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# Sparse Recovery for Compressive Sensing via Weighted $L_{p-q}$ Model

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**Abstract.** In this paper, we study weighted $L_{p-q}$ minimization model which comprises non-smooth, non-convex and non-Lipschitz quasi-norm $L_p(0 < p \leq 1)$ and $L_q(1 < q \leq 2)$ for recovering sparse signals. Based on the restricted isometry property (RIP) condition, we obtain exact sparse signal recovery result. We also obtain the theoretical bound for the weighted $L_{p-q}$ minimization model when measurements are depraved by the noises.

## 1. Introduction

Donoho [2] and Candès, Romberg and Tao [7] have initiated the area of compressed sensing (CS) in 2006. Since then an ample amount of work have been published on CS both in theoretical and applied fields. The aim of the compressed sensing is to recover a sparse signal $y \in \mathbb{R}^{\mathbb{M}}$ from very few non-adaptive linear measurements

$$z = \mathbf{A}y + \xi, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{\mathbb{N}\times\mathbb{M}}(N << M)$, $z \in \mathbb{R}^{\mathbb{N}}$ and $\xi \in \mathbb{R}^{\mathbb{N}}$ are the measurements matrix and additive noise respectively.
If the measurement matrix $\mathbf{A}$ satisfies some kinds of incoherence conditions such as mutual coherence ([9, 12]), restricted isometry property (RIP) ([5, 6]) etc., then stable and robust recovery can be obtained for sure by using constrained $L_1$- minimization ([7, 13]) given by

$$\min_{y\in\mathbb{R}^{\mathbb{M}}} \|y\|_1 \quad s.\,t. \quad \mathbf{A}y = z. \tag{2}$$

In this context, $L_1$ minimization problem works as a convex relaxation of $L_0$-minimization problem which is a NP-hard problem [1] and counts the nonzero entries.
In the meantime, Gribonaval and Nielsen [12] have proposed following $L_p(0 < p < 1)$- minimization (a non-convex recovery algorithm) to enhance sparsity

$$\min_{y\in\mathbb{R}^{\mathbb{M}}} \|y\|_p \quad s.\,t. \quad \mathbf{A}y = z. \tag{3}$$

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In practice, non-convex $L_p$ minimization is more challenging as compared to convex $L_1$ minimization. However, $L_p$ minimization provides reconstruction of the sparse signal from fewer number of measurements compared to $L_1$ minimization.
Recently, Esser et al. [8], for the first time, proposed and solved $L_{1-2}$ minimization model using the difference of convex functions algorithm. Yin et al. [10] presented a constrained $L_{1-2}$ minimization model given by

$$\min_{y\in\mathbb{R}^{\mathbb{M}}} \|y\|_1 - \|y\|_2 \quad s.\,t. \quad \mathbf{A}y = z \tag{4}$$

which included two kinds of norms $L_1$ and $L_2$ norms respectively and obtained sparse recovery result using RIP condition. Yin et al. [10] also obtained the theoretical bound using RIP condition when measurements are depraved by the noises for the minimization model

$$\min_{y\in\mathbb{R}^{\mathbb{M}}} \|y\|_1 - \|y\|_2 \quad s.\,t. \quad \|\mathbf{A}y - z\|_2 \le \eta. \tag{5}$$

The models proposed by [8] and [10], have been found to be more effective than $L_0, L_1$ and $L_p$ norm model in some sense with their respective approaches. Wang and Zhang [3] proposed $L_{1-p}(1 < p \le 2)$ minimization model for recovering sparse signal which was solved using projected neural network algorithms. Zhao et al. [14] represented a more general non-smooth, non-convex and non-Lipschitz sparse signal recovery model $L_{p-q}(0 < p \le 1,\ 1 < q \le 2)$ which is an extension of the models represented by [8] and Wang and Zhang [3]. Zhou and Yu [15] represented weighted $L_{p-1}(0 < p \le 1)$ minimization model for sparse signal recovery which is an extension of $L_p$ minimization method.
In the present paper, we study weighted $L_{p-q}$ minimization model for sparse signal recovery using RIP condition. We also establish a theoretical bound for the weighted $L_{p-q}$ minimization model using RIP condition. Some important corollaries are also obtain for the case $\alpha = 1$ for recovering sparse signal.
Remaining part of this paper is organized as follows: In section 2, we give some notations and definitions related to the presented work. In section 3, we propose our minimization model and provide theoretical results on weighted $L_{p-q}$ model which play a cruical role in finding the sparse signal recovery. In section 4, we prove an exact sparse recovery result based on RIP condition and also establish a theoretical bound for the weighted $L_{p-q}$ minimization model when measurements are depraved by the noises. In section 5, some important corollaries are obtain for the case $\alpha = 1$ for sparse signal recovery.

## 2. Notations and Preliminaries

Some useful notations are as follows:
Consider the column vector $y = (y_1, y_2, \cdots, y_M)^T$ and $z = (z_1, z_2, \cdots, z_M)^T$.
$\langle y, z\rangle = \sum_{k=1}^{M} y_k z_k$ is the inner product of $y$ and $z$. $\|y\| = \left(\sum_{k=1}^{M} |y_k|^2\right)^{\frac{1}{2}}$ is the Euclidean norm.
$\|y\|_p = \left(\sum_{k=1}^{M} |y_k|^p\right)^{\frac{1}{p}}$, $(0 < p \le 1)$ is the $L_p$ quasi-norm.
$\|y\|_q = \left(\sum_{k=1}^{M} |y_k|^q\right)^{\frac{1}{q}}$, $(1 < q \le 2)$ is the $L_q$ norm.
$\nabla g(y)$ is the gradient of $g$ at $y$.
Now, we give a following definition which is needed in obtaining our recoevry results in section 4.

**Definition 2.1 ([4]).** ***Ristricted Isometry Property (RIP)***
*For all s-sparse signals $y \in \mathbb{R}^{\mathbb{M}}$, the matrix $\mathbf{A} \in \mathbb{R}^{\mathbb{N}\times\mathbb{M}}$ is said to satisfy s-RIP with constant $\delta_s \in (0, 1)$ if the following inequality is true:*

$$(1 - \delta_s)\|y\|_2^2 \le \|\mathbf{A}y\|_2^2 \le (1 + \delta_s)\|y\|_2^2. \tag{6}$$

## 3. Formulation of Minimization Model

We consider the following non-smooth, non-convex and non-Lipschitz weighted $L_{p-q}$ minimization problem for sparse signal recovery:

$$g(y) = \min_{y\in\mathbb{R}^{\mathbb{M}}} \|y\|_p - \alpha\|y\|_q \quad s.\,t. \quad \mathbf{A}y = z, \tag{7}$$

where $0 < p \leq 1,\ 1 < q \leq 2$ and $0 \leq \alpha \leq 1$.
First, we prove a following lemma which is crucial in proving our Theorem 4.1.

**Lemma 3.1.** ($i$) *Let* $y \in \mathbb{R}^{\mathbb{M}},\ 0 \leq \alpha \leq 1,\ 0 < p \leq 1$ *and* $1 < q \leq 2$, *then*

$$(M^{\frac{1}{p}} - \alpha M^{\frac{1}{q}})\Big(\min_{i\in M}|y_i|\Big) \leq \|y\|_p - \alpha\|y\|_q \leq (M^{\frac{1}{p}-\frac{1}{q}} - \alpha)\|y\|_q. \tag{8}$$

*(ii) When* $\mathcal{K} = supp(\hat{y}) \subseteq M$ *and* $\|y\|_0 = s$, *then*

$$(s^{\frac{1}{p}} - \alpha s^{\frac{1}{q}})\Big(\min_{i\in M}|y_i|\Big) \leq \|y\|_p - \alpha\|y\|_q \leq (s^{\frac{1}{p}-\frac{1}{q}} - \alpha)\|y\|_q. \tag{9}$$

*Proof.* ($i$) We can easily find the right hand side of inequality (8) using Hölder's inequality and the norm inequality $\|y\|_p \leq M^{\frac{1}{p}-\frac{1}{q}}\|y\|_q$ for any $y \in \mathbb{R}^{\mathbb{M}}$. Thus, using the above fact, we get

$$\|y\|_p - \alpha\|y\|_q \leq (M^{\frac{1}{p}-\frac{1}{q}} - \alpha)\|y\|_q. \tag{10}$$

Now, we find left side of inequality (8). For $M = 1$, (8) is true. Now, for $M > s > 1$, we suppose $y_i \geq 0, i = 1, 2, \cdots M$, then

$$\nabla_{y_i} g(y_i) = |y_i|^{(p-1)}\Big(\sum_{k=1}^{M}|y_k|^p\Big)^{\frac{1}{p}-1} - \alpha\,|y_i|^{(q-1)}\Big(\sum_{k=1}^{M}|y_k|^q\Big)^{\frac{1}{q}-1} > 0,$$

where $|y_i|^{(p-1)}\Big(\sum_{k=1}^{M}|y_k|^p\Big)^{\frac{1}{p}-1} > 1$ and $|y_i|^{(q-1)}\Big(\sum_{k=1}^{M}|y_k|^q\Big)^{\frac{1}{q}-1} < 1$.
Hence, $g(y)$ is a monotonic increasing function with respect to $y_i$. Consequently,

$$g(y) \geq g\Big(\min_{i\in M} y_i, \cdots, \min_{i\in M} y_i\Big).$$

Thus,

$$\|y\|_p - \alpha\|y\|_q \geq (M^{\frac{1}{p}} - \alpha M^{\frac{1}{q}})\min_{i\in M}|y_i|.$$

($ii$) This part of the lemma can be proved by putting $y = y_s$ in the proof of part ($i$) of this lemma.
□

## 4. Recovery Results

**Theorem 4.1.** *Suppose the weighted* $L_{p-q}$ *has more than one s-sparsity solutions with* $s \leq \frac{1}{4}M$, *then it has a unique solution y with sparsity s if the vector y satisfying*

$$a(s) = \left(\frac{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}}{(s)^{\frac{1}{p}-\frac{1}{2}} + \alpha(s)^{\frac{1}{q}-\frac{1}{2}}}\right)^2 > 1 \tag{11}$$

*and a matrix* $\mathbf{A}$ *satisfies the condition*

$$\delta_{3s} + a(s)\delta_{4s} < a(s) - 1. \tag{12}$$

*Proof.* Let $\mathcal{K} = supp(\hat{y})$, then $|\mathcal{K}| = s$. Suppose $y$ and $\hat{y}$ are the two solutions of (7) with sparsity $s$. Now, we decompose $y$ as $y = \hat{y} + \mathbf{v}$ and $\mathbf{v} = \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^c}$ and thus we need to show that $\mathbf{v} = \theta$.
Now, we can write (7) as

$$\begin{aligned} \|y\|_p - \alpha\|y\|_q &= \|\hat{y} + \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\hat{y} + \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^c}\|_q \\ &\leq \|\hat{y}\|_p - \alpha\|\hat{y}\|_q. \end{aligned} \tag{13}$$

On the other hand,

$$\begin{aligned} &\|\hat{y} + \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\hat{y} + \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^c}\|_q \\ &= \|\hat{y} + \mathbf{v}_{\mathcal{K}}\|_p + \|\mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\hat{y} + \mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}}\|_q \\ &\geq \|\hat{y}\|_p - \|\mathbf{v}_{\mathcal{K}}\|_p + \|\mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\hat{y}\|_q - \alpha\|\mathbf{v}_{\mathcal{K}}\|_q - \alpha\|\mathbf{v}_{\mathcal{K}^c}\|_q. \end{aligned} \tag{14}$$

So, $\mathbf{v}$ must obey the following inequality

$$\|\mathbf{v}_{\mathcal{K}}\|_p + \alpha\|\mathbf{v}_{\mathcal{K}}\|_q \geq \|\mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\mathbf{v}_{\mathcal{K}^c}\|_q. \tag{15}$$

Arrange the elements in $\mathbf{v}_{\mathcal{K}^c}$ such as $\mathcal{K}^c = \mathcal{K}_1 \cup \mathcal{K}_2 \cdots$ with their absolute values, and divide $\mathcal{K}^c$ into $l$ subsets $\mathcal{K}_i (1 \leq i \leq l)$ , where each subsets contains $3s$ largest elements except $\mathcal{K}_l$ with less indices. Similarly, $\mathbf{v}_{\mathcal{K}_1}$ contains the $3s$ largest elements in $\mathbf{v}_{\mathcal{K}^c}$. According to the RIP condition for matrix $\mathbf{A}$ and the notation $\mathcal{K}_0 = \mathcal{K} \cup \mathcal{K}_1$, we have

$$\begin{aligned} 0 = \|\mathbf{A}\mathbf{v}\|_2 &= \left\|\mathbf{A}\mathbf{v}_{\mathcal{K}_0} + \sum_{i=2}^{l} \mathbf{A}\mathbf{v}_{\mathcal{K}_i}\right\|_2 \\ &\geq \|\mathbf{A}\mathbf{v}_{\mathcal{K}_0}\|_2 - \left\|\sum_{i=2}^{l} \mathbf{A}\mathbf{v}_{\mathcal{K}_i}\right\|_2 \\ &\geq \sqrt{1-\delta_{4s}}\|\mathbf{v}_{\mathcal{K}_0}\|_2 - \sqrt{1+\delta_{3s}}\sum_{i=2}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_2. \end{aligned} \tag{16}$$

Now, we will set an upper bound for $\sum_{i=2}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_2$. By using the division rule of $\mathcal{K}^c$, we can get $|\mathbf{v}_k| \leq |\mathbf{v}_r|$ for $k \in \mathcal{K}_i$ and $r \in \mathcal{K}_{i-1}, i \geq 2$. From Lemma (3.1)(i) and using condition $\|\mathbf{v}_{\mathcal{K}_{i-1}}\|_0 \leq 3s$, we have

$$|\mathbf{v}_k| \leq \min_{r\in\mathcal{K}_{i-1}} |\mathbf{v}_r| \leq \frac{\|\mathbf{v}_{\mathcal{K}_{i-1}}\|_p - \alpha\|\mathbf{v}_{\mathcal{K}_{i-1}}\|_q}{(3s)^{\frac{1}{p}} - \alpha(3s)^{\frac{1}{q}}} \tag{17}$$

where $\mathbf{v}_k$ and $\mathbf{v}_r$ are the $k^{th}$ and $r^{th}$ elements respectively in $\mathcal{K}^c$ and accordingly

$$\begin{aligned} \sum_{i=2}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_2 &\leq \sqrt{3s}\sum_{i=2}^{l}\max_{k\in\mathcal{K}_i}|\mathbf{v}_{\mathcal{K}}| \\ &\leq \frac{\sum_{i=1}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_p - \alpha\sum_{i=1}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_q}{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}}. \end{aligned} \tag{18}$$

Because $\mathcal{K}^c = \mathcal{K}_1 \cup \mathcal{K}_2 \cdots$, we note that

$$\sum_{i=1}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_p \leq \|\mathbf{v}_{\mathcal{K}^c}\|_p \quad \text{and} \quad \|\mathbf{v}_{\mathcal{K}^c}\|_q \leq \sum_{i=1}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_q. \tag{19}$$

Now, using (19) in (18), we get

$$\sum_{i=2}^{l} \|\mathbf{v}_{\mathcal{K}_i}\|_2 \leq \frac{\|\mathbf{v}_{\mathcal{K}^c}\|_p - \alpha\|\mathbf{v}_{\mathcal{K}^c}\|_q}{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}}. \tag{20}$$

Now, using (15), we have

$$\begin{aligned}\sum_{i=2}^{l} \|\mathbf{v}_{\mathcal{K}_i}\|_2 &\leq \frac{\|\mathbf{v}_{\mathcal{K}}\|_p + \alpha\|\mathbf{v}_{\mathcal{K}}\|_q}{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}} \\ &\leq \frac{(s)^{\frac{1}{p}-\frac{1}{2}} + \alpha(s)^{\frac{1}{q}-\frac{1}{2}}}{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}} \|\mathbf{v}_{\mathcal{K}}\|_2. \end{aligned} \tag{21}$$

Using (21) in (16), we get

$$\begin{aligned}\|\mathbf{A}\mathbf{v}\|_2 &\geq \sqrt{1-\delta_{4s}}\|\mathbf{v}_{\mathcal{K}_0}\|_2 - \sqrt{1+\delta_{3s}}\Big(\frac{(s)^{\frac{1}{p}-\frac{1}{2}} + \alpha(s)^{\frac{1}{q}-\frac{1}{2}}}{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}}\Big)\|\mathbf{v}_{\mathcal{K}}\|_2 \\ &\geq \sqrt{1-\delta_{4s}}\|\mathbf{v}_{\mathcal{K}_0}\|_2 - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\|\mathbf{v}_{\mathcal{K}}\|_2, \end{aligned} \tag{22}$$

where $a(s) = \left(\frac{(3s)^{\frac{1}{p}-\frac{1}{2}} - \alpha(3s)^{\frac{1}{q}-\frac{1}{2}}}{(s)^{\frac{1}{p}-\frac{1}{2}} + \alpha(s)^{\frac{1}{q}-\frac{1}{2}}}\right)^2$.
Using the inequality $\|\mathbf{v}_{\mathcal{K}}\|_2 \leq \|\mathbf{v}_{\mathcal{K}_0}\|_2$, (22) becomes

$$\begin{aligned}\|\mathbf{A}\mathbf{v}\|_2 &\geq \sqrt{1-\delta_{4s}}\|\mathbf{v}_{\mathcal{K}_0}\|_2 - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\|\mathbf{v}_{\mathcal{K}_0}\|_2 \\ &\geq \Big(\sqrt{1-\delta_{4s}} - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\Big)\|\mathbf{v}_{\mathcal{K}_0}\|_2 \end{aligned} \tag{23}$$

According to (12), we have

$$\Big(\sqrt{1-\delta_{4s}} - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\Big) > 0.$$

Hence, $\mathbf{v}_{\mathcal{K}_0} = \theta$ which implies $\mathbf{v}_{\mathcal{K}}$ and $\mathbf{v}_{\mathcal{K}_1}$ are zero vectors. Further, from the division rule of $\mathcal{K}^c$, we have $\mathbf{v}_{\mathcal{K}^c} = \theta$. This implies $\mathbf{v} = \theta$.
This completes the proof of the theorem. □

Now, we find the following stable recovery result of the weighted $L_{p-q}$ minimization model when the measurements are depraved by the noises.

**Theorem 4.2.** *Under the assumptions of Theorem (4.1) except that $z = \mathbf{A}y + \xi$ with $\|\xi\|_2 \leq \eta$ we have that the solution $y^{opt}$ to the variant of problem (7)*

$$\min_{y\in\mathbb{R}^{\mathrm{M}}} \|y\|_p - \alpha\|y\|_q \quad s.\,t. \quad \|\mathbf{A}y - z\|_2 \leq \eta \tag{24}$$

*obeys $\|y^{opt} - \hat{y}\| \leq C_s\eta$ for constant $C_s > 0$, where*

$$C_s = 2\Big[\frac{\sqrt{1+a(s)}}{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}\Big].$$

*Proof.* We have

$$\sum_{i=2}^{l} \|\mathbf{v}_{\mathcal{K}_i}\|_2 \leq \frac{\|\mathbf{v}_{\mathcal{K}}\|_2}{\sqrt{a(s)}} \tag{25}$$

and

$$\|A\mathbf{v}\|_2 \geq \left( \sqrt{1-\delta_{4s}} - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\right)\|\mathbf{v}_{\mathcal{K}_0}\|_2. \tag{26}$$

We also have

$$\|\mathbf{v}\|_2 = \sqrt{\|\mathbf{v}_{\mathcal{K}_0}\|_2^2 + \sum_{i=2}^{l}\|\mathbf{v}_{\mathcal{K}_i}\|_2^2}. \tag{27}$$

Using (25) in (27), we get

$$\begin{aligned} \|\mathbf{v}\|_2 &= \sqrt{\|\mathbf{v}_{\mathcal{K}_0}\|_2^2 + \frac{\|\mathbf{v}_{\mathcal{K}}\|_2^2}{a(s)}} \\ &\leq \sqrt{\left(1+\frac{1}{a(s)}\right)}\|\mathbf{v}_{\mathcal{K}_0}\|_2. \end{aligned} \tag{28}$$

Using (28) in (26), we get

$$\begin{aligned} \|\mathbf{A}\mathbf{v}\|_2 &\geq \frac{\left( \sqrt{1-\delta_{4s}} - \sqrt{1+\delta_{3s}}\frac{1}{\sqrt{a(s)}}\right)}{\sqrt{\left(1+\frac{1}{a(s)}\right)}}\|\mathbf{v}\|_2 \\ &= \left(\frac{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}{\sqrt{1+a(s)}}\right)\|\mathbf{v}\|_2. \end{aligned} \tag{29}$$

Since

$$\|\mathbf{A}y - z\|_2 = \|\xi\|_2 \leq \eta. \tag{30}$$

Now, using triangle inequality we can write

$$\begin{aligned} \|\mathbf{A}\mathbf{v}\|_2 &= \|(\mathbf{A}y - z) - (\mathbf{A}\hat{y} - z)\|_2 \\ &\leq \|\mathbf{A}y - z\|_2 + \|\mathbf{A}\hat{y} - z\|_2. \end{aligned} \tag{31}$$

Using (30), we get

$$\|\mathbf{A}\mathbf{v}\|_2 \leq 2\eta. \tag{32}$$

Now rewriting (29), we have

$$\|\mathbf{v}\|_2 \leq \left[\frac{\sqrt{1+a(s)}}{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}\right]\|A\mathbf{v}\|_2. \tag{33}$$

Using (32), we get

$$\begin{aligned} \|\mathbf{v}\|_2 &\leq \left[\frac{\sqrt{1+a(s)}}{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}\right]2\eta, \\ &\leq C_s\eta, \end{aligned} \tag{34}$$

where

$$C_s = 2\Big[\frac{\sqrt{1+a(s)}}{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}\Big].$$

□

**Remark 4.3.** *Since $a(s) > 1$, equation (11) implies*

$$s > \left(\frac{\alpha + 3^{\frac{1}{q}-\frac{1}{2}}}{3^{\frac{1}{p}-\frac{1}{2}-\alpha}}\right)^{\frac{pq}{p-q}}. \tag{35}$$

*If $\alpha = 1$ in (35), the sparsity $s$ becomes large as $q$ decreases for fixed $p$. In other words, we can say that the solution of the minimization problem (7) is sparser with large $s$ when the matrix $\mathbf{A}$ satisfies a RIP condition.*

## 5. Remark

**Corollary 5.1.** *If $\alpha = 1$ in the minimization model (7), model (7) has a unique solution $y$ with sparsity $s$ if the vector $y$ satisfying*

$$a(s) = \left(\frac{(3s)^{\frac{1}{p}-\frac{1}{2}} - (3s)^{\frac{1}{q}-\frac{1}{2}}}{(s)^{\frac{1}{p}-\frac{1}{2}} + (s)^{\frac{1}{q}-\frac{1}{2}}}\right)^2 > 1 \tag{36}$$

*and matrix $A$ satisfies the condition*

$$\delta_{3s} + a(s)\delta_{4s} < a(s) - 1. \tag{37}$$

**Corollary 5.2.** *Using the conditions (36), (37) and if $\alpha = 1$ in the minimization model (24), then the model (24) obeys $\|y^{opt} - \hat{y}\| \leq C_s\eta$ for constant $C_s > 0$ with $\|\xi\| \leq \eta$, where*

$$C_s = 2\Big[\frac{\sqrt{1+a(s)}}{\sqrt{a(s)(1-\delta_{4s})} - \sqrt{1+\delta_{3s}}}\Big].$$

## 6. Conclusions

In this paper, we study a non-smooth, non-convex and non-Lipschitz weighted difference $L_{p-q}$ minimization model. We establish exact sparse signal recovery using the RIP condition and also establish a theoretical bound for the weighted $L_{p-q}$ minimization model using the RIP condition when measurements are depraved by the noises. Our proposed non-smooth, non-convex and non-Lipschitz minimization model is more effective than $L_0$, $L_1$ and $L_p$ minimization model as the norm of this model is influenced by the models proposed by [8] and [10]. We also discuss the exact sparse recovery result and theoretical bound for the $L_{p-q}$ minimization model using RIP condition.

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