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Braided Galois Objects and Sweedler Cohomology of Certain Radford Biproducts

Bojana Femić[a]

[a]*Mathematical Institute of Serbian Academy of Sciences and Arts*

Abstract. We construct the group of H-Galois objects for a flat and cocommutative Hopf algebra in a braided monoidal category with equalizers provided that a certain assumption on the braiding is fulfilled. We show that it is a subgroup of the group of BiGalois objects of Schauenburg, and prove that the latter group is isomorphic to the semidirect product of the group of Hopf automorphisms of H and the group of H-Galois objects. Dropping the assumption on the braiding, we construct the group of H-Galois objects with normal basis. For H cocommutative we construct Sweedler cohomology and prove that the second cohomology group is isomorphic to the group of H-Galois objects with normal basis. We construct the Picard group of invertible H-comodules for a flat and cocommutative Hopf algebra H. We show that every H-Galois object is an invertible H-comodule, yielding a group morphism from the group of H-Galois objects to the Picard group of H. A short exact sequence is constructed relating the second cohomology group and the two latter groups, under the above mentioned assumption on the braiding. We show how our constructions generalize some results for modules over commutative rings, and some other known for symmetric monoidal categories. Examples of Hopf algebras are discussed for which we compute the second cohomology group and the group of Galois objects.

1. Introduction

The notion of a Hopf-Galois extension, defined in [18], is one of the pillars in the Hopf algebra theory. It is strongly related to algebraic geometry. A faithfully flat commutative Hopf-Galois extension for a Hopf algebra that is the coordinate algebra of an affine group scheme is a principle homogeneous space. Then faithfully flat not necessarily commutative Hopf-Galois extensions may be seen as a noncommutative analogue of this geometric concept. Hopf-Galois extensions arose from Hopf-Galois objects, also called Galois objects. The group of Galois objects over a commutative ring was introduced by Chase and Sweedler in 1969, [9]. It emerged as a generalization from the classical Galois field theory and the Galois theory for commutative rings developed in [8]. Galois objects in a closed symmetric monoidal category were studied in [19] in 1980. A recent construction was made in [31]. There, as for the category of modules in [5], the product in the group is induced by the cotensor product, which in categorical language is a particular equalizer. For the definition of Galois objects in a braided monoidal category one needs that equalizers are

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Email address: bfemic@mi.sanu.ac.rs (Bojana Femić)

preserved by certain tensor products. For this purpose Schauenburg introduced the notions of flatness and faithful flatness. It turns out that using them the construction of Galois objects from [19] is much simplified.

In this paper we construct the group of Galois objects as a subgroup of Schauenburg's group of biGalois-objects from [31]. We show that in order for this subgroup to exist one has to assume that the braiding Φ when acting between two H-Galois objects A and B is symmetric, meaning $\Phi_{A,B} = \Phi^{-1}_{B,A}$. The main result of this paper is Theorem 5.13 where the following short exact sequence is constructed, whose middle term is our mentioned group of Galois objects:

$$1 \longrightarrow \mathrm{H}^2(C; H, I) \xrightarrow{\zeta} \mathrm{Gal}(C; H) \xrightarrow{\xi} \mathrm{Pic}^{co}(C; H). \tag{1.1}$$

Here $\mathrm{H}^2(C; H, I)$ is the second Sweedler cohomology group, $\mathrm{Pic}^{co}(C; H)$ is the Picard group of invertible H-comodules, and H is a flat cocommutative Hopf algebra in a braided monoidal category C with equalizers such that the above assumption on the braiding holds. The group $\mathrm{H}^2(C; H, I)$ we construct by direct categorification of Sweedler cohomology for a cocommutative Hopf algebra over a commutative ring in [33]. We find that this construction requires the following restriction on the braiding to hold: $\Phi_{H,H} = \Phi^{-1}_{H,H}$. Though, by a smart observation of Schauenburg in [30, Corollary 5] this condition is already fulfilled for cocommutative H. The Picard group of invertible comodules $\mathrm{Pic}^{co}(C; H)$ we construct in the following way. Pareigis established in [27, Theorems 5.1 and 5.3] Morita Theorems for categories of modules in a monoidal category. Dualizing such characterized bimodules we obtain our notion of invertible comodules. The group of their isomorphism classes is our group $\mathrm{Pic}^{co}(C; H)$. On the other hand, it can be seen as a categorification of the Picard group for a coalgebra over a field introduced in [35]. The latter classifies Morita-Takeuchi equivalences, proposed by Takeuchi in [34] as equivalences of categories of comodules for coalgebras over a field, dually to Morita equivalences.

Galois objects which are isomorphic to H as H-comodules are called H-Galois objects with normal basis. We argue that for H cocommutative the property $\Phi_{H,H} = \Phi^{-1}_{H,H}$ implies that the assumption $\Phi_{A,B} = \Phi^{-1}_{B,A}$ is automatically fulfilled for all H-Galois objects with normal basis A and B, hence that such Galois objects induce a (sub)group $\mathrm{Gal}_{nb}(C; H)$. We prove in Theorem 5.9 that ζ from (1.1) corestricts to a group isomorphism $\mathrm{H}^2(C; H, I) \cong \mathrm{Gal}_{nb}(C; H)$. On the other hand, we prove that every H-Galois object is an invertible H-comodule, inducing a non-trivial morphism ξ in (1.1), when ζ is not surjective.

Our short exact sequence (1.1) is a generalization of the sequence of Álvarez and Vilaboa constructed in [1, Theorem 11] and [2, Proposition 0.3]. Whereas in the latter sequence the Hopf algebra should be finite and the category symmetric, the first restriction is not present in our case, and the second one is weakened by our assumption on the braiding.

More of a historical background of the sequence (1.1) we may resume as follows. The original idea for its construction was accomplished in 1976 in [24] for a commutative ring R and a finitely generated and projective Hopf algebra H with a bijective antipode, where the morphism from the group of Galois objects to the Picard group of invertible H-modules generalized that of [14, Theorem 2] for a group ring RG. A slightly more general construction was carried out in [10] in 1986. Following the latter, the corresponding morphism from the group of Galois objects to the Picard group of invertible modules was defined in a closed symmetric category in [2] in 2000. There was shown that the kernel of the morphism is isomorphic to the subgroup of Galois objects with a normal basis. Taking into account that this subgroup was proved to be isomorphic to Sweedler's second cohomology group in [1], generalizing the Normal Basis Theorem [33, Theorem 8.6], one gets the aforementioned short exact sequence. Harrison cohomology appearing in [24] is here replaced by Sweedler cohomology. This is consistent, since the Hopf algebra H in the first construction becomes a Hopf algebra H^* in the second one, and Harrison cohomology for H is isomorphic to Sweedler cohomology for H^*, [7, Proposition 9.2.3]. K-theoretical background for these exact sequences can be found in [7, (C.8), p. 470]. A version of the short exact sequence for commutative rings is [7, (10.25), p. 267] and how it emerges from the K-theoretical origin one can comprehend from the steps [7, (10.19)–(10.23), p. 265]. Our construction can be seen as a categorification of [7, (10.25), p. 267].

Apart from the above-mentioned main result Theorem 5.13, we prove generalizations to braided monoidal categories of the following important classical results. The Fundamental Theorem of Hopf modules for a flat Hopf algebra H in a braided monoidal category C with equalizers was proved in [20, Theorem 1.1]. It establishes an equivalence of categories $-\otimes H : C \rightleftarrows C_H^H : (-)^{coH}$ where C_H^H is the category of Hopf modules in C and $(-)^{coH}$ the functor that delivers H-coinvariants. We generalize this result proving that for an H-comodule algebra A there is an adjunction of categories $-\otimes A : C \rightleftarrows C_A^H : (-)^{coH}$ where C_A^H is the category of relative Hopf modules in C, and moreover, under the assumption that H is flat we prove in Theorem 3.11 that A is an H-Galois object if and only if the latter adjunction is an equivalence of categories. As a consequence we obtain that a flat Hopf algebra is itself an H-Galois object, and in particular faithfully flat. Moreover, we prove in Proposition 3.14 that for a flat H every H-comodule algebra morphism between two H-Galois objects is an isomorphism. The mentioned results are established in Section 3 and mainly present categorification of analogous results for modules over a commutative ring from [7].

Along the way of constructing the group of Galois objects, the other important results are Lemma 4.7 and Theorem 4.21. In the former we give sufficient conditions for the associativity of the cotensor product over flat coalgebras in a monoidal category with equalizers, generalizing and correcting an analogous result from [7] for colagebras over a commutative ring. In the latter we prove that for a flat and cocommutative Hopf algebra H the group of BiGalois objects of Schauenburg is isomorphic to the semidirect product of the group of Hopf automorphisms of H and the group of H-Galois objects. This is categorification of [32, Lemma 4.7].

The last Section is devoted to examples. For the Radford Hopf algebra H_ν and the Nichols Hopf algebra $E(n)$, using the fact that they both are Radford biproduct Hopf algebras, we compute the second Sweedler cohomology group and the groups of Galois and biGalois objects in the corresponding settings.

The outline of the paper is as follows. In Section 2 we give preliminaries on (faithful) flatness and preservation and inheritance of additional structure on equalizers. Section 3 is devoted to Hopf-Galois objects: we prove the mentioned adjunction and equivalences of categories for (relative) Hopf modules, characterizing H-Galois objects, we prove that flat H is an H-Galois object and thus faithfully flat, and that every H-comodule algebra morphism between H-Galois objects is an isomorphism. in Section 4 we construct the group of Galois objects. In Subsection 4.1 we study cotensor coproducts, and in Subsection 4.2 we construct the group itself, the subgroup of Galois objects with normal basis, and prove the semidirect product decomposition for the group of biGalois objects. In Section 5 we first construct Sweedler cohomology, then we prove how 2-cocycles twist the multiplication of Galois objects yielding an isomorphism between the second cohomology group and the group of Galois objects with normal basis. In Section 5.4 we construct the Picard group of invertible comodules and prove that Galois objects are invertible comodules. In Subsection 5.5 we finally construct our short exact sequence and compare it to the analogous sequence of Álvarez and Vilaboa. In the last Section we first recall Radford biproducts and Majid's bosonization, and then compute the second Sweedler cohomology group and the groups of Galois and biGalois objects for the Radford biproduct Hopf algebras H_ν and $E(n)$.

2. Preliminaries

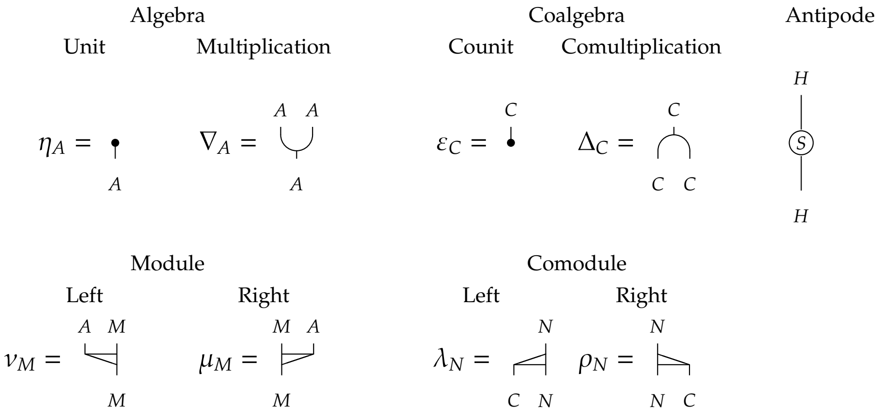
The reader is supposed to be familiarized with braided monoidal categories. For references we recommend [21] and [15], [16], [17] and [22]. The 7-tuple $(C, \otimes, a, I, l, r, \Phi)$ will denote a braided monoidal category, where C is a category, $\otimes : C \times C \to C$ is the tensor product functor, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is the associativity constraint, that satisfies Mac Lane's pentagonal axiom, I denotes the unit object, $l_X : I \otimes X \to X$ and $r_X : X \otimes I \to X$ are the left and right unity constraint, respectively, and $\Phi_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the braiding. In view of Mac Lane's Coherence Theorem we may (and we will do) assume that our braided monoidal category is strict, i.e., the associativity and unity constraints are identities in C. We will use the standard graphical calculus to work in braided monoidal categories. For two objects V, W in C the braiding between them and its inverse are denoted by

$\Phi_{V,W} =$ 

and $\Phi^{-1}_{V,W} =$ 

respectively. *We will assume throughout that $\mathcal{C}$ has equalizers.*

We refer to [6] for the definition of the following algebraic structures in braided monoidal categories: algebras, modules, coalgebras, comodules, Hopf algebras, module algebras and comodule algebras. The list of axioms for each of these structures, expressed in graphical calculus, may be found in [6, Page 159]. We fix some notation for the different structures: A is an algebra, C a coalgebra, H a Hopf algebra with antipode S, M an A-module and N a C-comodule.



2.1 Flatness: An object A in $\mathcal{C}$ is called *flat* if the functor $A \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers. If, in addition, it reflects isomorphisms, then A is called *faithfully flat*. By naturality of the braiding the functor $A \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers (resp. reflects isomorphisms) if and only if $- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}$ does it. The following statements for objects $A, B \in \mathcal{C}$ are easy to prove:

(i) *If A and B are flat, then so is $A \otimes B$.*
(ii) *If A and B are faithfully flat, then so is $A \otimes B$.*
(iii) *If the functor $A \otimes -$ reflects equalizers and $A \otimes B$ is faithfully flat, then B is faithfully flat.*

We will record in several results how the (co)equalizer of two morphisms with additional structures inherits these structures. The proofs of these results are standard.

2.2 Let $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$ be an equalizer in $\mathcal{C}$.

(i) If f and g are algebra morphisms, then E is an algebra and e is an algebra morphism. We call (E, e) an algebra pair.
(ii) If f and g are left (resp. right) H-module morphisms for an algebra H, then E is a left (resp. right) H-module and e is a left (resp. right) H-module morphism. The pair (E, e) is called an H-module pair. The dual statement follows for H-comodules *provided that H is a flat* coalgebra. Similarly, (E, e) is said to be an H-comodule pair.

Assume that A, B are H-comodule algebras, where H is flat, and let (E, e) be the equalizer of H-comodule algebra morphisms $f, g : A \longrightarrow B$. Then E inherits an H-comodule algebra structure such that e is a morphism of H-comodule algebras. It is important to stress that once the corresponding structure morphisms are induced on E, one does not need anymore the hypothesis on f and g to prove the compatibility conditions in each structure (the reader is encouraged to check it).

2.3 Let $A \in \mathcal{C}$ be an algebra and ${}_A\mathcal{C}$ the category of left A-modules. If $\mathcal{C}$ has equalizers, then ${}_A\mathcal{C}$ has equalizers, too - an equalizer in $\mathcal{C}$ of two morphisms in ${}_A\mathcal{C}$ is an equalizer in ${}_A\mathcal{C}$. Moreover, the forgetful functor $\mathcal{U} : {}_A\mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers. Both statements hold true for the category $\mathcal{C}^C$ of right C-comodules *if C is a flat coalgebra.*

2.4 Consider a commutative diagram of morphisms in C

$$\begin{array}{ccc} E_1 & \xrightarrow{e_1} & A_1 \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ E_2 & \xrightarrow[e_2]{} & A_2 \end{array}$$

and assume that e_2 is a monomorphism.

(i) If e_1, e_2 and f are right H-comodule morphisms and $e_2 \otimes H$ is a monomorphism, then $\overline{f}$ is an H-comodule morphism.
(ii) If e_1, e_2 and f are right H-module morphisms, then $\overline{f}$ is an H-module morphism.
(iii) If e_1, e_2 and f are algebra morphisms, then $\overline{f}$ is an algebra morphism.

As a consequence we have:

(a) Let $f : A \longrightarrow R$ and $g : B \longrightarrow R$ be algebra (resp. H-comodule) morphisms and assume that g (resp. $H \otimes g$) is a monomorphism. Let $h : A \longrightarrow B$ be such that $gh = f$. Then h is an algebra (resp. H-comodule) morphism.

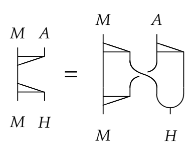
3. Hopf-Galois objects

In this section we recall from [31] the notion of Hopf-Galois object and present some results on it. The principal one is the characterization of a Hopf-Galois object in terms of a pair of functors being an equivalence, that generalizes the Fundamental Theorem of Hopf modules. As a consequence of this result we will obtain that a comodule algebra morphism between two Hopf-Galois objects is an isomorphism. This fact is an essential tool to establish the main results in [11].

3.1. *Hopf modules and relative Hopf modules*

We start by recalling from [31, Definition 3.7] the notion of relative Hopf module in C for a right comodule algebra A over a Hopf algebra H. We will show that there is an adjoint pair of functors between the category C_A^H of relative Hopf modules and C.

Definition 3.1 *Let H be a Hopf algebra in C and A a right H-comodule algebra. A right relative Hopf module (or an (A, H)-Hopf module) $M \in C$ is a right H-comodule and a right A-module such that the H-comodule structure of M is right A-linear, where $M \otimes H$ is endowed with the codiagonal A-module structure. The compatibility condition takes the form:*



We will denote by C_A^H the category whose objects are right relative Hopf modules and whose morphisms are A-linear H-colinear morphisms.

Definition 3.2 *Let M be a right H-comodule in C. The object of H-coinvariants of M is the equalizer:*

$$M^{coH} \xrightarrow{\ i\ } M \underset{M \otimes \eta_H}{\overset{\rho_M}{\rightrightarrows}} M \otimes H.$$

Actually, $(-)^{coH}$ defines a functor from $\mathcal{C}^H$ to $\mathcal{C}$. If $f : M \longrightarrow N$ is a morphism in $\mathcal{C}^H$, then $f^{coH} : M^{coH} \longrightarrow N^{coH}$ is induced by the commutativity of the square in the diagram:

$$\begin{array}{ccccc} M^{coH} & \xrightarrow{i_M} & M & & \\ \downarrow{\scriptstyle f^{coH}} & & \downarrow{\scriptstyle f} & & \\ N^{coH} & \xrightarrow{i_N} & N & \underset{N \otimes \eta_H}{\overset{\rho_N}{\rightrightarrows}} & N \otimes H \end{array}$$

The existence of f^{coH} is assured by the H-colinearity of f and the universal property of the equalizer (N^{coH}, i_N). Clearly, the functor $(-)^{coH}$ also acts on $\mathcal{C}^H_A$. Indeed, it is part of an adjoint pair as we see next.

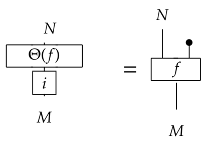
Proposition 3.3 *With A and H in $\mathcal{C}$ as above, $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}^H_A, N \mapsto N \otimes A$, is a left adjoint to $\mathcal{G} : \mathcal{C}^H_A \longrightarrow \mathcal{C}, M \mapsto M^{coH}$.*

Proof. Given $N \in \mathcal{C}$ we view $N \otimes A$ as a right H-comodule with coaction $\rho_{N \otimes A} = N \otimes \rho_A$ and a right A-module with action $\mu_{N \otimes A} = N \otimes \nabla_A$. The compatibility condition of $\mathcal{C}^H_A$ holds for $N \otimes A$ because A is an H-comodule algebra. The definition of $\mathcal{F}$ on morphisms is obvious.

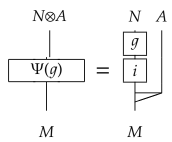
For $N \in \mathcal{C}$ and $M \in \mathcal{C}^H_A$ we define morphisms

$$\mathcal{C}^H_A(N \otimes A, M) \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathcal{C}(N, M^{coH})$$

as follows. Given $f \in \mathcal{C}^H_A(N \otimes A, M)$ its image $\Theta(f) \in \mathcal{C}(N, M^{coH})$ is defined as



whereas for $g \in \mathcal{C}(N, M^{coH})$ the image $\Psi(g) \in \mathcal{C}^H_A(N \otimes A, M)$ is given by

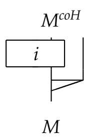


It is easy to check that $\Theta(f)$ and $\Psi(g)$ are well defined, that they are inverses of each other and naturality. □

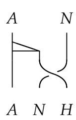
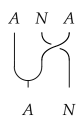
The unit of the adjunction $\alpha_N = \Theta_{N, N \otimes A}(id_{N \otimes A}) : N \to (N \otimes A)^{coH}$ is the unique morphism such that

$$i_{N \otimes A} \alpha_N = N \otimes \eta_A. \tag{3.2}$$

The counit of the adjunction $\beta_M = \Psi_{M^{coH}, M}(id_{M^{coH}}) : M^{coH} \otimes A \to M$ is given via

$\beta_M =$  (3.3)

Remark 3.4 Analogously, the pair of functors $(A \otimes -, (-)^{coH})$ is also an adjoint pair between the same categories, with counit $\beta' : A \otimes M^{coH} \longrightarrow M$ given by $\beta'_M = \beta_M \Phi_{A, M^{coH}}$. For $N \in \mathcal{C}$ we consider $A \otimes N$ as a relative Hopf module with the right H-comodule and A-module structures given by

$\rho_{A \otimes N} =$  and $\mu_{A \otimes N} =$  (3.4)

Using **2.2** and **2.3** one obtains:

Lemma 3.5 *If H is flat, then an equalizer in C of two morphisms in C_A^H is an equalizer in C_A^H. Moreover, the forgetful functor $\mathcal{U} : C_A^H \longrightarrow C$ preserves equalizers.*

When $A = H$ the category C_H^H is called the *category of Hopf modules*. In this setting, the Fundamental Theorem for Hopf modules was established in [20, Theorem 1.1]. We formulate it here.

Theorem 3.6 (Fundamental Theorem of Hopf modules) *Let H be a flat Hopf algebra in C. Then the pair of functors $-\otimes H : C \rightleftarrows C_H^H : (-)^{coH}$ establishes an equivalence. In particular, $M^{coH} \otimes H \cong M$ for all $M \in C_H^H$.*

3.2. Galois objects

In this section we will prove that the category of relative Hopf modules C_A^H admits an equivalence with C when A is an H-Galois object, generalizing the Fundamental Theorem of Hopf Modules.

Definition 3.7 *Let $H \in C$ be a Hopf algebra. A right H-comodule algebra A in C, with structure morphism ρ_A, is called an H-Galois object if the following two conditions are satisfied:*

(i) A is faithfully flat;

(ii) The canonical morphism $can : A \otimes A \xrightarrow{A\otimes\rho_A} A\otimes A\otimes H \xrightarrow{\nabla_A\otimes H} A\otimes H$ is an isomorphism.

Consider $A\otimes A$ and $A\otimes H$ as right H-comodules by the structure morphisms $A\otimes\rho_A$ and $A\otimes\Delta$, respectively. Then *can* is right H-colinear. Viewing $A \otimes A$ as a right A-module by the structure morphism $A \otimes \nabla$ and equipping $A \otimes H$ with the codiagonal structure, *can* is also right A-linear.

In [36, Lemma 1.1] the author proved that if H is a commutative Hopf algebra over a field K, then $A^{coH} \cong K$, for a commutative H-Galois object A. A similar statement, when working over a commutative ring R, was proved in [24, Lemma 2.9] for H^*-Galois objects, where H is now finite and cocommutative. We show next that this result for the category of vector spaces (and R-modules) extends to any braided monoidal category with equalizers even for a not necessarily commutative Hopf algebra. We will need first to prove this statement for H, although we will do it more generally for future purposes:

Lemma 3.8 *Let $H \in C$ be a Hopf algebra and $M \in C$. The diagram*

$$M \xrightarrow{M\otimes\eta_H} M\otimes H \underset{M\otimes H\otimes\eta_H}{\overset{M\otimes\Delta_H}{\rightrightarrows}} M\otimes H\otimes H$$

is an equalizer. Hence there is a natural isomorphism

$$\delta_M : M \longrightarrow (M\otimes H)^{coH} \tag{3.5}$$

satisfying $i_{M\otimes H}\delta_M = M\otimes\eta_H$.

Proof. Clearly $(M\otimes\Delta_H)(M\otimes\eta_H) = M\otimes\eta_H\otimes\eta_H = (M\otimes H\otimes\eta_H)(M\otimes\eta_H)$. Let $f : T \longrightarrow M\otimes H$ be a morphism in C such that $(M\otimes\Delta_H)f = (M\otimes H\otimes\eta_H)f$. Applying to this $M\otimes\varepsilon_H\otimes H$, we obtain $f = (M\otimes\varepsilon_H\otimes\eta_H)f$. We define $g : T \longrightarrow M$ by $g = (M\otimes\varepsilon_H)f$. From the above, $f = (M\otimes\eta_H)(M\otimes\varepsilon_H)f = (M\otimes\eta_H)g$. Moreover, g is the unique such a morphism because $M\otimes\eta_H$ is a monomorphism due to $(M\otimes\varepsilon_H)(M\otimes\eta_H) = id_M$.

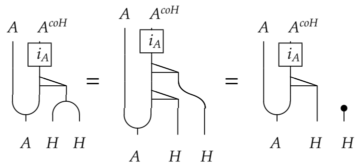
We know that $((M\otimes H)^{coH}, i_{M\otimes H})$ is an equalizer of the same pair of morphisms. Then $(M, M\otimes\eta_H)$ and $((M\otimes H)^{coH}, i_{M\otimes H})$ will be isomorphic as equalizers by a unique isomorphism $\delta_M : M \longrightarrow (M\otimes H)^{coH}$ such that $i_{M\otimes H}\delta_M = M\otimes\eta_H$. The naturality of δ_M is easy to check. □

Proposition 3.9 *Let A be an H-Galois object in $\mathcal{C}$. There is an isomorphism $\overline{\eta} : I \longrightarrow A^{coH}$ such that $i_A\overline{\eta} = \eta_A$. In particular,*

$$I \xrightarrow{\eta_A} A \underset{A\otimes\eta_H}{\overset{\rho_A}{\rightrightarrows}} A\otimes H \tag{3.6}$$

is an equalizer.

Proof. From the claim we get that (I, η_A) and (A^{coH}, i_A) are isomorphic as equalizers. Notice that $can(A \otimes i_A) : A \otimes A^{coH} \longrightarrow A \otimes H$ factors through $A \otimes H^{coH}$ since



By flatness of A and the preceding lemma the following diagram is an equalizer.

$$A\otimes I \xrightarrow{A\otimes\eta_H} A\otimes H \underset{A\otimes H\otimes\eta_H}{\overset{A\otimes\Delta_H}{\rightrightarrows}} A\otimes H\otimes H$$

Then $A \otimes H^{coH} \cong A \otimes I$. This assures the existence of $\varphi : A \otimes A^{coH} \longrightarrow A \otimes I$ such that

$$(A \otimes \eta_H)\varphi = can(A \otimes i_A). \tag{3.7}$$

Clearly, η_A factors through A^{coH} since it is H-colinear. Therefore there is $\overline{\eta} : I \longrightarrow A^{coH}$ with $i_A\overline{\eta} = \eta_A$. We show that $A \otimes \overline{\eta}$ is the inverse of φ,

$$(A \otimes \eta_H)\varphi(A \otimes \overline{\eta}) = can(A \otimes i_A)(A \otimes \overline{\eta}) = can(A \otimes \eta_A) = A \otimes \eta_H.$$

Then $\varphi(A \otimes \overline{\eta}) = id_A$. On the other hand,

$$can(A \otimes i_A)(A \otimes \overline{\eta})\varphi \overset{(3.7)}{=} (A \otimes \eta_H)\varphi(A \otimes \overline{\eta})\varphi = (A \otimes \eta_H)\varphi \overset{(3.7)}{=} can(A \otimes i_A).$$

Since $A \otimes i_A$ is a monomorphism, clearly so is $can(A \otimes i_A)$, and thus we obtain $(A \otimes \overline{\eta})\varphi = id_{A\otimes A^{coH}}$. This proves that $A \otimes \overline{\eta} : A \longrightarrow A \otimes A^{coH}$ is an isomorphism. By the faithful flatness of A we finally get that $\overline{\eta} : I \longrightarrow A^{coH}$ is an isomorphism. □

Remark 3.10 Our definition of H-Galois object is stronger than Schauenburg's one [31, Definition 3.1] in view of the preceding result. He defines an H-Galois object as an H-comodule algebra A such that (3.6) is an equalizer and $can : A \otimes A \to A \otimes H$ is an isomorphism. However, to define the group of H-biGalois objects he considers faithfully flat H-Galois objects. This is one reason to make our definition. Another one is that it allows to characterize when the adjunction from Proposition 3.3 is an equivalence, as we see next.

Theorem 3.11 *Let $A \in \mathcal{C}$ be a right H-comodule algebra and suppose that H is flat. The following statements are equivalent:*

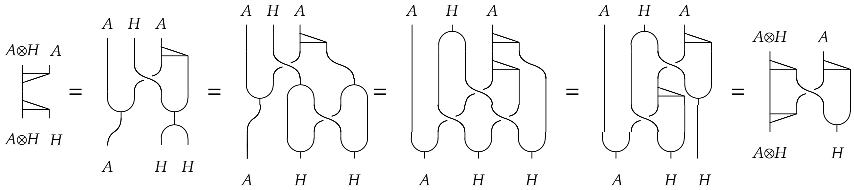
(i) A is a right H-Galois object;

(ii) The functors $-\otimes A : \mathcal{C} \rightleftarrows \mathcal{C}_A^H : (-)^{coH}$ establish an equivalence of categories.

Proof. $(i) \Rightarrow (ii)$ This is proved in [31]. The counit of the adjunction $\beta_M : M^{coH} \otimes A \longrightarrow M$ is given by $\beta_M = \mu(i \otimes A)$, where μ is the action of A on M (Diagram 3.3). It was shown in [31, Proposition 3.8] that β_M is an isomorphism. Recall from (3.2) that the unit of the adjunction $\alpha_N : N \longrightarrow (N \otimes A)^{coH}$ is induced by $N \otimes \eta_A$. In [31, Lemma 3.9] it is proved that α_N is an isomorphism. Notice that for this faithful flatness of A is needed.

(ii) $\Rightarrow$ (i) Being an equivalence of categories, the functor $-\otimes A : C \longrightarrow C_A^H$ preserves equalizers. By Lemma 3.5, A is flat. Suppose that $f\otimes A$ is an isomorphism in C for $f: M\longrightarrow N$ in C. Lying in C_A^H, it is then an isomorphism also in there. Then $\mathcal{G}(f\otimes A)=\mathcal{GF}(f)$ (and hence f) is an isomorphism, yielding that A is faithfully flat.

We finally prove that $can : A\otimes A\longrightarrow A\otimes H$ is an isomorphism. First of all, note that $A\otimes H\in C_A^H$. It is a right A-module by the codiagonal structure and an H-comodule with coaction $A\otimes\Delta$. We show that the compatibility condition is satisfied:



Let $\delta_A : A\longrightarrow (A\otimes H)^{coH}$ be the isomorphism from (3.5). Observe that we have an isomorphism

$$\nu : A\otimes A=\mathcal{F}(A)\stackrel{\mathcal{F}(\delta_A)}{\longrightarrow}\mathcal{F}((A\otimes H)^{coH})=\mathcal{FG}(A\otimes H)\stackrel{\beta_{A\otimes H}}{\longrightarrow} A\otimes H.$$

Now:

$$\nu=\beta_{A\otimes H}\mathcal{F}(\delta_A)=\beta_{A\otimes H}(\delta_A\otimes A)=\mu_{A\otimes H}(i_{A\otimes H}\otimes A)(\delta_A\otimes A)=\mu_{A\otimes H}(A\otimes\eta_H\otimes A)=can.$$

Thus can is an isomorphism. $\square$

Remark 3.12 (1) The above theorem is also true for the adjoint pair of functors

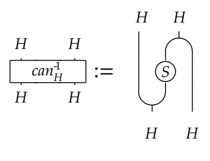
$$A\otimes - : C \rightleftarrows C_A^H : (-)^{coH}$$

(2) For the above theorem we were inspired by [7, Theorems 8.1.6 and 8.1.8], where a similar result is established for C the category of R-modules (R a commutative ring). Notice that, due to a transposition of letters when defining the unit of adjunction ψ_N in [7, Remark 8.1.4(3)], the author is led to state wrongly that ψ_N is an isomorphism if and only if $A^{co(H)}$ is trivial. To claim that ψ_N is an isomorphism A faithfully flat is needed.

By Theorem 3.6 for a flat Hopf algebra H the functors $-\otimes H : C \rightleftarrows C^H : (-)^{coH}$ are an equivalence of categories. By the above theorem we get:

Corollary 3.13 *A flat Hopf algebra $H\in C$ is an H-Galois object, in particular faithfully flat.*

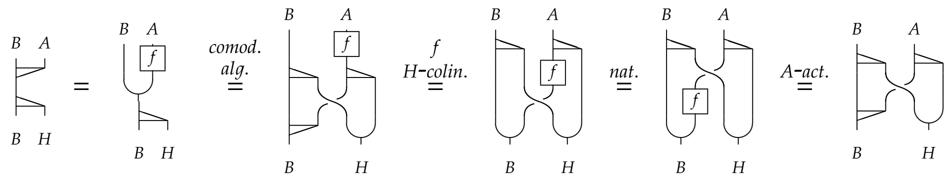
The inverse of can_H is:



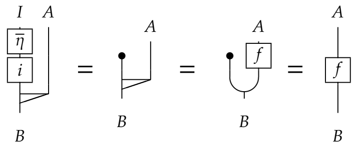
The next proposition will be very useful in the future to check that a comodule algebra morphism between two Galois objects is an isomorphism. It is a generalization to any braided monoidal category of a result from [7] for the category of modues over a commutative ring.

Proposition 3.14 *Let $H\in C$ be a flat Hopf algebra. An H-comodule algebra morphism $f: A\longrightarrow B$ between two H-Galois objects A and B is an isomorphism.*

Proof. As an H-Galois object, B is a right H-comodule. Equip it with the right A-module structure given by $\mu_B := \nabla_B(B \otimes f)$. With these structures, B lies in $\mathcal{C}^H_A$:



Having that A is an H-Galois object, by the preceding theorem the counit β of the adjunction $-\otimes A : \mathcal{C} \rightleftarrows \mathcal{C}^H_A : (-)^{coH}$, given in (3.3), is an isomorphism. Let $\overline{\eta} : I \longrightarrow B^{coH}$ be the isomorphism from Proposition 3.9. Then we obtain that

$\beta_B(\overline{\eta} \otimes A) =$ 

is an isomorphism. □

4. The group of Galois and biGalois objects

In this section we will construct, under an assumption on the braiding, the group of H-Galois objects for a flat and cocommutative Hopf algebra H. Our construction relies on the construction of the group of biGalois objects due to Schauenburg [31]. We will also construct, dropping the assumption on the braiding, the group of H-Galois objects with a normal basis.

4.1. The cotensor product

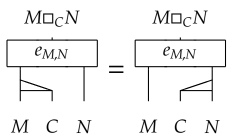
We record in this first subsection several properties of the cotensor product, that will give the group law in the set of isomorphism classes of Galois and biGalois objects.

Definition 4.1 *Let $C \in \mathcal{C}$ be a coalgebra, M a right C-comodule and N a left C-comodule. The cotensor product of M and N is the equalizer*

$$M\square_C N \xrightarrow{\ e\ } M \otimes N \underset{M\otimes\lambda_N}{\overset{\rho_M \otimes N}{\rightrightarrows}} M \otimes C \otimes N. \tag{4.8}$$

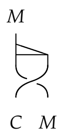
where ρ_M and λ_N are the structure morphisms of M and N respectively.

In braided diagrams the equalizer property of $M\square_C N$ reads as:



For morphisms $f : M \longrightarrow M'$ in $\mathcal{C}^C$ and $g : N \longrightarrow N'$ in ${}^C\mathcal{C}$, $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ induces $f\square_C g : M\square_C N \longrightarrow M'\square_C N'$ in $\mathcal{C}$. Thus we have a bifunctor $-\square_C- : \mathcal{C}^C \times {}^C\mathcal{C} \to \mathcal{C}$.

Remark 4.2 When C is cocommutative every right C-comodule M is a left C-comodule (indeed a C-bicomodule) via $\lambda_M = \Phi_{C,M}^{-1}\rho_M$. In braided diagrams:



(4.9)

We will work with right C-comodules and convert them into left ones using (4.9). Thus we will be able to make cotensor products of two right C-comodules when C is cocommutative.

Lemma 4.3 *Let M be a right C-comodule, N a left C-comodule and X a flat object in $\mathcal{C}$. There are natural isomorphisms of equalizers*

$$\theta_{M,N,X} : (M\square_C N)\otimes X \to M\square_C(N\otimes X) \quad \text{and} \quad \kappa_{M,N,X} : X\otimes(M\square_C N) \to (X\otimes M)\square_C N.$$

Here, the structure of left C-comodule of $N\otimes X$ is inherited from N, that is, $\lambda_{N\otimes X} = \lambda_N \otimes X$.

Proof. The universal property of the equalizers $(M\square_C(N\otimes X), e_{M,N\otimes X})$ and $((X\otimes M)\square_C N, e_{X\otimes M,N})$ gives the existence of $\theta_{M,N,X}$ and $\kappa_{M,N,X}$ such that the diagrams

$$\begin{array}{ccc} (M\square_C N)\otimes X & \xrightarrow{e_{M,N}\otimes X} & (M\otimes N)\otimes X \\ \downarrow{\scriptstyle \theta_{M,N,X}} & & \downarrow{\scriptstyle \alpha_{M,N,X}} \\ M\square_C(N\otimes X) & \xrightarrow{e_{M,N\otimes X}} & M\otimes(N\otimes X) \end{array} \quad (4.10) \qquad \begin{array}{ccc} X\otimes(M\square_C N) & \xrightarrow{X\otimes e_{M,N}} & X\otimes(M\otimes N) \\ \downarrow{\scriptstyle \kappa_{X,M,N}} & & \downarrow{\scriptstyle \alpha^{-1}_{X,M,N}} \\ (X\otimes M)\square_C N & \xrightarrow{e_{X\otimes M,N}} & (X\otimes M)\otimes N \end{array} \quad (4.11)$$

commute. Flatness of X makes $((M\square_C N)\otimes X, e_{M,N}\otimes X)$ into an equalizer. By its universal property, $\alpha^{-1}_{M,N,X}e_{M,N\otimes X}$ induces a morphism $\theta'_{M,N,X} : M\square_C(N\otimes X) \longrightarrow (M\square_C N)\otimes X$, which is the inverse of θ. The naturality of θ follows from the naturality of α and the universal property of the equalizer. Similarly, $\kappa_{X,M,N}$ is an isomorphism. □

Remark 4.4 Let C, D and E be coalgebras in $\mathcal{C}$, where C and E are flat. For $M \in {}^C\mathcal{C}^D$ and $N \in {}^D\mathcal{C}^E$ it can be proved that $M\square_D N$ is a left C- and a right E-comodule by $\kappa^{-1}_{C,M,N}(\lambda_M\square_D N)$ and $\theta^{-1}_{M,N,E}(M\square_D\rho_N)$ respectively. Moreover, the equalizer morphism $e_{M,N}$ becomes C-E-bicolinear. If F is another flat coalgebra and $X\in\mathcal{C}^F$, the morphism $\theta_{M,N,X} : (M\square_D N)\otimes X \to M\square_D(N\otimes X)$ is of C-F-bicomodules. The same is true for $\kappa_{X,M,N} : X\otimes(M\square_D N) \to (X\otimes M)\square_D N$ if X is now a left F-comodule.

We will say that $M \in {}^C\mathcal{C}^D$ is *coflat* in $\mathcal{C}^D$ if the morphism $\theta_{M,N,X}$ is an isomorphism. This definitioin is taken from [25, Page 202] although formulated in the opposite category. Nevertheless, our definition is stronger since we are assuming that $\mathcal{C}$ has equalizers and that the coalgebras C, D and any other one involved are flat to assure that the category of bicomodules also has equalizers.

In view of (4.9) and the preceding considerations we have:

Corollary 4.5 *Let C be a flat cocommutative coalgebra in $\mathcal{C}$. For two right C-comodules M and N we have that $M\square_C N$ is a C-bicomodule with right structure morphism $\theta^{-1}_{M,N,C}(M\square_C\rho_N)$ and left structure morphism $\kappa^{-1}_{C,M,N}(\lambda_M\square_C N)$, where $\lambda_M = \Phi^{-1}_{C,M}\rho_M$.*

The following result is easy to prove.

Lemma 4.6 *For a left C-comodule M its structure morphism $\lambda_M : M \longrightarrow C\otimes M$ factors through $\overline{\lambda}_M : M \longrightarrow C\square_C M$. This gives a natural isomorphism with inverse $\overline{\pi}_M : C\square_C M \longrightarrow M$ induced by $\varepsilon\otimes M$. If additionally $M \in {}^C\mathcal{C}^D$ (and C and D are flat), then this is an isomorphism of C-D-bicomodules. Analogously, it is $M\square_D D \cong M$ as C-D-bicomodules.*

We next point out two sufficient conditions for the cotensor product to be associative.

Lemma 4.7 *Let C and D be flat coalgebras in $\mathcal{C}$ and $M \in \mathcal{C}^C$ flat. For every $N \in {}^C\mathcal{C}^D$ and $L \in {}^D\mathcal{C}$ it is $M\Box_C(N\Box_D L) \cong (M\Box_C N)\Box_D L$ as equalizers if one of the following two conditions is satisfied:*

(i) L is flat;
(ii) M is coflat in $\mathcal{C}^C$.

If, in addition, $M \in {}^E\mathcal{C}^C$ and $L \in {}^D\mathcal{C}^F$, where E and F are flat coalgebras, then this isomorphism is of E-F-bicomodules.

Proof. We view $N\Box_D L$ as a left C-comodule and $M\Box_C N$ as a right D-comodule with the structures from Remark 4.4. Consider the diagram

$$\begin{array}{ccccc}
(M\Box_C N)\Box_D L & \xrightarrow{e_{M\Box_C N,L}} & (M\Box_C N)\otimes L & \overset{\rho_{M\Box_C N}\otimes L}{\underset{(M\Box_C N)\otimes\lambda_L}{\rightrightarrows}} & (M\Box_C N)\otimes D\otimes L \\
\downarrow{\scriptstyle e_{M,N}\Box_D L} & & \downarrow{\scriptstyle e_{M,N}\otimes L} & & \downarrow{\scriptstyle e_{M,N}\otimes D\otimes L} \\
(M\otimes N)\Box_D L & \xrightarrow{e_{M\otimes N,L}} & (M\otimes N)\otimes L & \overset{\rho_{M\otimes N}\otimes L}{\underset{(M\otimes N)\otimes\lambda_L}{\rightrightarrows}} & (M\otimes N)\otimes D\otimes L \\
{\scriptstyle (\rho_M\otimes N)\Box_D L}\downdownarrows{\scriptstyle (M\otimes\lambda_N)\Box_D L} & & {\scriptstyle (\rho_M\otimes N)\otimes L}\downdownarrows{\scriptstyle (M\otimes\lambda_N)\otimes L} & & {\scriptstyle (\rho_M\otimes N)\otimes D\otimes L}\downdownarrows{\scriptstyle (M\otimes\lambda_N)\otimes D\otimes L} \\
(M\otimes C\otimes N)\Box_D L & \xrightarrow{e_{M\otimes C\otimes N,L}} & (M\otimes C\otimes N)\otimes L & \overset{\rho_{M\otimes C\otimes N}\otimes L}{\underset{(M\otimes C\otimes N)\otimes\lambda_L}{\rightrightarrows}} & (M\otimes C\otimes N)\otimes D\otimes L
\end{array}$$

The three rows are equalizers. Assume that L is flat. Then by Lemma 4.3 the second and – since D is flat too – the third column are equalizers. The same conclusion holds if we suppose that M is coflat in $\mathcal{C}^C$, because then $M\Box_C(N\otimes L) \cong (M\Box_C N)\otimes L$ and $M\Box_C(N\otimes D\otimes L) \cong (M\Box_C N)\otimes D\otimes L$. Note that all inner squares commute. Then by the equalizer version of Lemma 3×3 (see [4, Exercise 2.2.3.13]), the first column is an equalizer too.

We show that $(M\Box_C(N\Box_D L), e_{M,N\Box_D L})$ and $((M\Box_C N)\Box_D L, e_{M,N}\Box_D L)$ are isomorphic as equalizers. For this purpose we consider the diagram

$$\begin{array}{ccccc}
M\Box_C(N\Box_D L) & \xrightarrow{e_{M,N\Box_D L}} & M\otimes(N\Box_D L) & \overset{\rho_M\otimes(N\Box_D L)}{\underset{M\otimes\lambda_{N\Box_D L}}{\rightrightarrows}} & M\otimes C\otimes(N\Box_D L) \\
\downarrow{\scriptstyle \omega_{M,N,L}} & \boxed{1} & \downarrow{\scriptstyle \kappa_{M,N,L}} & & \downarrow{\scriptstyle \kappa_{M\otimes C,N,L}} \\
(M\Box_C N)\Box_D L & \xrightarrow{e_{M,N}\Box_D L} & (M\otimes N)\Box_D L & \overset{(\rho_M\otimes N)\Box_D L}{\underset{(M\otimes\lambda_N)\Box_D L}{\rightrightarrows}} & (M\otimes C\otimes N)\Box_D L
\end{array}$$

Since M and C are flat, $\kappa_{M,N,L}$ and $\kappa_{M\otimes C,N,L}$ are isomorphisms (Lemma 4.3). The right square obviously commutes with upper lines. It commutes with lower lines as well, because $\lambda_{N\Box_D L}$ is induced by $\lambda_N\Box_D L$. Knowing that both rows are equalizers, $\kappa_{M,N,L}e_{M,N\Box_D L}$ induces $\omega_{M,N,L}$ so that the left square commutes. