-step) and the Maximization step (the M-step). Suppose the complete dataset consists of $X = (\mathcal{Y}, Z)$ where only $\mathcal{Y}$ is observed and $Z_i = [X_i|\delta_i = 0]$ for $i \in S_{c_2}$. The complete data likelihood function is

$$L_c(\lambda) = \frac{\alpha^n}{\theta^{n\alpha}} \Big(\prod_{i \in S_{c_1}} y_i^{\alpha-1} \Big) \Big(\prod_{i \in S_{c_2}} Z_i^{\alpha-1} \Big) \exp\left\{ -\frac{1}{2} \left(\sum_{i \in S_{c_1}} u_i^2 + \sum_{i \in S_{c_2}} \Big(\frac{Z_i}{\theta} \Big)^{2\alpha} \right) \right\} \Big(\prod_{i \in S_2} \frac{1}{\Phi(-v_i)} \Big),$$

In the E-step of $(h+1)$-th iteration, we require to compute the pseudo log-likelihood function $Q(\lambda; \lambda_{(h)})$. It can be obtained by replacing any function of Z_i, say $g(Z_i)$, in the logarithm of L_c with $E(g(Z_i)|Z_i > c_i, \lambda_{(h)})$ for $i \in S_{c_2}$, where $\lambda_{(h)}$ is a vector of the h-th iteration values of the parameters α and θ. In the M-step, the expected complete data log-likelihood $Q(\lambda; \lambda_{(h)})$ is maximize with respect to λ to determine $\lambda_{(h+1)}$. So, taking the first-order derivatives of the function Q with respect to α and θ, we obtain, respectively,

$$\frac{n}{\alpha} + \sum_{i \in S_{c_1}} (1 - u_i^2) \ln y_i + \sum_{i \in S_{c_2}} \tilde{A}_i - \sum_{i \in S_{c_2}} \frac{\tilde{C}_i}{\theta^{2\alpha}} + \sum_{i=1}^{n} (1 - \nu_i) v_i h(v_i) \ln \tau_i = 0, \tag{4}$$

$$n - \sum_{i \in S_{c_1}} u_i - \sum_{i \in S_{c_2}} \frac{\tilde{B}_i}{\theta^{2\alpha}} + \sum_{i=1}^{n} (1 - \nu_i) v_i h(v_i) = 0, \tag{5}$$

where $\tilde{A}_i = E\big(\ln Z_i | Z_i > c_i, \lambda_{(h)} \big)$, $\tilde{B}_i = E\big(Z_i^{2\alpha} | Z_i > c_i, \lambda_{(h)} \big)$, and $\tilde{C}_i = E\big(Z_i^{2\alpha} \ln Z_i | Z_i > c_i, \lambda_{(h)} \big)$. These expectations can be easily calculated using [3]. We refer to **Appendix I** for more details. Therefore estimate $\lambda_{(h+1)}$ is obtained numerically by solving the system of nonlinear equations (4) and (5). We repeat the two steps as necessary until convergence is achieved to the desired level of accuracy.

3.3. Asymptotic variances and covariance of the MLEs

The asymptotic variance-covariance matrix of the MLEs is derived using the missing information principle from [22]. Computing inverse observed information matrix, we can obtain asymptotic variances and covariance of the MLEs, where the observed information matrix, $I_{\mathcal{Y}}(\lambda)$ equals the complete information matrix $I_X(\lambda)$ minus the missing information matrix $I_Z(\lambda)$. The complete information matrix and the missing information matrix are given by

$$I_X(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\log L_c(\lambda)\right], \quad I_Z(\lambda) = -\sum_{i\in S_{c_2}} E\Big[\frac{\partial^2}{\partial\lambda^2}\log f_{Z_i|C_i}(z_i|z_i > c_i, \lambda)\Big]$$

So the observed information can be obtained as $I_{\mathcal{Y}}(\lambda) = I_X(\lambda) - I_Z(\lambda)$. If we denote V as the asymptotic variance-covariance matrix for $\hat{\lambda} = (\hat{\alpha}, \hat{\theta})$, then the estimate of V can be obtained as

$$\hat{V} = \begin{bmatrix}\hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & \hat{V}_{22}\end{bmatrix} = I_{\mathcal{Y}}^{-1}(\hat{\lambda}).$$

The calculation of $\hat{V}$ is presented in **Appendix II**.

3.4. Approximate confidence intervals

There are several ways to construct confidence intervals which vary in ease of calculation and accuracy. As an application of previous subsections, we can construct ACIs for parameters α and θ using the asymptotic normality of the MLEs. Therefore, for $0 < \gamma < 1$, the $100(1-\gamma)\%$ ACIs for α and θ are respectively given by

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\hat{V}_{11}}, \quad \text{and} \quad \hat{\theta} \pm z_{\gamma/2}\sqrt{\hat{V}_{22}},$$

where $z_{\gamma/2}$ is the upper $\gamma/2$th percentile point of the standard normal distribution.

In addition, the bootstrap technique is an alternative to construct confidence intervals for the unknown parameters α and θ. In the following, we use the parametric bootstrap method which was used by some authors such as [23, 24]. Here are the main steps of using the parametric bootstrap to compute confidence intervals for the parameters α and θ as follows:

Step 1 Given the original LTRC sample of size n, calculate $\hat{\lambda} = (\hat{\alpha}, \hat{\theta})$.

Step 2 Using the MLE $\hat{\lambda} = (\hat{\alpha}, \hat{\theta})$ as the true value of the parameter, within the same sampling framework of LTRC data, generate a sample of size n.

Step 3 Based on the bootstrap sample obtained above, calculate $\hat{\lambda}^* = (\hat{\alpha}^*, \hat{\theta}^*)$, the MLE for $\lambda = (\alpha, \theta)$, in the same way as described in Subsection 3.1.

Step 4 Repeat Steps 2 and 3 $B-1$ times. Then denote the MLEs by $\hat{\lambda}_1^*, \hat{\lambda}_2^*, \dots, \hat{\lambda}_B^*$, where $\hat{\lambda}_i^* = (\hat{\alpha}_i^*, \hat{\theta}_i^*)$ is the MLE of λ based on the i-th bootstrap sample, $i = 1, 2, \dots, B$.

Step 5 To construct a bootstrap-p confidence interval(BCI), arrange $\hat{\alpha}_i^*, i = 1, 2, \dots, B$ in an ascending order to obtain the bootstrap samples as $\hat{\alpha}_{(1)}^*, \hat{\alpha}_{(2)}^*, \dots, \hat{\alpha}_{(B)}^*$. Then $\left(\hat{\alpha}_{(\lfloor B\gamma/2\rfloor)}^*, \hat{\alpha}_{(\lfloor B-B\gamma/2\rfloor)}^*\right)$ is a two-sided $100(1-\gamma)\%$ BCI for α, where $\lfloor x\rfloor$ is the largest integer less than or equal to x. The BCI for θ is obtained in an analogous manner.

To improvement of the precision of the percentile bootstrap confidence interval, we can further use the following bootstrap bias correction technique. For a model parameter, say α, a two-sided $100(1-\gamma)\%$ parametric bias-corrected bootstrap confidence interval (BCI_a) is specified by

$$\hat{\alpha} - b_\alpha \pm z_{\gamma/2}\sqrt{v_\alpha},$$

where b_α and v_α are respectively the bootstrap bias and bootstrap variance for MLE $\hat{\alpha}$ and are defined as

$$b_\alpha = \bar{\hat{\alpha}}^* - \hat{\alpha} \text{ and } v_\alpha = \frac{1}{B-1}\sum_{i=1}^{B}(\hat{\alpha}_i^* - \bar{\hat{\alpha}}^*)^2,$$

with $\bar{\hat{\alpha}}^* = \sum_{i=1}^{B}\hat{\alpha}_i^*/B$. The parametric BCIa for θ can be constructed in a similar way.

4. Bayesian inference

The Bayesian approach in statistical inference provides an alternative choice for parameters estimation. In this section, we first consider the Bayesian estimates and the associated credible intervals of the unknown parameters of GHN under LTRC data. Then, the Bayesian prediction problem concerning the lifetime of an individual unit censored at time c_i is investigated. Finally, a discussion about how to estimate the expected number of future failures within a fixed interval is presented.

4.1. Prior information and posterior inference

In this subsection, we provide the necessary assumptions about prior distributions. Recently, the two-parameter GHN distribution has been studied in different frameworks of data, from a Bayesian viewpoint. Ahmadi and Yousefzadeh [2], Ahmadi et al. [3] and Abd El-Raheem [4] considered Gamma prior distributions for the unknown parameters α and θ. Ahmadi and Ghafouri[5] used Gamma and inverse Gamma(IG) distributions as prior distributions for α and θ respectively.

In this paper, we develop the Bayesian set-up by considering the idea of Kottas [25] regarding the choice of prior distributions. Our prior knowledge about the true values of α and θ are, respectively, expressed via Uniform $(0, \psi)$ and IG(a, β) with the PDFs

$$\pi(\alpha;\psi) = \frac{1}{\psi}I_{(0,\psi)}(\alpha), \quad \psi > 0, \tag{6}$$

$$\pi(\theta; a, \beta) = \frac{\beta^a}{\Gamma(a)}\Big(\frac{1}{\theta}\Big)^{a+1}e^{-\beta/\theta}, \quad a, \beta > 0. \tag{7}$$

In order to incorporate uncertainty about the prior distributions, the hierarchical Bayesian approach is utilized as well. This approach models the lack of information on the hyper-parameters of the prior distributions through other prior distributions on these hyper-parameters. In this regard, it is assumed that ψ and β have respectively, conjugate priors Pareto(a_ψ, b_ψ) and Gamma(a_β, b_β) with the PDFs

$$\psi \sim \pi(\psi; a_\psi, b_\psi) = \frac{a_\psi b_\psi^{a_\psi}}{\psi^{a_\psi+1}}I_{(b_\psi,\infty)}, \quad \beta \sim \pi(\beta; a_\beta, b_\beta) = \frac{b_\beta^{a_\beta}}{\Gamma(a_\beta)}\beta^{a_\beta-1}e^{-b_\beta\beta}. \tag{8}$$

Utilizing likelihood function (1) and prior distributions (6)-(8), the joint posterior density function of α, β, θ and ψ given the data $\{O_i = (y_i, \nu_i, \delta_i); i = 1, \cdots n\}$ is obtained as

$$\pi(\alpha,\beta,\theta,\psi|O_i) \propto \frac{\alpha^m\beta^{a+a_\beta-1}}{\psi^{a_\psi+2}\theta^{m\alpha+a+1}}\exp\Big\{-\frac{1}{2}\sum_{i\in S_{c_1}}u_i^2 - \Big(b_\beta + \frac{1}{\theta}\Big)\beta\Big\}\Big(\prod_{i\in S_{c_1}}y_i^\alpha\Big)$$
$$\times\Big(\prod_{i\in S_{c_2}}\Phi(-u_i)\Big)\Big(\prod_{i\in S_2}\frac{1}{\Phi(-v_i)}\Big)I_{(0,\psi)}(\alpha)I_{(b_\psi,\infty)}(\psi). \tag{9}$$

It is obvious that, under the squared error loss function, there is no closed-form for the expression of Bayesian estimates of α and θ. Numeric computation can be used but are not recommended due to the large error and sensitivity to the sample. Instead of it, we use Gibbs sampling that is one popular MCMC approach. In order to construct a Gibbs sampler for the model (9), we need to calculate the full conditional distributions. Utilizing (9) the full conditional density function of α is obtained as

$$\pi(\alpha|\beta,\theta,\psi,O_i) \propto \alpha^m \exp\Big\{-\sum_{i\in S_{c_1}} u_i^2/2\Big\}\Big(\prod_{i\in S_{c_1}} u_i\Big)\Big(\prod_{i\in S_{c_2}} \Phi(-u_i)\Big)\Big(\prod_{i\in S_2} \frac{1}{\Phi(-v_i)}\Big) I_{(0,\psi)}(\alpha). \tag{10}$$

The full conditional distribution of β is

$$\pi(\beta|\alpha,\theta,\psi,O_i) \propto \beta^{a+a_\beta-1} e^{-\left(b_\beta+\frac{1}{\theta}\right)\beta} \equiv \text{Gamma}(a+a_\beta, b_\beta+\frac{1}{\theta}),$$

Finally, we can get the full conditional density functions of θ and ψ, respectively, as

$$\pi(\theta|\alpha,\beta,\psi,O_i) \propto \frac{1}{\theta^{m\alpha+a+1}} \exp\Big\{-\frac{1}{2}\sum_{i\in S_{c_1}} u_i^2 - \frac{\beta}{\theta}\Big\}\Big(\prod_{i\in S_{c_2}} \Phi(-u_i)\Big)\Big(\prod_{i\in S_2} \frac{1}{\Phi(-v_i)}\Big), \tag{11}$$

and

$$\pi(\psi|\alpha,\beta,\theta,O_i) \propto \frac{1}{\psi^{a_\psi+2}} I_{(\max(\alpha,b_\psi),\infty)}(\psi) \equiv \text{Pareto}(a_\psi+1, \max(\alpha,b_\psi)).$$

Note that the full conditional densities (10) and (11) are not in the form of well known distributions. In the following, we implement the slice sampler to generate a sample from the full conditional distribution (10). Let $\eta_i = \ln(y_i/\theta)$ for $i \in S_{c_1} \cup S_{c_2}$ and $\xi_i = \ln(\tau_i/\theta)$ for $i \in S_2$.

Algorithm 1: slice sampler approach for (10)

Step 1 For each $i \in S_{c_1}$,

i) Generate W_{i0} from Uniform$(0,u_i)$ and W_{i1} from Uniform $(0, e^{-u_i^2/2})$.

ii) Set $L_i^{(1)} = \max\Big\{0, \frac{1}{2\eta_i}\ln\Big(-2\ln(W_{i1})\Big)\Big\}$ and $U_i^{(1)} = \frac{1}{\eta_i}\ln(W_{i0})$, if $\eta_i < 0$ and $L_i^{(1)} = \max\Big\{0, \frac{1}{\eta_i}\ln(W_{i0})\Big\}$ and $U_i^{(1)} = \frac{1}{2\eta_i}\ln\Big(-2\ln(W_{i1})\Big)$ otherwise.

Step 2 For each $i \in S_{c_2}$,

i) Generate W_{i2} from Uniform$(0,\Phi(-u_i))$.

ii) Set $L_i^{(2)} = \max\Big\{0, \frac{1}{\eta_i}\ln\Big(\Phi^{-1}(1-W_{i2})\Big)\Big\}$ and $U_i^{(2)} = \infty$, if $\eta_i < 0$ and $L_i^{(2)} = 0$ and $U_i^{(2)} = \frac{1}{\eta_i}\ln\Big(\Phi^{-1}(1-W_{i2})\Big)$ otherwise.

Step 3 For each $i \in S_2$,

i) Generate W_{i3} from Uniform$(0, 1/\Phi(-v_i))$.

ii) For $W_{i3} \le 2$, set $L_i^{(3)} = 0$ and $U_i^{(3)} = \infty$. For $W_{i3} > 2$, set $L_i^{(3)} = 0$ and $U_i^{(3)} = \frac{1}{\xi_i}\ln\Big(\Phi^{-1}(1-1/W_{i3})\Big)$, if $\xi_i < 0$ and $L_i^{(3)} = \max\Big\{0, \frac{1}{\xi_i}\ln\Big(\Phi^{-1}(1-1/W_{i3})\Big)\Big\}$ and $U_i^{(3)} = \infty$ otherwise.

Step 4 Generate W_* from Uniform(0, 1), and compute $\alpha = \Big[W_* U_*^{m+1} + (1-W_*)L_*^{m+1}\Big]^{1/(m+1)}$, where $L_* = \max\Big\{L_i^{(k)} : 1 \le i \le n,\ k = 1,2,3\Big\}$ and $U_* = \min\Big\{U_i^{(k)} : 1 \le i \le n,\ k = 1,2,3\Big\}$.

Note our objective is to generate a sample from PDF $f(\alpha) \propto \alpha^m I_{(L_*,U_*)}(\alpha)$ in **Step 4** of **Algorithm 1**. The inverse-transform method has been used in this respect. In order to generate a sample from the full conditional distribution (11), Metropolis-Hastings (M-H) algorithm can be utilized with the normal proposal distribution $N(\theta, S_\theta)$. In general, S_θ is not known and the choice of it is an important issue. Let $\ell_\theta = \ln \pi(\theta|\alpha, \beta, \psi, O_i)$. The second derivative of ℓ_θ with respect to θ can be specified as follows:

$$\frac{\partial^2 \ell_\theta}{\partial \theta^2} = \frac{m\alpha + a + 1}{\theta^2} - \frac{2\beta}{\theta^3} - \frac{\alpha}{\theta^2}\Big[(1+2\alpha)\sum_{i\in S_{c_1}} u_i^2 - \sum_{i\in S_{c_2}} u_i h(u_i)\Big[\Big(u_i - h(u_i)\Big)\alpha u_i - \alpha - 1\Big] + \sum_{i\in S_2} v_i h(v_i)\Big[\Big(v_i - h(v_i)\Big)\alpha v_i - \alpha - 1\Big]\Big]. \tag{12}$$

One choice for S_θ is $[-\frac{\partial^2 \ell_\theta}{\partial \theta^2}]^{-1}$ evaluated at the posterior mode of full condition posterior (11), say $\tilde{\theta}$. The posterior mode $\tilde{\theta}$ can be evaluated by usual optimization methods.

Algorithm 2: M-H sampling

Step 1 Set initial values $\alpha^{(0)}, \beta^{(0)}, \theta^{(0)}, \psi^{(0)}$.

Step 2 Calculate the posterior mode $\tilde{\theta}$ of $\pi(\theta|\alpha^{(0)}, \beta^{(0)}, \psi^{(0)}, O_i)$.

Step 3 Using (12), evaluate S_θ at the posterior mode $\tilde{\theta}$.

Step 4 For $j = 1, 2, ..., N$, repeat the following steps:

i) For given $\alpha^{(j-1)}, \beta^{(j-1)}, \theta^{(j-1)}$ and $\psi^{(j-1)}$, generate $\alpha^{(j)}$ using **Algorithm 1**.

ii) Generate $\theta^{(j)}$ as follows:

- Generate new candidate parameter value η from $N(\theta^{(j-1)}, S_\theta)$.
- Set $\theta' = |\eta|$.
- Calculate $\tau = \min\left\{1, \frac{\pi(\theta'|\alpha^{(j)}, \beta^{(j-1)}, \psi^{(j-1)}, O_i)}{\pi(\theta|\alpha^{(j)}, \beta^{(j-1)}, \psi^{(j-1)}, O_i)}\right\}$.
- Set $\theta^{(j)} = \theta'$ with probability τ otherwise set $\theta^{(j)} = \theta^{(j-1)}$.

iii) Generate $\beta^{(j)}$ from Gamma$(a + a_\beta, b_\beta + \frac{1}{\theta^{(j)}})$.

iv) Generate $\psi^{(j)}$ from Pareto$(a_\psi + 1, \max(\alpha^{(j)}, b_\psi))$.

Using the generated random samples from **Algorithm 2**, the Bayesian estimates of α and θ under the squared error loss function can be computed as

$$\hat{\alpha}_B = \frac{1}{N-M}\sum_{j=M}^{N} \alpha^{(j)}, \quad \text{and} \quad \hat{\theta}_B = \frac{1}{N-M}\sum_{j=M}^{N} \theta^{(j)}$$

respectively, where M is burn-in period.

In addition, to construct the credible interval of α, we sort all the $\alpha^{(j)}, j = M+1, M+2, \ldots, N$, in an ascending sequence, as $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-M)}$. Then for $0 < \gamma < 1$, a $100(1-\gamma)\%$ credible interval of α is specified by

$$\Big(\alpha_{(k)}, \alpha_{(k+N-M-\lfloor (N-M)\gamma+1\rfloor)}\Big), \quad k = 1, 2, \ldots, \lfloor (N-M)\gamma \rfloor.$$

Therefore, the $100(1-\gamma)\%$ HPD credible interval of α can be obtained as the k_*-th one satisfying

$$\alpha_{(k_*+N-M-\lfloor (N-M)\gamma+1\rfloor)} - \alpha_{(k_*)} \leq \alpha_{(k+N-M-\lfloor (N-M)\gamma+1\rfloor)} - \alpha_{(k)}$$

for all $k = 1, 2, \ldots, \lfloor (N-M)\gamma \rfloor$. The HPD credible interval of θ can be constructed in a similar way.

4.2. Prediction for the remaining lifetime of a censored unit

The prediction of lifetime of a censored unit is an important problem in reliability theory and can be applied in industrial applications. Here, the point and interval predict are of interest. Let T_i be the failure time of a unit which censored at time $c_i, i \in S_{c_2}$. The conditional survival function of T_i is

$$S(t|T_i > c_i; \alpha, \theta) = 1 - F(t|T_i > c_i, \alpha, \theta) = \frac{\Phi\left(-(\frac{t}{\theta})^{\alpha}\right)}{\Phi\left(-(\frac{c_i}{\theta})^{\alpha}\right)}, \tag{13}$$

The conditional PDF of T_i corresponding to (13) is

$$f(t|T_i > c_i; \alpha, \theta) = \frac{\phi\left(-(\frac{t}{\theta})^{\alpha}\right)}{\Phi\left(-(\frac{c_i}{\theta})^{\alpha}\right)}\left(\frac{\alpha}{t}\right)\left(\frac{t}{\theta}\right)^{\alpha}. \tag{14}$$

Based on the joint posterior distribution (9), the predictive density of T_i and the predictive survival function given data are as

$$f^*(t|c_i) = E_{posterior}\Big[f(t|T_i > c_i; \alpha, \theta)\Big], \quad S^*(t|c_i) = E_{posterior}\Big[S(t|T_i > c_i; \alpha, \theta)\Big],$$

respectively. Suppose $\{(\alpha^{(j)}, \theta^{(j)}), j = M, M+1, \ldots, N\}$ are samples obtained from posterior distribution (9), using **Algorithm 2**, then the simulation consistent estimators of $f^*(t|c_i)$ and $S^*(t|c_i)$ can be obtained as

$$\widehat{f^*}(t|c_i) = \frac{1}{N-M}\sum_{j=M}^{N} f\left(t|T_i > c_i; \alpha^{(j)}, \theta^{(j)}\right), \tag{15}$$

$$\widehat{S^*}(t|c_i) = \frac{1}{N-M}\sum_{j=M}^{N} S\left(t|T_i > c_i; \alpha^{(j)}, \theta^{(j)}\right), \tag{16}$$

respectively. Utilizing (14) and (15), the Bayesian point predictor of T_i under the squared error loss function can be expressed as

$$\widehat{T}_{iB} = \int_{c_i}^{\infty} t\widehat{f^*}(t|c_i)\,dt = \frac{1}{N-M}\sum_{j=M}^{N}\frac{2^{\frac{1}{2\alpha^{(j)}}-1}\theta^{(j)}}{\sqrt{\pi}\Phi\left(-(\frac{c_i}{\theta^{(j)}})^{\alpha^{(j)}}\right)}\Gamma\left[\frac{1}{2\alpha^{(j)}} + \frac{1}{2}, \frac{1}{2}\left(\frac{c_i}{\theta^{(j)}}\right)^{2\alpha^{(j)}}\right].$$

From (13) and (16), the Bayesian predictive bounds of a two-sided interval with cover $1-\gamma$, for the value of T_i, may be obtained by solving the following two equations for the lower bound, L and upper bound, U:

$$\widehat{S^*}(L|c_i) = 1 - \frac{\gamma}{2}, \quad \widehat{S^*}(U|c_i) = \frac{\gamma}{2}.$$

4.3. The cumulitive number of failures in an interval

Let $c^* = \max\{c_i, i \in S_{c_2}\}$, where c_i denotes censored time of the i-th unit, $i \in S_{c_2}$. For the fixed interval $[L, R]$, $c^* < L$ and $L < R$, we define $Z_i = 1$ if the i-th unit fails in $[L, R]$ and 0 otherwise. Thus the random variable $\mathcal{J} = \sum_{i \in S_{c_2}} Z_i$, describes the number of future failures in the interval $[L, R]$, out of m units which belong to S_{c_2}. It is immediate that

$$E(\mathcal{J}; \alpha, \theta) = \sum_{i \in S_{c_2}} Pr(Z_i = 1) = \left[\Phi\left(\left(\frac{R}{\theta}\right)^{\alpha}\right) - \Phi\left(\left(\frac{L}{\theta}\right)^{\alpha}\right)\right]\sum_{i \in S_{c_2}}\frac{1}{\Phi\left(-\left(\frac{c_i}{\theta}\right)^{\alpha}\right)}.$$

Therefore using the samples $\{(\alpha^{(j)}, \theta^{(j)}), j = M, M+1, \ldots, N\}$, the Bayesian estimate of $E(\mathcal{J}; \alpha, \theta)$ under the squared error loss function can be specify by $\widehat{\mathcal{E}}_B = \frac{1}{N-M}\sum_{j=M}^{N} E(\mathcal{J}; \alpha^{(j)}, \theta^{(j)})$.

Table 1: MSEs and biases (in parentheses) of ML, EM, and Bayesian estimates of $\alpha = 2.17, 5$ and $\theta = 20$.

α	n	Trunc. (%)	Cen. (%)	Estimation α			Estimation θ		
				MLE, EM	BE 1	BE 2	MLE, EM	BE 1	BE 2
2.17	30	20	19.32	0.2044	0.2007	0.1204	2.0557	2.1763	0.7113
				(0.1217)	(0.1071)	(0.0237)	(-0.0841)	(0.1320)	(-0.0189)
		40	14.67	0.2117	0.2082	0.1089	2.0377	2.0963	0.6624
				(0.1415)	(0.1295)	(0.0348)	(-0.0054)	(0.1186)	(-0.0029)
		60	10.13	0.2566	0.2504	0.1163	2.2389	2.2814	0.6383
				(0.1600)	(0.1470)	(0.0430)	(0.0608)	(0.0989)	(0.0079)
	60	20	19.33	0.0875	0.0889	0.0633	1.0541	1.0955	0.5820
				(0.0599)	(0.0563)	(0.0084)	(0.0046)	(0.1106)	(0.0178)
		40	14.55	0.0893	0.0909	0.0599	1.0508	1.0715	0.5471
				(0.0634)	(0.0602)	(0.0110)	(-0.0342)	(0.0300)	(-0.0415)
		60	9.75	0.0999	0.1021	0.0603	1.2324	1.2585	0.5822
				(0.0630)	(0.0605)	(0.0090)	(-0.0217)	(0.0019)	(-0.0553)
	100	20	19.31	0.0472	0.0491	0.0385	0.6229	0.6406	0.4265
				(0.0348)	(0.0336)	(0.0030)	(0.0093)	(0.0729)	(0.0145)
		40	14.49	0.0494	0.0515	0.0383	0.6326	0.6422	0.4184
				(0.0342)	(0.0326)	(0.0031)	(-0.0345)	(0.0036)	(-0.0458)
		60	9.76	0.0567	0.0594	0.0403	0.7473	0.7676	0.4624
				(0.0420)	(0.0399)	(0.0080)	(-0.0069)	(0.0060)	(-0.0458)
5	30	20	18.44	0.9905	0.9749	0.5968	0.3849	0.3886	0.2911
				(0.3036)	(0.2751)	(0.0576)	(-0.0376)	(0.0353)	(-0.0174)
		40	13.87	1.1333	1.1203	0.6570	0.3852	0.3880	0.2852
				(0.3279)	(0.3023)	(0.0668)	(-0.0293)	(0.0152)	(-0.0416)
		60	9.54	1.1854	1.1562	0.6475	0.4208	0.4267	0.2978
				(0.3411)	(0.3095)	(0.0629)	(-0.0180)	(-0.0041)	(-0.0619)
	60	20	18.43	0.4239	0.4309	0.3153	0.1947	0.1952	0.1695
				(0.1628)	(0.1525)	(0.0403)	(-0.0251)	(0.0110)	(-0.0191)
		40	13.77	0.4470	0.4544	0.3282	0.1933	0.1961	0.1660
				(0.1525)	(0.1438)	(0.0265)	(-0.0227)	(0.0002)	(-0.0357)
		60	9.18	0.4885	0.4994	0.3478	0.2113	0.2159	0.1782
				(0.1578)	(0.1502)	(0.0173)	(-0.0070)	(0.0017)	(-0.0423)
	100	20	18.35	0.2335	0.2407	0.1937	0.1151	0.1161	0.1059
				(0.0857)	(0.0787)	(0.0125)	(-0.0144)	(0.0068)	(-0.0126)
		40	13.86	0.2469	0.2565	0.2022	0.1190	0.1210	0.1087
				(0.0913)	(0.0874)	(0.0171)	(-0.0199)	(0.0038)	(-0.0038)
		60	9.23	0.2718	0.2843	0.2205	0.1266	0.1295	0.1147
				(0.0942)	(0.0889)	(0.0109)	(-0.0099)	(-0.0047)	(-0.0335)

5. Numerical computations

In this section, a simulation study was mainly performed to compare how the different methods work for different sample sizes and truncation rates. The performance of all estimates has been compared numerically in terms of their mean squared errors (MSEs), average biases, and interval estimates in terms of coverage probabilities and average widths of two-sided confidence intervals. The underlying failure time was independently generated from a GHN distribution. Two choices of the shape and scale parameters are made: $\alpha = 2.17$, as the symmetric density function, and $\alpha = 5$, as the positively skewed density function as well as scale parameters $\theta = 20$ and 40.

The total sample size n was chosen to be 30, 60, and 100. For the fixed truncation percentages 20%, 40% and 60%, we get the censoring rates between 10% and 80%. To form a LTRC data, following Balakrishnan and Mitra [14], we set the truncation time between the year of installation and the truncation point of 1980, as to mimic the dataset used by Hong et al. [13]. Based on a fixed truncation rate, the installation years were simulated by unequal probability with-replacement sampling from an arbitrary set of years. As stated in [14], unequal probabilities were assigned to different years as follows: for the period 1960-1979, a probability of 0.1 was attached to each of the first six years and a probability of 0.04 was attached to each of the remaining years of this period; for the period 1980–1995, a probability of 0.15 was attached to each of the first five years, and the remaining probability was distributed equally over the remaining years of this period. We also fixed 2008 as the year of censoring. Right censoring occurs when the lifetime exceeds the censoring time 2008. Censoring rates were computed for all scenarios and different truncation rates.

In order to solve the nonlinear equations and obtain the estimates of the unknown parameters using the ML method and EM algorithm, the *nleqslv* package was applied. We employed the moment estimates for

Table 2: MSEs and biases (in parentheses) of ML, EM, and Bayesian estimates of $\alpha = 2.17, 5$ and $\theta = 40$.

α	n	Trunc. (%)	Cen. (%)	Estimation α			Estimation θ		
				MLE, EM	BE1	BE2	MLE, EM	BE1	BE2
2.17	30	20	66.89	0.5888 (0.2579)	0.5991 (0.2903)	0.2586 (0.1011)	19.7219 (-0.1685)	18.9725 (-0.1271)	5.2953 (-0.2482)
		40	56.96	0.4387 (0.1828)	0.4356 (0.1915)	0.2308 (0.0503)	19.8532 (0.1648)	18.9389 (0.0502)	5.7563 (-0.0893)
		60	47.07	0.4088 (0.1599)	0.4045 (0.1447)	0.2164 (0.0149)	14.1672 (0.0396)	14.2905 (-0.2752)	5.2608 (-0.1625)
	60	20	66.87	0.2080 (0.1045)	0.2128 (0.1293)	0.1268 (0.0441)	16.1017 (0.2012)	15.4310 (0.2779)	5.4166 (0.0642)
		40	56.88	0.1719 (0.0764)	0.1745 (0.0885)	0.1161 (0.0116)	8.9498 (0.0294)	8.9130 (0.0139)	4.5873 (-0.0134)
		60	46.98	0.1809 (0.0802)	0.1858 (0.0830)	0.1230 (0.0062)	6.2288 (0.0509)	6.3370 (-0.0572)	3.8035 (-0.0314)
	100	20	66.95	0.1068 (0.0497)	0.1102 (0.0658)	0.0759 (0.0154)	8.8607 (0.1818)	8.8626 (0.2527)	4.4265 (0.1693)
		40	56.85	0.0929 (0.0501)	0.0973 (0.0604)	0.0708 (0.0092)	5.2533 (0.0143)	5.2571 (0.0158)	3.5090 (0.0269)
		60	46.98	0.0985 (0.0510)	0.1034 (0.0555)	0.0760 (0.0021)	3.8116 (0.0370)	3.8535 (-0.0182)	2.8448 (-0.0051)
5	30	20	78.86	9.2358 (1.1764)	9.3042 (1.2233)	2.8673 (0.5057)	7.6333 (-0.3052)	8.1385 (0.1019)	4.3084 (-0.0011)
		40	63.37	2.6804 (0.5786)	2.7278 (0.5844)	1.3461 (0.1850)	3.6359 (-0.1965)	3.7574 (-0.0288)	2.7429 (-0.0628)
		60	48.09	1.7154 (0.4012)	1.7341 (0.3840)	0.9771 (0.0761)	2.4432 (-0.1068)	2.5294 (-0.0378)	2.0279 (-0.0623)
	60	20	78.79	1.4819 (0.4257)	1.5637 (0.4660)	0.8826 (0.1937)	3.5104 (-0.1606)	3.6646 (0.0702)	2.7087 (0.0570)
		40	56.85	0.9215 (0.2741)	0.9468 (0.2893)	0.6111 (0.0900)	1.7062 (-0.0305)	1.7498 (0.0617)	1.5118 (0.0447)
		60	48.13	0.6423 (0.1844)	0.6593 (0.1863)	0.4523 (0.0275)	1.1634 (-0.0401)	1.1858 (0.0008)	1.0695 (-0.0181)
	100	20	78.71	0.6794 (0.2355)	0.7202 (0.2627)	0.4784 (0.1088)	1.8967 (-0.0839)	1.9614 (0.0607)	1.6582 (0.0675)
		40	63.40	0.4568 (0.1632)	0.4753 (0.1761)	0.3456 (0.0573)	0.9992 (-0.0343)	1.0145 (0.0224)	0.9361 (0.0160)
		60	48.04	0.3539 (0.0997)	0.3646 (0.1008)	0.2822 (0.0062)	0.7207 (-0.0317)	0.7274 (-0.0062)	0.6854 (-0.0190)

the parameters as starting values. They were obtained from the pseudo-complete data. See [16] for more details. We stopped iterations in the EM algorithm when the maximum of absolute difference of estimates in $(h+1)$-th and h-th iteration was less than 1×10^{-4}. Moreover, to obtain the bootstrap confidence intervals, we used $B = 5000$ bootstrap samples and follow the procedure described in Subsection 3.3.

In the Bayesian context, we chose the values of the hyper-parameters as follows. Regarding the prior for ψ, we simplify by setting $a_\psi = 2$, yielding a Pareto distribution with infinite variance for ψ. To choose b_ψ we obtain the CDF of the marginal prior distribution of parameter α as follows:

$$\begin{aligned}\Pi^*_\alpha(p) &= \int_0^p \int_\psi \pi(\alpha;\psi)\pi(\psi;2,b_\psi)\,d\psi\,d\alpha \\ &= \int_0^p \int_{\max\{\alpha,b_\psi\}}^\infty \frac{2b_\psi^2}{\psi^4}\,d\psi\,d\alpha = \begin{cases} \frac{2p}{3b_\psi} & 0<p<b_\psi \\ 1-\frac{b_\psi^2}{3p^2} & p\geq b_\psi \end{cases}.\end{aligned}$$

If we assume m_α is the median of the marginal prior distribution α, then $b_\psi = \frac{4}{3}m_\alpha$. We substitute m_α with the MLE $\hat{\alpha}$. The prior distribution of θ is simplified by setting $a = 2$, resulting in infinite prior variance for θ. Also, the hyper-parameter β is characterized by improper prior. In this regard we set $a_\beta = 0$ and $b_\beta = 0$. For the first prior (Prior 1), we considered $a_\psi = 2, b_\psi = \frac{4}{3}\hat{\alpha}, a = 2, a_\beta = 0, b_\beta = 0$, where $\hat{\alpha}$ is the MLE of α based on current sample. The second prior (Prior 2) is different for the input parameters of GHN distribution. Let a_0 be the considered values 2.17 and 5. For $\alpha = a_0$ and $\theta = 20$, we consider $a_\psi = 2, b_\psi = a_0, a = \frac{406}{3}, a_\beta = 8060, b_\beta = 3$. Also, for $\alpha = a_0$ and $\theta = 40$, we set $a_\psi = 2, b_\psi = a_0, a = \frac{204.5}{3}, a_\beta = 8060, b_\beta = 3$.

Table 3: Coverage probabilities and average widths(in parentheses) of 95% CIs for true values $\alpha = 2.17, 5$ and $\theta = 20$.

	n	Trunc. (%)	α=2.17, θ=20						α=5, θ=20					
			Cen. (%)	ACL	BCL	BCLa	HPD1	HPD2	Cen. (%)	ACL	BCL	BCLa	HPD1	HPD2
α	30	20	19.32	94.38 (1.57)	91.22 (1.76)	96.1 (1.77)	92.62 (1.50)	95.62 (1.38)	18.44	95.46 (3.49)	90.84 (3.40)	97.06 (4.02)	93.96 (3.37)	96.74 (3.12)
		40	14.67	95.06 (1.61)	90.82 (1.83)	96.78 (1.84)	93.56 (1.54)	96.94 (1.37)	13.87	94.36 (3.61)	90.04 (4.14)	96.72 (4.17)	93.02 (3.47)	96.12 (3.19)
		60	10.13	94.92 (1.72)	90.64 (1.97)	96.72 (1.99)	92.74 (1.63)	97.28 (1.39)	9.54	94.96 (3.79)	91.18 (4.34)	96.96 (4.37)	93.68 (3.61)	96.52 (3.29)
	60	20	19.33	94.58 (1.08)	92.84 (1.14)	95.80 (1.14)	93.32 (1.05)	95.44 (0.97)	18.43	94.84 (2.39)	92.18 (2.56)	96.34 (2.57)	93.08 (2.33)	95.38 (2.20)
		40	14.55	94.86 (1.10)	93.18 (1.17)	96.02 (1.17)	92.82 (1.06)	95.72 (0.97)	13.77	95.16 (2.47)	92.62 (2.64)	96.14 (2.64)	93.38 (2.39)	95.30 (2.23)
		60	9.75	95.20 (1.16)	93.34 (1.25)	96.20 (1.25)	92.42 (1.11)	96.16 (0.99)	9.18	94.96 (2.59)	93.20 (2.77)	95.90 (2.78)	95.90 (2.48)	95.00 (2.32)
	100	20	19.31	94.70 (0.82)	93.64 (0.85)	95.42 (0.85)	92.40 (0.79)	94.32 (0.75)	18.35	95.14 (1.83)	93.44 (1.90)	95.96 (1.90)	92.80 (1.74)	94.04 (1.67)
		40	14.49	95.22 (0.84)	93.90 (0.87)	95.62 (0.87)	92.18 (0.80)	94.68 (0.75)	13.86	95.00 (1.89)	93.72 (1.96)	95.68 (1.97)	92.58 (1.80)	94.40 (1.72)
		60	9.76	95.14 (0.89)	94.08 (0.93)	95.76 (0.93)	91.92 (0.84)	94.40 (0.77)	9.23	95.52 (1.98)	94.06 (2.06)	96.10 (2.07)	92.08 (1.86)	94.08 (1.77)
θ	30	20	19.32	93.72 (5.45)	93.24 (5.45)	93.60 (5.44)	94.58 (5.84)	98.64 (4.31)	18.44	93.26 (2.35)	93.40 (2.36)	93.18 (2.36)	94.14 (2.48)	96.38 (2.34)
		40	14.67	94.02 (5.55)	93.94 (5.52)	93.70 (5.52)	94.84 (5.81)	98.68 (4.32)	13.87	93.52 (2.39)	93.06 (2.39)	93.40 (2.39)	94.44 (2.49)	96.38 (2.34)
		60	10.13	93.58 (5.91)	94.08 (5.87)	93.44 (5.86)	93.86 (6.10)	99.00 (4.41)	9.54	93.42 (2.08)	93.16 (2.47)	93.06 (2.47)	93.90 (2.57)	96.48 (2.39)
	60	20	19.33	93.66 (3.90)	93.52 (3.90)	93.66 (3.90)	93.82 (4.00)	97.00 (3.42)	18.43	93.94 (1.68)	94.14 (1.69)	94.22 (1.69)	94.16 (1.72)	95.42 (1.67)
		40	14.55	94.36 (3.99)	94.20 (3.98)	94.10 (3.98)	94.20 (4.05)	97.48 (3.43)	13.77	94.62 (1.71)	94.38 (1.71)	94.58 (1.71)	94.56 (1.74)	94.68 (1.68)
		60	9.75	94.08 (4.27)	94.02 (4.25)	93.88 (4.25)	93.56 (4.28)	97.34 (3.55)	9.18	94.50 (1.78)	94.48 (1.78)	94.32 (1.78)	94.00 (1.79)	95.56 (1.73)
	100	20	19.31	94.58 (3.04)	94.40 (3.03)	94.44 (3.03)	94.30 (3.06)	96.22 (2.78)	18.35	94.34 (1.32)	94.16 (1.32)	94.28 (1.32)	94.26 (1.32)	94.92 (1.30)
		40	14.49	94.56 (3.11)	94.44 (3.11)	94.38 (3.11)	93.82 (3.11)	96.52 (2.81)	13.86	94.72 (1.34)	94.70 (1.34)	94.66 (1.34)	94.28 (1.34)	95.40 (1.31)
		60	9.76	94.42 (3.32)	94.38 (3.31)	94.26 (3.31)	93.74 (3.28)	96.64 (2.91)	9.23	94.88 (1.38)	94.88 (1.38)	94.74 (1.38)	94.14 (1.37)	95.34 (1.34)

It may be noted that the hyper-parameters of the second prior are selected in such a way that the mean of Uniform$(0, \psi)$ is the same as the true value of α. And the mean of Pareto$(2, b_\psi)$ is the same as ψ. With regard to θ, the mean of IG prior is the same as the true value of θ and the mean of Gamma prior is the same as β. To implement **Algorithm 2**, we ran the iterative process up to $N = 10000$ iterations by discarding the first $M = 1000$ iterations as burn-in-period. We repeated the whole procedure 10000 times.

Tables 1 and 2 presente the biases and MSEs of the ML, EM, and Bayesian estimates of α and θ. Since the numbers are reported by four decimal places, MLE and EM estimates have exactly the same results in terms of biases and MSEs. Hence, one column has been considered in the tables to show the results of these two methods. We use the notations "BE 1" and "BE 2" to refer to the Bayesian estimates under Prior 1 and Prior 2, respectively. From Tables 1 and 2 it is observed that the biases and MSEs of BE 2 are smaller than those of MLE, EM, and BE 1. Monte Carlo studies have shown, however, that its finite-sample performance for θ can be poor when the sample size is low and censoring rate is high. For a fixed truncation rate, the biases and MSEs decrease with an increase in sample sizes for all cases. Note that by fixing the truncation rate and increasing sample size, the censoring rate remains approximately constant. According to the results in columns BE 1 and BE 2, one can deduce that MSEs and biases depend on the choice of priors.

Tables 2 and 3 display coverage probabilities and average widths of 95% intervals for α and θ. The HPD credible intervals under Prior 1 and Prior 2 are referred to as "HPD 1" and "HPD 2", respectively. Results for α show that BCI_a has higher coverage probability than the other intervals in most cases. Moreover, the average width of HPD 2 is shorter than the other intervals. For θ, HPD 2 gives better results in terms of coverage probability and average width. In bootstrap approach, the coverage probability of BCI_a is higher, however, it has a larger average width of confidence interval. In Bayesian approach, HPD 2 has a better performance than HPD 1, in coverage probability and average width. On the other hand, the performance

Table 4: Coverage probabilities and average widths(in parentheses) of 95% CIs for true values $\alpha = 2.17, 5$ and $\theta = 40$.

	n	Trunc. (%)	α=2.17, θ=40						α=5, θ=40					
			Cen. (%)	ACL	BCL	BCLa	HPD1	HPD2	Cen. (%)	ACL	BCL	BCLa	HPD1	HPD2
α	30	20	66.89	91.78 (2.39)	85.96 (3.24)	95.54 (3.78)	88.96 (2.12)	94.84 (1.95)	78.86	96.62 (6.60)	79.62 (7.40)	98.48 (7.80)	90.72 (5.92)	97.72 (5.28)
		40	56.96	94.54 (2.25)	91.16 (2.71)	97.12 (2.78)	92.02 (2.02)	96.68 (1.89)	63.37	94.84 (4.96)	87.02 (6.74)	98.00 (6.93)	91.98 (4.62)	96.64 (4.27)
		60	47.07	94.16 (2.28)	92.14 (2.58)	96.24 (2.64)	91.16 (2.04)	95.68 (1.90)	48.09	94.34 (4.39)	89.88 (5.28)	96.98 (5.34)	92.22 (4.14)	96.12 (3.85)
	60	20	66.87	94.18 (1.59)	91.56 (1.79)	96.88 (1.80)	92.28 (1.48)	96.08 (1.37)	78.79	94.76 (3.84)	87.62 (5.05)	97.96 (5.17)	92.42 (3.69)	96.34 (3.42)
		40	56.88	94.38 (1.52)	93.34 (1.64)	95.90 (1.65)	92.80 (1.43)	95.36 (1.34)	56.85	94.76 (3.29)	90.34 (3.76)	96.88 (3.78)	92.78 (3.17)	95.94 (2.96)
		60	46.98	94.40 (1.56)	93.40 (1.65)	95.62 (1.66)	92.50 (1.46)	95.38 (1.37)	48.13	95.30 (2.97)	92.78 (3.23)	96.62 (3.24)	93.62 (2.88)	95.76 (2.70)
	100	20	66.95	94.16 (1.20)	93.26 (1.28)	95.50 (1.28)	91.94 (1.13)	94.92 (1.06)	78.71	94.26 (2.82)	90.12 (3.27)	97.10 (3.29)	92.74 (2.76)	95.48 (2.58)
		40	56.85	95.16 (1.16)	93.94 (1.21)	96.04 (1.21)	92.48 (1.10)	95.01 (1.04)	63.40	94.76 (2.48)	92.70 (2.67)	96.60 (2.68)	93.26 (2.41)	95.32 (2.28)
		60	46.98	95.16 (1.20)	93.76 (1.23)	95.72 (2.34)	92.40 (1.12)	94.58 (1.06)	48.04	94.94 (2.26)	93.54 (2.38)	96.04 (2.38)	93.36 (2.19)	94.96 (2.08)
θ	30	20	66.89	86.40 (20.58)	87.62 (26.25)	87.88 (26.84)	84.56 (16.10)	94.26 (10.77)	78.86	86.46 (9.15)	84.26 (10.76)	89.74 (10.53)	82.78 (8.01)	90.22 (7.03)
		40	56.96	92.40 (16.49)	91.62 (18.89)	93.42 (22.42)	86.80 (13.34)	94.28 (9.71)	63.37	90.82 (6.85)	89.56 (7.23)	92.72 (7.25)	85.80 (5.89)	89.88 (5.54)
		60	47.07	93.94 (14.16)	92.38 (15.34)	94.44 (16.17)	87.38 (12.05)	93.58 (9.10)	48.09	92.24 (5.82)	91.20 (6.01)	93.58 (6.02)	86.12 (4.98)	89.08 (4.78)
	60	20	66.87	92.14 (14.79)	92.32 (16.54)	93.42 (17.55)	87.42 (11.89)	94.14 (9.08)	78.79	90.86 (6.61)	89.26 (7.07)	93.24 (7.05)	85.42 (5.60)	88.72 (5.30)
		40	56.88	93.84 (11.34)	92.98 (11.98)	94.42 (11.98)	87.20 (9.17)	92.26 (7.78)	56.85	93.24 (4.92)	92.58 (5.05)	93.84 (5.03)	86.56 (4.07)	88.88 (3.99)
		60	46.98	95.40 (9.81)	94.80 (10.15)	95.96 (10.16)	88.06 (8.08)	92.98 (7.08)	48.13	94.10 (4.16)	93.62 (4.21)	94.66 (4.21)	87.80 (3.43)	89.14 (3.39)
	100	20	66.95	94.58 (11.32)	93.84 (12.02)	94.86 (12.05)	88.74 (9.02)	93.54 (7.73)	78.71	92.68 (5.15)	91.98 (5.37)	94.22 (5.34)	86.54 (4.27)	88.68 (4.16)
		40	56.85	94.76 (8.67	94.10 (8.94)	95.38 (8.92)	87.78 (6.98)	91.36 (6.35)	63.40	93.68 (3.81)	93.18 (3.87)	94.50 (3.87)	86.70 (3.10)	88.18 (3.08)
		60	46.98	95.30 (7.55)	94.54 (7.69)	95.76 (7.67)	87.42 (6.13)	90.32 (5.70)	48.04	93.86 (3.23)	93.90 (3.26)	94.48 (3.26)	87.56 (2.63)	88.44 (2.62)

of the BCI is unsatisfactory when sample sizes are small or even moderately large. The simulations also show, for a fixed truncation rate, the average width decreases with an increase in sample sizes for all cases. We have also obtained coverage probabilities and average widths of 90% intervals for α and θ under different truncation and censoring rate, which we don't report here for reasons of brevity. The results are similar to those described above.

The summary for the 10000 simulation runs for $(\alpha, \theta) = (5, 20)$ is graphically illustrated in Figures 1 and 2. These figures are a confirmation of the above results about point estimation. It is observed that the medians of the boxplots are close to the input parameters $(5, 20)$. From dispersions of the boxplots shown in Figures 1 and 2, it is found BE 2 provides the most precise results than the other estimates based on different methods.

6. Data analysis

To illustrate the practical usefulness of the proposed method, we apply it to the LTRC data from [13]. The dataset includes 710 transformer lifetimes from an energy company with 62 failures. Although the original data are not available, their article provides a subset of the data containing 286 observations with 39 failures, which is available in Appendix II of [26]. The company's data records were collected between 1980 and 2008. Those were installed before 1980 must be viewed as transformers sampled from truncated distribution. Hence, the data are left truncated and right censored. The censoring and truncation rates of these data are 86.4% and 58.39%, respectively.

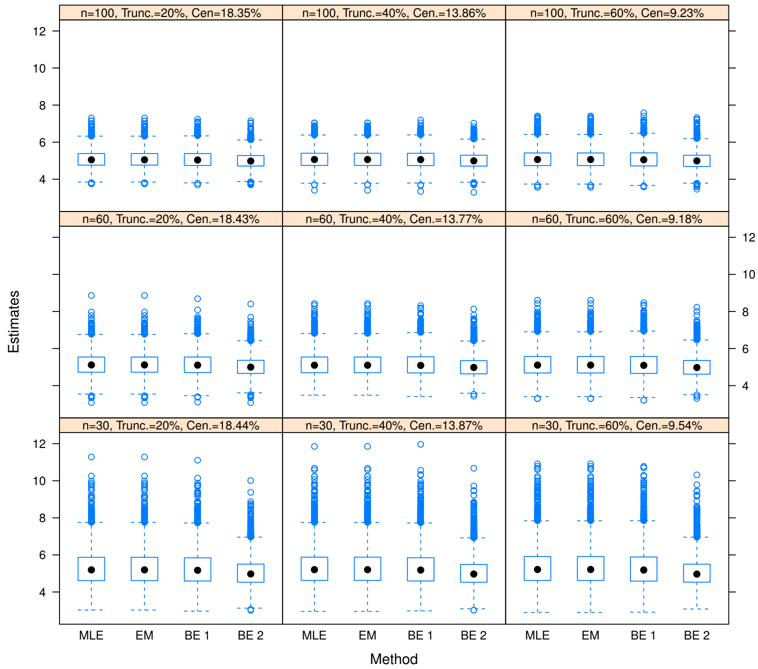

Figure 1: Boxplot for estimates of α under different methods, for $(\alpha, \theta) = (5, 20)$.

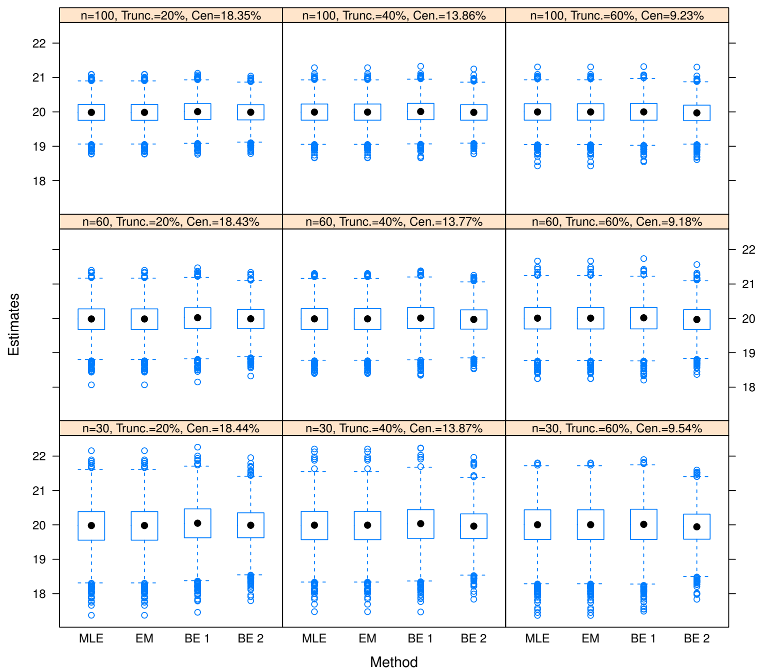

Figure 2: Boxplot for estimates of θ under different methods, for $(\alpha, \theta) = (5, 20)$.

Table 5: Estimates of α and θ for the transformer lifetime data.

Method	Confidence level	α	θ
MLE,EM		0.8359	168.3678
BE1		0.8505	174.6761
ACI	90%	(0.5986, 1.0730)	(96.8764, 239.8593)
	95%	(0.5532, 1.1184)	(83.1806, 253.5551)
BCI	90%	(0.6430, 1.1372)	(115.1415, 274.2115)
	95%	(0.6157, 1.2094)	(109.0838, 303.6409)
BCIa	90%	(0.5564, 1.0614)	(76.7201, 245.7937)
	95%	(0.5080, 1.1098)	(60.5251, 261.9887)
HPD1	90%	(0.6526, 1.0496)	(112.6480, 237.3729)
	95%	(0.6256, 1.1012)	(105.1294, 256.9416)

Table 6: The Bayesian estimates of survival probability of a transformer that will survive till the time point c+k, provided the transform censored at time c.

c	c+10	c+20	c+30	c+40	c+50
35	0.9384(0.91,0.96)	0.8799(0.83,0.92)	0.8243(0.76,0.89)	0.7714(0.68,0.85)	0.7212(0.61,0.82)
40	0.9380(0.91,0.96)	0.8791(0.83,0.92)	0.8231(0.75,0.89)	0.7698(0.68,0.88)	0.7191(0.60,0.82)
44	0.9376(0.91,0.96)	0.8784(0.83,0.92)	0.8219(0.75,0.89)	0.7683(0.67,0.86)	0.7174(0.60,0.83)
48	0.9372(0.91,0.96)	0.8775(0.82,0.93)	0.8207(0.74,0.89)	0.7668(0.66,0.86)	0.7155(0.59,0.83)
57	0.9362(0.91,0.96)	0.8755(0.82,0.93)	0.8178(0.73,0.89)	0.7631(0.65,0.86)	0.7112(0.57,0.83)

Hong et al. [13] have fitted the Weibull distribution to their lifetime data by a graphical method and Balakrishnan and Mitra [14] have used a lognormal distribution to model these lifetime data. Note that GHN distribution discussed in the current manuscript is a special case of GG distribution with pdf $f(x) = \frac{\eta}{\beta\Gamma(r)}(x/\beta)^{r\eta-1}\exp\{-(x/\beta)^{\eta}\}$ when $r = 1/2, \eta = 2\alpha, \beta = 2^{1/(2\alpha)}\theta$. We used the likelihood-ratio test to choose the best model between two nested models GG and GHN distributions. To obtain the MLEs of the unknown parameters of GG distribution we follow the estimation procedures in [27, 28]. The MLEs and log-likelihood value by fitting the GG distribution are $(\hat{r}, \hat{\eta}, \hat{\beta}) = (10.07, 0.19, 0.0010)$ and -233.23, respectively. The log-likelihood value of GHN distribution based on the MLEs given in Table 5 is -234.55. The likelihood-ratio test yields a p-value of 0.1 by a Chi-squared distribution with one degree of freedom. Hence, for any usual significance level, this analysis confirms that the extension from the GHN distribution to the GG distribution is not statistically significant for modeling the given data set. So the GHN distribution is an appropriate model for the electric power-transformers data.

