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EP Elements, Partial Isometries and Solutions of Some Equations

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Abstract. In this paper, we firstly present new charactrizations of EP elements, partial isometries. Next, we investigate the general solutions of some equations. Finally, we discuss the relation between the consistency of certain equations and SEP elements.

1. Introduction

Throughout this article, R will denote an associative ring with an identity and involution, i,e., a ring R with mapping $a \to a^*$ satisfying

$$(a+b)^* = a^* + b^*, (ab)^* = b^*a^*, (a^*)^* = a,$$

for all $a, b \in R$.

An element $a \in R$ is said to be group invertible [5, 9], if there is $a^\# \in R$ such that

$$aa^\# = a^\# a, a = aa^\# a, a^\# = a^\# aa^\#.$$

The element $a^\#$ is called group inverse of a and it is uniquely determined by above equations (see [9]). We denote the set of all group invertible elements in R by $R^\#$. Clearly, $a^\#$ coincides with the ordinary inverse a^{-1} of a, if a is invertible [12].

An element $a^\dagger$ is said to be the Moore-Penrose inverse (or MP-inverse) of a [3], if satisfying the following conditions:

$$(a^\dagger a)^* = a^\dagger a, (aa^\dagger)^* = aa^\dagger, a^\dagger aa^\dagger = a^\dagger, aa^\dagger a = a.$$

If $a^\dagger$ exists, then it is unique, see [1-4]. We write $R^\dagger$ for the set of all Moore-Penrose inverse of R.

An element $a \in R$ is called EP [6-8] if $a \in R^\# \cap R^\dagger$ and $a^\# = a^\dagger$. We denote the set of all EP elements of R by R^{EP}. An element $a \in R$ is said to be symmetric if satisfying $a^* = a$, and a is called normal if satisfying $aa^* = a^*a$. And is said to be 2-normal if $a^2a^* = a^*a^2$. An element a is said to be a partial isometry if $a = aa^*a$. We write R^{PI} for the set of all partial isometry elements of R. In [11], an element a is called a strongly EP elements if $a \in R^{EP}$ is partial isometry. We denote the set of all strongly EP elements of R by R^{SEP}. This article considers the characterizations of EP elements, from the perspective of the results of equations. Let

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$a \in R^{\#} \cap R^{\dagger}$ and $\chi_a = \left\{a, a^*, a^{\dagger}, a^{\#}, (a^{\#})^*, (a^{\dagger})^*\right\}$. We show that $a \in R^{PI}$ if and only if the equation $aa^*xa = xa$ has at least one solution in χ_a. We also prove that $a \in R^{EP}$ if and only if the general solution of the equation $a^*xa - a^{\#}ay = 0$ is given by $x = -(a^{\dagger})^*Pa^{\dagger} + u - aa^{\dagger}uaa^{\dagger}$, $y = -a^{\dagger}aPa^{\dagger}a + \gamma - a^{\dagger}a\gamma$. And $a \in R^{EP}$ if and only if the general solution of the equation $a^*xa - a^{\dagger}ay = 0$ is given by $x = -(a^{\dagger})^*Pa^{\dagger} + u - aa^{\dagger}uaa^{\dagger}$, $y = -a^{\dagger}aPaa^{\dagger} + \gamma - a^{\dagger}a\gamma$ for all $P, u, \gamma \in R$.

2. Some new charactrizations of related generalized inverses

In this section, we will give several charactrizations of an EP elements, partial isometry, strongly EP elements, normal and symmetric elements. We begin with the following lemma.

Lemma 2.1. *Let $a \in R^{\#} \cap R^{\dagger}$. Then* (1) $(a^{\dagger})^*a^2 \in R^{\dagger}$ *and* $[(a^{\dagger})^*a^2]^{\dagger} = a^{\dagger}a^{\#}a^*$.
(2) $(a^{\dagger})^*a^2 \in R^{\#}$ *and* $[(a^{\dagger})^*a^2]^{\#} = (a^{\#})^2a^*aa^{\#}$.

Proof. (1) *Since*

$$((a^{\dagger})^*a^2)(a^{\dagger}a^{\#}a^*) = (a^{\dagger})^*(a^2a^{\dagger}a^{\#})a^* = ((a^{\dagger})^*aa^{\#})a^* = (a^{\dagger})^*a^* = aa^{\dagger},$$

$$(a^{\dagger}a^{\#}a^*)((a^{\dagger})^*a^2) = a^{\dagger}a^{\#}a^{\dagger}a^3 = a^{\dagger}a,$$

$$((a^{\dagger})^*a^2)(a^{\dagger}a^{\#}a^*)((a^{\dagger})^*a^2) = aa^{\dagger}(a^{\dagger})^*a^2 = (a^{\dagger})^*a^2,$$

$$(a^{\dagger}a^{\#}a^*)((a^{\dagger})^*a^2)(a^{\dagger}a^{\#}a^*) = a^{\dagger}a(a^{\dagger}a^{\#}a^*) = a^{\dagger}a^{\#}a^*,$$

this infers $(a^{\dagger})^*a^2 \in R^{\dagger}$ *and* $[(a^{\dagger})^*a^2]^{\dagger} = a^{\dagger}a^{\#}a^*$.

Similarly, we can show (2).

Observing the proof of Lemma 2.1, we have the following proposition.

Proposition 2.2. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equavilent:*
(1) $a \in R^{EP}$; (2) $(a^{\dagger})^*a^2 \in R^{EP}$; (3) $a^{\dagger}a^{\#}a^* = (a^{\#})^2a^*aa^{\#}$.

Proposition 2.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then* (1) $a \in R^{PI}$ *if and only if* $((a^{\dagger})^*a^2)^{\dagger} = a^{\dagger}a^{\#}a^{\dagger}$.
(2) $a \in R^{SEP}$ *if and only if* $(a^{\#})^2a^*aa^{\#} = a^{\dagger}a^{\#}a^{\dagger}$.

Proof. (1) *" $\Rightarrow$ " Since $a \in R^{PI}, a^{\dagger} = a^*$. Hence $((a^{\dagger})^*a^2)^{\dagger} = a^{\dagger}a^{\#}a^{+}$ by Lemma 2.1.*
" $\Leftarrow$ " Assume that $((a^{\dagger})^*a^2)^{\dagger} = a^{\dagger}a^{\#}a^{\dagger}$. Then, by Lemma 2.1, we have $a^{\dagger}a^{\#}a^* = a^{\dagger}a^{\#}a^{\dagger}$. Pre-multiplying the equality by $a^{\dagger}a^3$, one has $a^* = a^{\dagger}$. Hence $a \in R^{PI}$.

(2) *" $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then $a \in R^{PI}$ and $a \in R^{EP}$. It follows from Lemma 2.1 and Proposition 2.2 that $(a^{\#})^2a^*aa^{\#} = a^{\dagger}a^{\#}a^* = ((a^{\dagger})^*a^2)^{\dagger} = a^{\dagger}a^{\#}a^{\dagger}$.*

" $\Leftarrow$ " If $(a^{\#})^2a^*aa^{\#} = a^{\dagger}a^{\#}a^{\dagger}$, then

$$(a^{\#})^2a^*aa^{\#} = a^{\dagger}a^{\#}a^{\dagger} = a^{\dagger}a^{\#}a^{\dagger}aa^{\dagger} = (a^{\#})^2a^*aa^{\#}aa^{\dagger} = (a^{\#})^2a^*.$$

This gives

$$aa^{\#} = (a^{\dagger})^*a^*aa^{\#} = ((a^{\dagger})^*aa^{\#}))a^*aa^{\#} = (a^{\dagger})^*a^2(a^{\#})^2a^*aa^{\#} = (a^{\dagger})^*a^2(a^{\#})^2a^* = aa^{\dagger}.$$

Hence $a \in R^{EP}$. By Proposition 2.2 and Lemma 2.1, we have $((a^{\dagger})^*a^2)^{\dagger} = a^{\dagger}a^{\#}a^* = (a^{\#})^2a^*aa^{\#} = a^{\dagger}a^{\#}a^{\dagger}$, this gives $a \in R^{PI}$ by (1). *Thus $a \in R^{SEP}$.*

Similarly, we have the following proposition.

Proposition 2.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then* (1) $a \in R^{PI}$ *if and only if* $((a^{\dagger})^*a^2)^{\dagger} = a^*a^{\#}a^*$.
(2) $a \in R^{SEP}$ *if and only if* $(a^{\#})^2a^*aa^{\#} = a^*a^{\#}a^*$.

Proposition 2.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $((a^{\dagger})^*a^2)^{\dagger} = a^{\#}a^{\#}a^*$.*

Proof. *" $\Rightarrow$ " It is an immediate result of Lemma 2.1, because $a^{\dagger} = a^{\#}$.*
" $\Leftarrow$ " Assume that $((a^{\dagger})^*a^2)^{\dagger} = a^{\#}a^{\#}a^*$. Then by Lemma 2.1, we have $a^{\dagger}a^{\#}a^* = a^{\#}a^{\#}a^*$. Post-multiplying the equality by $(a^{\dagger})^*a^2$, one yields $a^{\dagger}a = a^{\#}a$. Hence $a \in R^{EP}$.

Recall that $a \in R$ is normal if $aa^* = a^*a$. And a is said to be 2-normal if $a^2a^* = a^*a^2$. Clearly $a \in R^\# \cap R^\dagger$ is normal if and only if $a^\# a^* = a^* a^\#$.

Proposition 2.6. *Let $a \in R^\# \cap R^\dagger$. Then a is normal if and only if $((a^\dagger)^* a^2)^\dagger = a^\# a^* a^\dagger$.*
Proof. *" $\Rightarrow$ " Since a is normal, $a \in R^{EP}$ and $a^*a^\dagger = a^\dagger a^*$ by [11, Lemma 1.3.2]. Hence, by Lemma 2.1.*

$$((a^\dagger)^*a^2)^\dagger = a^\dagger a^\# a^* = a^\dagger a^* a^\# = a^\# a^* a^\dagger.$$

" $\Leftarrow$ " Suppose that $((a^\dagger)^*a^2)^\dagger = a^\# a^* a^\dagger$. Then $a^\dagger a^\# a^* = a^\# a^* a^\dagger$ by Lemma 2.1, this gives

$$(1 - a^\dagger a)a^\# a^* a^\dagger = 0.$$

Post-multiplying the equality by $a(a^\dagger a^\# a)^*$, one obtains $(1 - a^\dagger a)a^\# = 0$. Hence $a \in R^{EP}$, this gives

$$a^\# a^* = a^\#(a^* a^\dagger a) = (a^\# a^* a^\dagger)a = (a^\dagger a^\# a^*)a = a^\# a^\# a^* a,$$

$$aa^* = a^2(a^\# a^*) = a^2(a^\# a^\# a^* a) = aa^\# a^* a = a^* a.$$

Hence a is normal.

Proposition 2.7. *Let $a \in R^\# \cap R^\dagger$. Then a is 2-normal if and only if $((a^\dagger)^*a^2)^\dagger = a^*(a^\#)^2$.*
Proof. *" $\Rightarrow$ " Assume that a is 2-normal. Then $a^*(a^\#)^2 = (a^\#)^2 a^*$. By Lemma 2.1, $((a^\dagger)^*a^2)^\dagger = a^\dagger a^\# a^* = a^\dagger a (a^\#)^2 a^* = a^\dagger a a^* (a^\#)^2 = a^*(a^\#)^2$.*
" $\Leftarrow$ " If $((a^\dagger)^*a^2)^\dagger = a^*(a^\#)^2$, then $a^\dagger a^\# a^* = a^*(a^\#)^2$ by Lemma 2.1, it follows that $a^*(a^\#)^2(1 - aa^\dagger) = 0$. Pre-multiplying the equality by $a^3(a^\dagger)^*$, one has $a(1 - aa^\dagger) = 0$. Hence $a \in R^{EP}$, this gives $a^* = a^\dagger a a^* = a^2 a^\dagger a^\# a^* = a^2 a^*(a^\#)^2$ by Lemma 2.1, one has $a^*a^2 = a^2a^*(a^\#)^2a^2 = a^2a^*aa^\# = a^2a^*$. Thus a is 2-normal.

Proposition 2.8. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $((a^\dagger)^*a^2)^\dagger = a^\dagger(a^\#)^2$.*
Proof. *" $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then $a^\# = a^*$, this infers $((a^\dagger)^*a^2)^\dagger = a^\dagger a^\# a^* = a^\dagger(a^\#)^2$.*
" $\Leftarrow$ " Suppose that $((a^\dagger)^*a^2)^\dagger = a^\dagger(a^\#)^2$, then $a^\dagger a^\# a^* = a^\dagger(a^\#)^2$ by Lemma 2.1. Pre-multiplying the equality by a^3, one obtains $aa^* = aa^\#$. Hence $a \in R^{SEP}$ by [10,Theorem 2.3].

Proposition 2.9. *Let $a \in R^\# \cap R^\dagger$. Then a is symmetric if and only if $((a^\dagger)^*a^2)^\dagger = a^\#$.*
Proof. *" $\Rightarrow$ " Assume that a is symmetric. Then $a = a^*$. It follows from Lemma 2.1 that $((a^\dagger)^*a^2)^\dagger = a^\dagger a^\# a^* = a^\dagger a^\# a = a^\dagger a^2 (a^\#)^2 = a^\dagger a a^* (a^\#)^2 = a^*(a^\#)^2 = a(a^\#)^2 = a^\#$.*
" $\Leftarrow$ " Suppose that $((a^\dagger)^*a^2)^\dagger = a^\#$. Then $a^\dagger a^\# a^* = a^\#$ by Lemma 2.1. Pre-multiplying the equality by a^3. yielding $aa^* = a^2$, this implies $a^\# = a^\dagger a^\# a^* = a^\dagger(a^\#)^2 aa^* = a^\dagger(a^\#)^2a^2 = a^\dagger a^\# a$. Hence $a \in R^{EP}$. Now we have $a = a^2a^\# = a^2a^\dagger a^\# a^* = aa^\# a^* = a^\dagger a a^* = a^*$. Thus a is symmetric.

Lemma 2.10. *Let $a \in R^\# \cap R^\dagger$. Then $a^\dagger a^\dagger \in R^\dagger$ and $(a^\dagger a^\dagger)^\dagger = aa^*(a^\#)^*a$.*
Proof. *A routine verification shows that:*

$$(a^\dagger a^\dagger)(aa^*(a^\#)^*a) = a^\dagger a^*(a^\#)^*a = a^\dagger a,$$

$$(aa^*(a^\#)^*a)(a^\dagger a^\dagger) = aa^*(a^\#)^*a^\dagger = aa^\dagger,$$

$$(a^\dagger a^\dagger)(aa^*(a^\#)^*a)(a^\dagger a^\dagger) = a^\dagger aa^\dagger a^\dagger = a^\dagger a^\dagger,$$

$$(aa^*(a^\#)^*a)(a^\dagger a^\dagger)(aa^*(a^\#)^*a) = aa^\dagger aa^*(a^\#)^*a = aa^*(a^\#)^*a.$$

Hence $a^\dagger a^\dagger \in R^\dagger$ and $(a^\dagger a^\dagger)^\dagger = aa^*(a^\#)^*a$.

Corollary 2.11. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if $a^\dagger a^\dagger \in R^{EP}$.*
Proof. *" $\Rightarrow$ " Since $a \in R^{EP}$, $aa^\dagger = a^\dagger a$, this gives $(a^\dagger a^\dagger)(a^\dagger a^\dagger)^\dagger = a^\dagger a = aa^\dagger = (a^\dagger a^\dagger)^\dagger(a^\dagger a^\dagger)$. Hence $a^\dagger a^\dagger \in R^{EP}$.*
" $\Leftarrow$ " Assume that $a^\dagger a^\dagger \in R^{EP}$, Then $a^\dagger a^\dagger(a^\dagger a^\dagger)^\dagger = (a^\dagger a^\dagger)^\dagger(a^\dagger a^\dagger)$. By Lemma 2.10, one gets $a^\dagger a = aa^\dagger$. Hence $a \in R^{EP}$.

Corollary 2.12. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a^{\dagger}a^{\dagger}a \in R^{EP}$ and $(a^{\dagger}a^{\dagger}a)^{\dagger} = a^*(a^{\#})^*a$.*
Proof. Noting that $a^* = a^{\dagger}aa^*$. Then by Lemma 2.10, it follows that

$$(a^{\dagger}a^{\dagger}a)(a^*(a^{\#})^*a) = (a^{\dagger}a^{\dagger})(aa^*(a^{\#})^*a) = a^{\dagger}a,$$

$$(a^*(a^{\#})^*a)(a^{\dagger}a^{\dagger}a) = a^{\dagger}((aa^*(a^{\#})^*a)(a^{\dagger}a^{\dagger}))a = a^{\dagger}aa^{\dagger}a = a^{\dagger}a.$$

Hence $a^{\dagger}a^{\dagger}a \in R^{EP}$ and $(a^{\dagger}a^{\dagger}a)^{\dagger} = a^*(a^{\#})^*a$.

Corollary 2.13. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a^{\dagger}a^{\dagger}(a^{\dagger})^* \in R^{EP}$ and $(a^{\dagger}a^{\dagger}(a^{\dagger})^*)^{\dagger} = a^*aa^*(a^{\#})^*a$.*
Proof. Noting that $a^{\dagger} = a^{\dagger}(a^{\dagger})^*a^*$. Then by Lemma 2.10, we have

$$(a^{\dagger}a^{\dagger}(a^{\dagger})^*)(a^*aa^*(a^{\#})^*a) = (a^{\dagger}a^{\dagger}(a^{\dagger})^*a^*)(aa^*(a^{\#})^*a) = (a^{\dagger}a^{\dagger})(aa^*(a^{\#})^*a) = a^{\dagger}a,$$

$$(a^*aa^*(a^{\#})^*a)(a^{\dagger}a^{\dagger}(a^{\dagger})^*) = a^*((aa^*(a^{\#})^*a)(a^{\dagger}a^{\dagger}))(a^{\dagger})^* = a^*aa^{\dagger}(a^{\dagger})^* = a^{\dagger}a.$$

Hence $a^{\dagger}a^{\dagger}(a^{\dagger})^* \in R^{EP}$ and $(a^{\dagger}a^{\dagger}(a^{\dagger})^*)^{\dagger} = a^*aa^*(a^{\#})^*a$.

Corollary 2.14. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $(a^{\dagger})^*a^{\dagger}a^{\dagger} \in R^{EP}$ and $((a^{\dagger})^*a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*aa^*$.*
Proof. By Lemma 2.10, one has

$$((a^{\dagger})^*a^{\dagger}a^{\dagger})(aa^*(a^{\#})^*aa^*) = (a^{\dagger})^*((a^{\dagger}a^{\dagger})(aa^*(a^{\#})^*a))a^* = (a^{\dagger})^*a^{\dagger}aa^* = aa^{\dagger},$$

and

$$(aa^*(a^{\#})^*aa^*)((a^{\dagger})^*a^{\dagger}a^{\dagger}) = (aa^*(a^{\#})^*a)(a^*(a^{\dagger})^*a^{\dagger}a^{\dagger}) = (aa^*(a^{\#})^*a)(a^{\dagger}a^{\dagger}) = aa^{\dagger}.$$

Hence $(a^{\dagger})^*a^{\dagger}a^{\dagger} \in R^{EP}$ and $((a^{\dagger})^*a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*aa^*$.

Also, we have the following corollary.

Corollary 2.15. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $aa^{\dagger}a^{\dagger} \in R^{EP}$ and $(aa^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*$.*

Theorem 2.16. *Let $a \in R^{\#} \cap R^{\dagger}$. Then*
(1) $a \in R^{PI}$ if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = aa^{\dagger}(a^{\#})^*a$;
(2) $a \in R^{PI}$ if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*(a^{\dagger})^*$;
(3) $a \in R^{EP}$ if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = a^{\dagger}a^3$;
(4) $a \in R^{SEP}$ if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = a^*a^3$;
(5) $a \in R^{SEP}$ if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = (a^{\#})^*a$;
(6) a is normal if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = a^*a(a^{\#})^*a$;
(7) a is 2-normal if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = a^*a^2(a^{\#})^*$;
(8) a is symmetric if and only if $(a^{\dagger}a^{\dagger})^{\dagger} = aa^*$.
Proof. (1) $'' \Rightarrow ''$ Since $a \in R^{PI}$, $a^* = a^{\dagger}$. Hence $(a^{\dagger}a^{\dagger})^{\dagger} = aa^{\dagger}(a^{\#})^*a$ by Lemma 2.10.
$'' \Leftarrow ''$ Assume that $(a^{\dagger}a^{\dagger})^{\dagger} = aa^{\dagger}(a^{\#})^*a$. Then $aa^*(a^{\#})^*a = aa^{\dagger}(a^{\#})^*a$ by Lemma 2.10. Post-multiplying the equality by $a^{\dagger}a^*$, one gets $aa^* = aa^{\dagger}$. Hence $a \in R^{PI}$ by [10, Theorem 2.1].
(2) $'' \Rightarrow ''$ Suppose that $a \in R^{PI}$. Then $(a^{\dagger})^* = a$, it follows that $(a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*(a^{\dagger})^*$ by Lemma 2.10.
$'' \Leftarrow ''$ If $(a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*(a^{\dagger})^*$, then $aa^*(a^{\#})^*a = aa^*(a^{\#})^*(a^{\dagger})^*$ by Lemma 2.10. Pre-multiplying the equality by $(a^{\dagger})^*a^*a^{\dagger}$, one gets $a = (a^{\dagger})^*$. Hence $a \in R^{PI}$.
(3) $'' \Rightarrow ''$ Since $a \in R^{EP}$, $a^{\#} = a^{\dagger}$. Hence, by Lemma 2.10, $(a^{\dagger}a^{\dagger})^{\dagger} = aa^*(a^{\#})^*a = aa^*(a^{\dagger})^*a = a^2 = a^{\dagger}a^3$.
$'' \Leftarrow ''$ Assume that $(a^{\dagger}a^{\dagger})^{\dagger} = a^{\dagger}a^3$. Then $aa^*(a^{\#})^*a = a^{\dagger}a^3$, by Lemma 2.10. Pre-multiplying the equality by $1 - aa^{\dagger}$, one has $(1 - aa^{\dagger})a^{\dagger}a^3 = 0$. Post-multiplying the last equality by $(a^{\#})^2a^{\dagger}$, one obtains $(1 - aa^{\dagger})a^{\dagger} = 0$. Hence $a \in R^{EP}$.
(4) $'' \Rightarrow ''$ If $a \in R^{SEP}$, then $a^{\#} = a^{\dagger} = a^*$. By 3), we have $(a^{\dagger}a^{\dagger})^{\dagger} = a^{\dagger}a^3 = a^*a^3$.
$'' \Leftarrow ''$ Assume that $(a^{\dagger}a^{\dagger})^{\dagger} = a^*a^3$. Then $aa^*(a^{\#})^*a = a^*a^3$, by Lemma 2.10. This gives $aa^*(a^{\#})^*a = a^{\dagger}a^2a^*(a^{\#})^*a$. Post-multiplying the equality by $a^{\dagger}a^{\dagger}a$, one has $a = a^{\dagger}a^2$, Hence $a \in R^{EP}$, it follows that $a^2 = aa^*(a^{\dagger})^*a = aa^*(a^{\#})^*a = a^*a^3$. Hence $a \in R^{SEP}$ by [10, Theorem 2.3].
(5) $'' \Rightarrow ''$ Assume that $a \in R^{SEP}$. Then $(a^{\#})^* = a$ and $(a^{\dagger}a^{\dagger})^{\dagger} = a^{\dagger}a^3$ by 3), this infers $(a^{\dagger}a^{\dagger})^{\dagger} = a^2 = (a^{\#})^*a$.
$'' \Leftarrow ''$ If $(a^{\dagger}a^{\dagger})^{\dagger} = (a^{\#})^*a$, then $aa^*(a^{\#})^*a = (a^{\#})^*a$ by Lemma 2.10. Post-multiplying the equality by $a^{\dagger}a^{\dagger}$, one has

$aa^\dagger = (a^\#)^*a^\dagger$, this gives $a^* = a^*aa^\dagger = a^*(a^\#)^*a^\dagger = a^\dagger$, so $aa^\dagger = (a^\#)^*a^*$. Hence $aa^\# = aa^\dagger$, this infers $a \in R^{EP}$. Thus $a \in R^{SEP}$.
(6) $'' \Rightarrow ''$ If a is normal, then $aa^* = a^*a$. Hence $(a^\dagger a^\dagger)^\dagger = a^*a(a^\#)^*a$ by Lemma 2.10.
$'' \Leftarrow ''$ Assume that $(a^\dagger a^\dagger)^\dagger = a^*a(a^\#)^*a$, then $aa^*(a^\#)^*a = a^*a(a^\#)^*a$ by Lemma 2.10. Post-multiplying the equality by $a^\dagger a^* a^\dagger$, one obtains $aa^*a^\dagger = a^*$, Hence $a \in R^{EP}$. Now we have $a^*a = aa^*a^\dagger a = aa^*$. Therefore a is normal.
(7) $'' \Rightarrow ''$ Assume that a is 2-normal. Then $a^*a^2 = a^2a^* = (a^2a^*)aa^\dagger = a^*a^3a^\dagger$. Pre-multiplying the equality by $a^\#(a^\dagger)^*$, one has $a = a^2a^\dagger$. Hence $a \in R^{EP}$. By (3), $(a^\dagger a^\dagger)^\dagger = a^\dagger a^3 = a^2$. Noting that $a^*a^2(a^\#)^* = a^2a^*(a^\dagger)^* = a^2$. Hence $(a^\dagger a^\dagger)^\dagger = a^*a^2(a^\#)^*$.
$'' \Leftarrow ''$ Suppose that $(a^\dagger a^\dagger)^\dagger = a^*a^2(a^\#)^*$, then $aa^*(a^\#)^*a = a^*a^2(a^\#)^*$. Pre-multiplying the equality by $1 - a^\dagger a$, one has $(1 - a^\dagger a)aa^*(a^\#)^*a = 0$. Post-multiplying the last equality by $a^\dagger a^\dagger a$, yields $(1 - a^\dagger a)a = 0$. Hence $a \in R^{EP}$, this gives $a^2 = aa^*(a^\dagger)^*a = aa^*(a^\#)^*a = a^*a^2(a^\#)^*$. Now we have $a^2a^* = a^*a^2(a^\#)^*a^* = a^*a^2(a^\dagger)^*a^* = a^*a^3a^\dagger = a^*a^2$. Thus a is 2-normal.
(8) $'' \Rightarrow ''$ Assume that a is symmetric, Then $a^* = a$ and a is EP. By 3), $(a^\dagger a^\dagger)^\dagger = a^\dagger a^3 = a^2 = aa^*$.
$'' \Leftarrow ''$ Suppose that $(a^\dagger a^\dagger)^\dagger = aa^*$, then $aa^*(a^\#)^*a = aa^*$. Pre-multiplying the equality by $a^\dagger a^\dagger$, one yields $a^\dagger a = a^\dagger a^*$, this gives $a^\dagger a = a^\dagger a^2 a^\dagger$, so $a = a^2a^\dagger$. Hence $a \in R^{EP}$, this infers $aa^* = aa^*(a^\dagger)^*a = a^2$, $a = a^\dagger a^2 = a^\dagger aa^* = a^*$. Hence a is symmetric.

Noting that $a^\dagger a^3 \in R^{EP}$ with $(a^\dagger a^3)^\dagger = a^\dagger a^\#$. Hence Theorem 2.16 implies the following corollary.

Corollary 2.17. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if $a^\dagger a^\dagger = a^\dagger a^\#$.*

Since $aa^\dagger(a^\#)^*a \in R^\dagger$ with $(aa^\dagger(a^\#)^*a)^\dagger = a^\dagger a^*$. Theorem 2.16 gives the following corollary.

Corollary 2.18. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if $a^\dagger a^\dagger = a^\dagger a^*$.*

It is easy to see that $a^*a^3 \in R^{EP}$ with $(a^*a^3)^\dagger = a^\dagger a^\# a^\dagger (a^\dagger)^*$. Hence we have the following corollary.

Corollary 2.19. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $a^\dagger a^\dagger = a^\dagger a^\# a^\dagger (a^\dagger)^*$.*

According to [10], $aa^* \in R^{EP}$ with $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$. Hence we have.

Corollary 2.20. *Let $a \in R^\# \cap R^\dagger$. Then a is symmetric if and only if $a^\dagger a^\dagger = (a^\dagger)^*a^\dagger$.*

Also we have $(a^\#)^*a \in R^{EP}$ with $((a^\#)^*a)^\dagger = a^\dagger a^* a^\dagger a$, this leads to the following corollary.

Corollary 2.21. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $a^\dagger a^\dagger = a^\dagger a^* a^\dagger a$.*

3. Partial isometries and Solutions of some Equations

In this section, we establish the relation among partial isometry, EP, SEP and the solutions of equation in χ_a, and investigate the general solutions of some equations to give charactrizations of partial isometries, strongly EP elements and EP elements.
Let $a \in R^{PI}$. Then $aa^*(a^\dagger)^*a = (a^\dagger)^*a$, this induces us to construct the following equation.

$$aa^*xa = xa. \tag{1}$$

Proposition 3.1. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the equation* (1) *has at least one solution in χ_a.*
Proof. $'' \Rightarrow ''$ Assume that $a \in R^{PI}$. Then $a = (a^\dagger)^*$, which implies $x = (a^\dagger)^*$ is a solution.
$'' \Leftarrow ''$ 1) If $x = a$ is a solution, then $aa^*a^2 = a^2$, this gives $aa^*a = aa^*a^2a^\# = a^2a^\# = a$. Hence $a \in R^{PI}$;
2) If $x = a^\#$, then $aa^*a^\#a = a^\#a$. Post-multiplying by a, one gets $aa^*a = a$. Hence $a \in R^{PI}$;
3) If $x = a^\dagger$, then $aa^*a^\dagger a = a^\dagger a$. Post-multiplying the equality by $a^\dagger$, one has $aa^*a^\dagger = a^\dagger$. Hence $a \in R^{PI}$ by [10, Theorem 2.3];
4) If $x = a^*$, then $aa^*a^*a = a^*a$, this gives $aa^*a^* = a^*$, and so $a^2a^* = a$. Hence $a \in R^{PI}$ by [10, Theorem 2.3];
5) If $x = (a^\#)^*$, then $aa^*(a^\#)^*a = (a^\#)^*a$, this infers $(a^\dagger a^\dagger)^\dagger = (a^\#)^*a$ by Lemma 2.10. Hence $a \in R^{PI}$ by Theorem 2.16.
6) If $x = (a^\dagger)^*$, then $aa^*(a^\dagger)^*a = (a^\dagger)^*a$. Post-multiplying the equality by $a^\#$, one obtains $a = (a^\dagger)^*$. Hence $a \in R^{PI}$.

Now we change the equation (1) as follows

$$aa^*xa = ax. \tag{2}$$

We have the following proposition, which proof is easy.

Proposition 3.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the equation* (2) *has at least one solution in χ_a.*

Pre-multiplying the equation (2) by $a^\dagger$, we have the following equation.

$$a^*xa = a^\dagger ax. \tag{3}$$

Proposition 3.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the equation* (3) *has at least one solution in χ_a.*

Remark 3.4. *The equation* (3) *can be generalized as follows*

$$a^*xa - a^\dagger ay = 0. \tag{4}$$

Clearly, the general solution of the equation (4) is given as follows

$$\begin{cases} x = -(a^\dagger)^*Pa^\dagger + u - aa^\dagger uaa^\dagger \\ \quad y = -a^\dagger aPa^\dagger a + \gamma - a^\dagger a\gamma \end{cases}, \textit{where } P, u, \gamma \in R. \tag{5}$$

Corollary 3.5. *Let $a \in R^\dagger$. Then $a \in R^{PI}$ if and only if the general solution of the equation* (4) *is given by*

$$\begin{cases} x = -(a^\dagger)^*Pa^* + u - aa^\dagger uaa^\dagger \\ \quad y = -a^\dagger aPa^\dagger a + \gamma - a^\dagger a\gamma \end{cases}, \textit{where } P, u, \gamma \in R. \tag{6}$$

Proof. *" $\Rightarrow$ " Assume that $a \in R^{PI}$, then $a^\dagger = a^*$. Hence the general solution* (5) *of the equation* (4) *is equivalent to* (6).
" $\Leftarrow$ " If (6) *is the general solution of* (4), *then $a^*(-(a^\dagger)^*Pa^* + u - aa^\dagger uaa^\dagger)a - a^\dagger a(-a^\dagger aPa^\dagger a + \gamma - a^\dagger a\gamma) = 0$, that is $a^\dagger aPa^*a = a^\dagger aPa^\dagger a$ for any $P \in R$. Especially, choose $P = 1$, we have $a^*a = a^\dagger a$. Hence $a \in R^{PI}$.*

Now we modify the equation (4) *as follows.*

$$a^*xa - a^\# ay = 0. \tag{7}$$

Proposition 3.6. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the general solution of the equation* (7) *is given by* (5).
Proof. " $\Rightarrow$ " Since $a \in R^{EP}$, $a^\dagger a = a^\# a$. Hence, by Remark 3.4 the general solution of (7) *is given by* (5).
" $\Leftarrow$ " If the general solution of (7) *is given by* (5), *then*

$$a^*(-(a^\dagger)^*Pa^\dagger + u - aa^\dagger uaa^\dagger)a - a^\# a(-a^\dagger aPa^\dagger a + \gamma - a^\dagger a\gamma) = 0,$$

this gives $a^\dagger aPa^\dagger a = a^\# aPa^\dagger a$ for each $P \in R$. Especially, choose $P = 1$, we have $a^\dagger a = a^\# a$. Hence $a \in R^{EP}$.

The equation (7) *can be changed as follows.*

$$a^*x(a^\dagger)^* - a^\# ay = 0. \tag{8}$$

Proposition 3.7. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the general solution of the equation* (8) *is given by* (5).
Proof. " $\Rightarrow$ " Since $a \in R^{SEP}$, $a = (a^\dagger)^*$ and $a^\dagger a = a^\# a$. Hence, the equation (8) *is equivalent to the equation* (4), *we are done.*
" $\Leftarrow$ " If the general solution of the equation (8) *is given by* (5), *then*

$$a^*(-(a^\dagger)^*Pa^\dagger + u - aa^\dagger uaa^\dagger)(a^\dagger)^* - a^\# a(-a^\dagger aPa^\dagger a + \gamma - a^\dagger a\gamma) = 0,$$

that is $a^\dagger aPa^\dagger(a^\dagger)^* = a^\# aPa^\dagger a$ for all $a \in R$. Choose $P = 1$, we have $a^\dagger(a^\dagger)^* = a^\# a$, so $a = aa^\# a = aa^\dagger(a^\dagger)^* = (a^\dagger)^*$, this infers $a \in R^{PI}$. Now $a^\dagger a = a^\dagger(a^\dagger)^* = a^\# a$, hence $a \in R^{EP}$. Therefore $a \in R^{SEP}$.

Proposition 3.8. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the general solution of the equation* (4) *is given by*

$$\begin{cases} x = -(a^{\dagger})^{*}Pa^{\dagger} + u - aa^{\dagger}uaa^{\dagger} \\ \quad y = -a^{\dagger}aPaa^{\dagger} + \gamma - a^{\dagger}a\gamma \end{cases}, \text{where } P, u, \gamma \in R. \tag{9}$$

Proof. " $\Rightarrow$ " *Assume that $a \in R^{EP}$, then $a^{\dagger}a = aa^{\dagger}$. Hence* (5) *equivalent to* (9).
" $\Leftarrow$ " *If the general solution of the equation* (4) *is given by* (9), *then*

$$a^{*}(-(a^{\dagger})^{*}Pa^{\dagger} + u - aa^{\dagger}uaa^{\dagger})a - a^{\dagger}a(-a^{\dagger}aPaa^{\dagger} + \gamma - a^{\dagger}a\gamma) = 0,$$

this gives $a^{\dagger}aPa^{\dagger}a = a^{\dagger}aPaa^{\dagger}$ for each $P \in R$. Choose $P = a^{*}$, one yields $a^{*}a^{\dagger}a = a^{*}$, this leads to $a = a^{\dagger}a^{2}$. Hence $a \in R^{EP}$.

4. Consistency of certain equations and SEP elements

In this section, we characterize EP elements and strongly EP elements by the consistency of related equation.
Let $a \in R^{\#} \cap R^{\dagger}$. Then we consider the following equation.

$$(a^{\#})^{2}a^{*}xa^{\#} = a^{\dagger}a^{\#}a^{\dagger}. \tag{10}$$

Theorem 4.1. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if the equation* (10) *is consistent and the general solution is given by*

$$x = a + u - aa^{\dagger}uaa^{\dagger}, \tag{11}$$

where $u \in R$.
Proof. " $\Rightarrow$ " *Assume that $a \in R^{SEP}$. Then $(a^{\#})^{2}a^{*}aa^{\#} = a^{\dagger}a^{\#}a^{\dagger}$ by Proposition 2.3. Hence the equation* (10) *is consistent and* (11) *is the solution of the equation* (10).

Now let $x = x_0$ be any solution of the equation (10). *Then*

$$(a^{\#})^{2}a^{*}x_0a^{\#} = a^{\dagger}a^{\#}a^{\dagger}.$$

Since $a \in R^{SEP}$, $(a^{\dagger})^{*} = a$ and $a^{\dagger} = a^{\#}$. Hence

$$\begin{aligned} aa^{\dagger}x_0aa^{\dagger} &= (a^{\dagger})^{*}a^{*}x_0a^{\#}(a^{2}a^{\dagger}) = (a^{\dagger})^{*}a^{2}((a^{\#})^{2}a^{*}x_0a^{\#})a^{2}a^{\dagger} \\ &= (a^{\dagger})^{*}a^{2}(a^{\dagger}a^{\#}a^{\dagger})a^{2}a^{\dagger} = (a^{\dagger})^{*}(a^{2}a^{\dagger}a^{\#})a^{\dagger}a^{2}a^{\dagger} \\ &= (a^{\dagger})^{*}aa^{\#}a^{\dagger}a^{2}a^{\dagger} = (a^{\dagger})^{*}a^{\dagger}a^{2}a^{\dagger} = aa^{\#}a^{2}a^{\#} = a. \end{aligned}$$

This implies $x_0 = a + x_0 - aa^{\dagger}x_0aa^{\dagger}$. Thus the general solution of the equation (10) *is given by* (11).
" $\Leftarrow$ " *If the general solution of the equation* (10) *is given by* (11), *then*

$$(a^{\#})^{2}a^{*}(a + u - aa^{\dagger}uaa^{\dagger})a^{\#} = a^{\dagger}a^{\#}a,$$

i.e. $(a^{\#})^{2}a^{*}aa^{\#} = a^{\dagger}a^{\#}a^{\dagger}$. Thus $a \in R^{SEP}$ by Proposition 2.3.

Proposition 4.2. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the equation* (10) *is consistent. In this case, the general solution is given by*

$$x = (a^{\dagger})^{*} + u - aa^{\dagger}uaa^{\dagger}, \tag{12}$$

where $u \in R$.
Proof. Assume that $a \in R^{EP}$. Then $(a^{\#})^{2}a^{*}(a^{\dagger})^{*}a^{\#} = (a^{\#})^{2}a^{\dagger}aa^{\#} = (a^{\#})^{2}a^{\dagger} = a^{\dagger}a^{\#}a^{\dagger}$, which implies the equation (10) *is consistent, and* (12) *is the solution of the equation* (10).

Also, for any solution x_0 of the equation (10), *we have*

$$x_0 = (a^\dagger)^* + x_0 - aa^\dagger x_0 aa^\dagger$$

Thus (12) *is the general solution of the equation* (10).
" $\Leftarrow$ " If the equation (10) is consistent, then we have $a^\dagger a^\# a^\dagger = (a^\#)^2 a^* d a^\#$ for some $d \in R$. This gives $a^\dagger a^\# a^\dagger a^\dagger a = a^\dagger a^\# a^\dagger$ because $a^\# = a^\# a^\dagger a$. Hence

$$aa^\dagger a^\dagger a = (a^3 a^\dagger a^\#) a^\dagger a^\dagger a = a^3 a^\dagger a^\# a^\dagger = aa^\dagger.$$

Thus $a \in R^{EP}$.

Proposition 4.3. *Let $a \in R^\# \cap R^\dagger$. Then the general solution of the following equation is given by* (11).

$$a^\dagger a^\# a^\# a^\dagger x aa^\dagger = a^\dagger a^\# a^\dagger. \tag{13}$$

Proof. It is routine.

Corollary 4.4. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the equation* (10) *and* (13) *have the same solution.*

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References

[1] R.E. Harte, M. Mbekhta, On generalized inverses in C^* – algebras, Stud Math. 103 (1992) 71-77.
[2] J. J. Koliha, D.Djordjević, D.Cvetković, Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426 (2007) 371-381.
[3] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406-413.
[4] J.J. Koliha, The Drazin and Moore-Penrose inverse in C^* – algebras, Math. Proc. Royal Irish Acad. 99A (1999) 17-27.
[5] D.Mosić, D. S.Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (2009) 732-745.
[6] J. J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72 (2002) 137-152.
[7] D. Mosić, D.S Djordjević, J.J. Koliha, EP elements in rings, Linear Algebra Appl. 431 (2009) 527-535.
[8] D. Mosić, D.S Djordjević, Further results on partial isometries and EP elements in rings with involution, Math. Compute. Model. 54 (2011) 460-465.
[9] A. Ben-Israel, T. N. E.Greville, Generalized Inverses: Theory and Applications, (2nd edition), Springer, New York, 2003.
[10] D. Mosić, D.S. Djordjević, Partial isometries and EP elements in rings with involution, Electronic. J. Linear Algebra. 18 (2009) 761-772.
[11] D. Mosić, Generalized inverses, Faculty of Sciences and Mathematics, University of Niš, 2018.
[12] R. Zhao, H. Yao, J. Wei, Characterizations of partial isometries and two special kinds of EP elements, Czechislovak. Math. Journal. 70 (145) (2020) 539-551.
[13] W. Xing Chen, On EP elements, Normal elements and Partial isometries in rings with involution, Electronic. J. Linear Algebra. 23 (2012) 553-561.