Table 5 reports point and interval estimates of α and θ using methods discussed in the previous sections. The approximate and bootstrap confidence intervals as well as the corresponding HPD credible intervals are computed at levels 90% and 95%. Since the true value of α is unknown, the Bayesian estimates are obtained only under Prior 1. Note that the determination of the hyper-parameters in prior distributions (6)-(8) is a separate work that one can consider for a real dataset. Table 5 depicts HPD 1 is shorter than the other confidence intervals, for both cases α and θ.

For maintenance purposes, the prediction of the remaining lifetime of the censored transformers is an important issue. We obtain the probability of survival at various times $c + k$ for $k > 0$, given that the unit censored at the time point c. Table 6 presents the Bayesian estimates of the conditional survival probabilities and the associated 95% HPD credible intervals under Prior 1 for $k = 10, 20, 30$ and 40. It is observed that for a fixed c, the conditional survival probabilities decrease with increasing k and also the width of HPD credible intervals increases. For a fixed k, older transformers are less likely to survive than younger ones, however, their probabilities are close. Moreover, the width of HPD credible intervals increases with c. These results are depicted in Figure 3 for $c = 35$ and 48.

We also compute the Bayesian estimates and the associated 95% HPD credible intervals of the expected number of transformers failing in future fixed intervals as discussed in Subsection 4.2. The results are presented in Table 7.

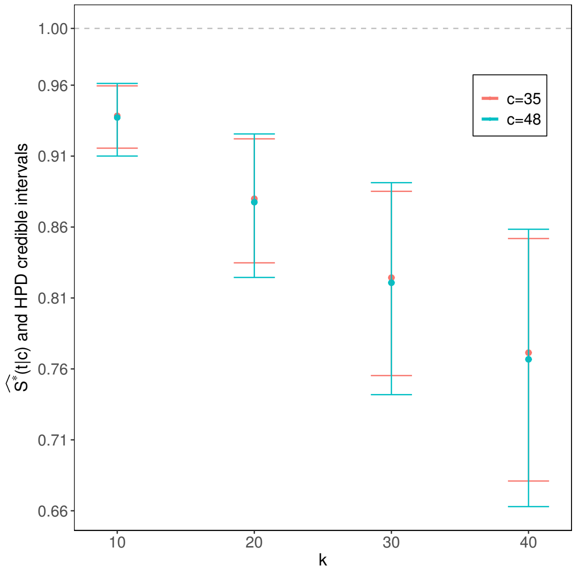

Figure 3: The Bayesian estimates of the conditional survival function and the associated 95% HPD credible intervals for $c = 35, 48$ and $k = 10, 20, 30, 40$.

Table 7: The Bayesian estimates and the associated 95% HPD credible intervals of the expected number of transformers failing in future fixed interval $[L, R]$.

$[L, R]$	[70,80]	[80-90]	[90,100]
$\widehat{\mathcal{E}}_B$	13	12	11
HPD1(95%)	(8, 18)	(7, 17)	(7, 16)

7. Concluding remarks

In this study, we have considered both frequentist and Bayesian inference of the unknown parameters of GHN distribution based on LTRC data. In this paper in addition to the classical approaches, we conducted the analysis based on the hierarchical Bayesian approach that has some advantages to classical Bayes. The Bayesian estimates and HPD credible intervals of the unknown parameters are obtained using Gibbs sampling procedure. We have also discussed some other Bayesian scenarios facing LTRC data, namely the prediction for the remaining lifetime and the Bayesian estimate of the cumulative number of failures during a specific interval. We have then conducted a simulation study to assess the performance of all the proposed methods to estimate the unknown parameters α and θ and a real dataset analysis has been presented to illustrate all the methods of inference developed in this paper. The simulation results demonstrate that the Bayesian estimates based on Prior 2 perform better than other estimates in terms of bias and MSE. Compared with confidence intervals, it is observed that the confidence intervals obtained using the parametric bias-corrected bootstrap method and the HPD credible intervals obtained under Prior 2 are quite satisfactory. Although in this paper, we have considered lifetime data exceed a threshold, the same approach can be extended to random left truncation and right censoring and we plan to investigate this problem as our future work.

References

[1] K. Cooray, M.M.A. Ananda. A generalization of the half-normal distribution with applications to lifetime data, Comm. Statist. Theory Methods 37 (2008) 1323–1337.
[2] K. Ahmadi, F. Yousefzadeh. Estimation for the parameters of generalized half-normal distribution based on progressive type-I interval censoring, Comm. Statist. Simulation Comput. 44 (2015) 2671–2695.

[3] K. Ahmadi, M. Rezaei, F. Yousefzadeh. Estimation for the generalized half-normal distribution based on progressive type-II censoring, J. Stat. Comput. Simul. 85 (2015) 1128–1150.
[4] A.M. Abd El-Raheem. Inference and optimal design of multiple constant-stress testing for generalized half-normal distribution under type-II progressive censoring, J. Stat. Comput. Simul. 89 (2019) 3075–3104.
[5] K. Ahmadi, S. Ghafouri. Reliability estimation in a multicomponent stress-strength model under generalized half-normal distribution based on progressive type-II censoring, J. Stat. Comput. Simul. 89 (2019) 2505–2548.
[6] F. Sultana, Y.M. Tripathi. Estimation and prediction for the generalized half normal distribution under hybrid censoring, J. Test Eva., 48 (2020) 1071–1094.
[7] L. Wang. Estimation for the parameters of the generalized half-normal distribution, J. Stat. Comput. Simul. 88 (2018) 629–645.
[8] L. Wang, Y. Shi. Estimation for constant-stress accelerated life test from generalized half-normal distribution, J. Syst. Eng. Elect. 28 (2017) 810–816.
[9] F.Z. Dogru, Y.M. Bulut, O. Arslan. Optimal B-robust estimators for the parameters of the generalized half-normal distribution, REVSTAT 15 (2017) 455–471.
[10] J. Mazucheli, S. Dey. Bias-corrected maximum likelihood estimation of the parameters of the generalized half-normal distribution, J. Stat. Comput. Simul. 88 (2018) 1027–1038.
[11] A.M. Abd El-Raheem. Estimation and optimal plans of multiple ramp-stress accelerated life tests for the generalized half-normal distribution, J. Egyp. Math. Soc. 26(2018)347–364.
[12] A.M. Abd El-Raheem. Optimal design of multiple accelerated life testing for generalized half-normal distribution under type-I censoring, J. Comput. Appl. Math. 368 (2020) 112539, 19 pp.
[13] Y. Hong, W.Q. Meeker, J.D. McCalley. Prediction of remaining life of power transformers based on left truncated and right censored lifetime data, Ann. Appl. Stat. 3 (2009) 857–879.
[14] N. Balakrishnan, D. Mitra. Likelihood inference for lognormal data with left truncation and right censoring with an illustration, J. Statist. Plann. Inference 141 (2011) 3536–3553.
[15] N. Balakrishnan, D. Mitra. Left truncated and right censored Weibull data and likelihood inference with an illustration, Comput. Statist. Data Anal. 56 (2012) 4011–4025.
[16] N. Balakrishnan, D. Mitra. Likelihood inference based on left truncated and right censored data from a Gamma distribution, IEEE Trans. Reliab. 62 (2013) 679–688.
[17] N. Balakrishnan, D. Mitra. Em-based likelihood inference for some lifetime distributions based on left truncated and right censored data and associated model discrimination, South African Statist. J. 48 (2014) 125–171.
[18] D. Kundu, D. Mitra. Bayesian inference of Weibull distribution based on left truncated and right censored data, Comput. Stat. Data Anal. 99 (2016) 38–50.
[19] D. Kundu, D. Mitra, A. Ganguly. Analysis of left truncated and right censored competing risks data, Comput. Stat. Data Anal. 108 (2017) 12–26.
[20] A. Dempster, N. Laird, D. Rubin. Maximum likelihood from incomplete data via the EM algorithm, J. Roy. Statist. Soc. Ser. B 39 (1997) 1–38.
[21] G. McLachlan, T. Krishnan. The EM Algorithm and Extensions, Wiley, New York, 2008.
[22] T.A. Louis. Finding the observed information matrix when using the EM algorithm, J. Roy. Statist. Soc. Ser. B 44 (1982) 226–233.
[23] B. Efron. The Jackknife, the Bootstrap, and Other Resampling Plans. Society for Industrial and Applied Mathematics, Philadelphia(PA), 1982.
[24] F.M. Dekking, C. Kraaikamp, H.P. Lopuhaa, L.E. Meester. A Modern Introduction to Probability and Statistics: Understanding why and how. Springer, London, 2005.
[25] A. Kottas. Nonparametric Bayesian survival analysis using mixtures of Weibull distributions, J. Statist. Plann. Inference, 136 (2006) 578–596.
[26] T. Emura, S. Shiu. Estimation and model selection for left-truncated and right-censored lifetime data with application to electric power transformers analysis, Comm. Statist. Simulation Comput. 45 (2016) 171–3189.
[27] O. Gomès, C. Combes, A. Dussauchoy. Parameter estimation of the generalized gamma distribution, Math. Comput. Simulation 79 (2008) 955–963.
[28] X. Shang, H.K.T. Ng. On parameter estimation for the generalized gamma distribution based on left-truncated and right-censored data, Comput. Math. Methods 3 (2021) Paper No. e1091, 22 pp.

Appendix I

Assuming $t_i^{(h)} = (c_i/\theta_{(h)})^{\alpha_{(h)}}$, from the Eqs. (10)-(12) of Ahmadi et al. [3], we have

$$\tilde{A}_i = \frac{1}{2\sqrt{\pi}\Phi\left(-t_i^{(h)}\right)}\Bigg\{\left(\ln(\theta_{(h)}) + \frac{\ln(2)}{2\alpha_{(h)}}\right)\Gamma\left[\frac{1}{2}, \frac{1}{2}\left(t_i^{(h)}\right)^2\right]$$
$$- \frac{1}{2\alpha_{(h)}}\Big[\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,2^{j+\frac{1}{2}}}\left(t_i^{(h)}\right)^{2j+1}\Big(\frac{2\ln\left(t_i^{(h)}\right) - \ln(2)}{j+\frac{1}{2}} - \frac{1}{(j+\frac{1}{2})^2}\Big) - \Gamma\Big(\frac{1}{2}\Big)\psi\Big(\frac{1}{2}\Big)\Big]\Bigg\},$$
$$\tilde{B}_i = \frac{2^{\alpha/\alpha_{(h)}-1}\left(\theta_{(h)}\right)^{2\alpha}}{\sqrt{\pi}\Phi\left(-t_i^{(h)}\right)}\Gamma\left[\frac{\alpha}{\alpha_{(h)}} + \frac{1}{2}, \frac{1}{2}\left(t_i^{(h)}\right)^2\right],$$

$$\tilde{C}_i = \frac{2^{\alpha/\alpha_{(h)}-1}\left(\theta_{(h)}\right)^{2\alpha}}{\sqrt{\pi}\Phi\left(-t_i^{(h)}\right)}\Bigg\{\left(\ln\left(\theta_{(h)}\right) + \frac{\ln(2)}{2\alpha_{(h)}}\right)\Gamma\left[\frac{\alpha}{\alpha_{(h)}} + \frac{1}{2}, \frac{1}{2}\left(t_i^{(h)}\right)^2\right] + \frac{1}{2\alpha_{(h)}}\Bigg[\Gamma\left(\frac{\alpha}{\alpha_{(h)}} + \frac{1}{2}\right)\psi\left(\frac{\alpha}{\alpha_{(h)}} + \frac{1}{2}\right)$$
$$- \sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,2^{j+\alpha/\alpha_{(h)}+\frac{1}{2}}}\left(t_i^{(h)}\right)^{2j+2\alpha/\alpha_{(h)}+1}\left(\frac{2\ln(t_i^{(h)}) - \ln(2)}{j+\alpha/\alpha_{(h)}+\frac{1}{2}} - \frac{1}{\left(j+\alpha/\alpha_{(h)}+\frac{1}{2}\right)^2}\right)\Bigg]\Bigg\},$$

where $\Gamma(.)$ and $\psi(.)$ are gamma and digamma functions, respectively. Also, $\Gamma[a,b]$ is the upper incomplete gamma function defined as $\Gamma[a,b] = \int_b^{\infty} t^{a-1}e^{-t}\,dt$.

Appendix II

Let (l,k)-th element of $I_X(\lambda)$ be $a_{lk}(\alpha,\theta)$, for $l,k = 1,2$, then one has following expressions as

$$a_{11}(\alpha,\theta) = \frac{n}{\alpha^2} + 2\sum_{i\in S_1}E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln^2\left(\frac{X_i}{\theta}\right)\right) + 2\sum_{i\in S_2}E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln^2\left(\frac{X_i}{\theta}\right)\Big|X_i > \tau_i\right)$$
$$- \sum_{i=1}^{n}(1-\nu_i)v_ih(v_i)(\ln v_i)^2\left[1 - (v_i - h(v_i))\,v_i\right],$$
$$a_{22}(\alpha,\theta) = -\frac{\alpha}{\theta^2}\Bigg\{n - (2\alpha+1)\left[\sum_{i\in S_1}E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\right) + \sum_{i\in S_2}E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\Big|X_i > \tau_i\right)\right]$$
$$- \sum_{i=1}^{n}(1-\nu_i)v_ih(v_i)\left[(v_i - h(v_i))\alpha v_i - \alpha - 1\right]\Bigg\},$$
$$a_{12}(\alpha,\theta) = a_{21}(\alpha,\theta) = \frac{1}{\theta}\Bigg\{n - \sum_{i\in S_1}\left[E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\right) + 2\alpha E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln\left(\frac{X_i}{\theta}\right)\right)\right]$$
$$- \sum_{i\in S_2}\left[E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\Big|X_i > \tau_i\right) + 2\alpha E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln\left(\frac{X_i}{\theta}\right)\Big|X_i > \tau_i\right)\right]$$
$$+ \sum_{i=1}^{n}(1-\nu_i)v_ih(v_i)\Big[1 + \left[1 - (v_i - h(v_i))v_i\right]\ln v_i\Big]\Bigg\}.$$

For $i \in S_1$, using Lemma 2.3 from Wang [7], the following expectations can be obtained:

$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\right) = 1,$$
$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln\left(\frac{X_i}{\theta}\right)\right) = \frac{1}{2\alpha}\left(\psi\left(\frac{3}{2}\right)+\ln 2\right),$$
$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln^2\left(\frac{X_i}{\theta}\right),\right) = \frac{1}{4\alpha^2}\left[\left(\psi\left(\frac{3}{2}\right)+\ln 2\right)^2 + \frac{\pi^2}{2} - 4\right].$$

Moreover, for $i \in S_2$, it is easy to see that

$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\Big| X_i > \tau_i\right) = \frac{1}{\sqrt{\pi}\Phi(-v_i)}\Gamma\left[\frac{3}{2}, \frac{v_i^2}{2}\right], \tag{17}$$

$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln\left(\frac{X_i}{\theta}\right)\Big| X_i > \tau_i\right) = \frac{1}{\sqrt{\pi}\Phi(-v_i)}\left\{\frac{\ln 2}{2\alpha}\Gamma\left[\frac{3}{2}, \frac{v_i^2}{2}\right] + \frac{1}{2\alpha}\left[\Gamma\left(\frac{3}{2}\right)\psi\left(\frac{3}{2}\right) - \sum_{j=0}^{\infty}\frac{(-1)^j v_i^{2j+3}}{j!\,2^{j+3/2}}\left(\frac{2\ln v_i - \ln 2}{j+\frac{3}{2}} - \frac{1}{\left(j+\frac{3}{2}\right)^2}\right)\right]\right\}, \tag{18}$$

$$E\left(\left(\frac{X_i}{\theta}\right)^{2\alpha}\ln^2\left(\frac{X_i}{\theta}\right)\Big| X_i > \tau_i\right) = \frac{1}{4\sqrt{\pi}\alpha^2\Phi(-v_i)}\left\{(\ln 2)^2\Gamma\left[\frac{3}{2}, \frac{v_i^2}{2}\right] + 2(\ln 2)\Gamma\left(\frac{3}{2}\right)\psi\left(\frac{3}{2}\right) - \sum_{j=0}^{\infty}\frac{(-1)^j v_i^{2j+3}}{j!\,2^{j+3/2}}\left[\frac{(2\ln v_i)^2 - (\ln 2)^2}{j+\frac{3}{2}} - \frac{4\ln v_i}{\left(j+\frac{3}{2}\right)^2} + \frac{2}{\left(j+\frac{3}{2}\right)^3}\right] + \Gamma\left(\frac{3}{2}\right)\left[\left(\psi\left(\frac{3}{2}\right)\right)^2 + \frac{\pi^2}{2} - 4\right]\right\}. \tag{19}$$

Now, we shall find the elements of the missing information matrix. Denote the 2×2 matrix $I_Z(\lambda)$ as follows

$$I_Z(\lambda) = \begin{bmatrix} b_{11}(\alpha,\theta) & b_{12}(\alpha,\theta) \\ b_{21}(\alpha,\theta) & b_{22}(\alpha,\theta) \end{bmatrix},$$

then one has

$$b_{11}(\alpha,\theta) = \frac{m}{\alpha^2} + \sum_{i=1}^{n}(1-\delta_i)\left\{2E\left(\left(\frac{Z_i}{\theta}\right)^{2\alpha}\ln^2\left(\frac{Z_i}{\theta}\right)\Big| Z_i > c_i\right) + t_i h(t_i)(\ln t_i)^2\left[1-(t_i - h(t_i))\,t_i\right]\right\}, \tag{20}$$

$$b_{22}(\alpha,\theta) = -\frac{\alpha}{\theta^2}\left\{m - \sum_{i=1}^{n}(1-\delta_i)\left[(1+2\alpha)E\left(\left(\frac{Z_i}{\theta}\right)^{2\alpha}\Big| Z_i > c_i\right) - t_i h(t_i)\left[(t_i - h(t_i))\alpha t_i - \alpha - 1\right]\right]\right\}, \tag{21}$$

$$b_{12}(\alpha,\theta) = b_{21}(\alpha,\theta) = \frac{1}{\theta}\left\{m - \sum_{i=1}^{n}(1-\delta_i)\left[2\alpha E\left(\left(\frac{Z_i}{\theta}\right)^{2\alpha}\ln\left(\frac{Z_i}{\theta}\right)\Big| Z_i > c_i\right) + E\left(\left(\frac{Z_i}{\theta}\right)^{2\alpha}\Big| Z_i > c_i\right) + t_i h(t_i)\left[1 + \left[1-(t_i - h(t_i))t_i\right]\ln t_i\right]\right]\right\}, \tag{22}$$

where $t_i = (c_i/\theta)^\alpha$. In order to calculate the expectations contained in Eqs.(20)-(22), we can utilize from Eqs.(17)-(19). They can be calculated by replacing notation v_i with t_i, for $i \in S_{c_2}$.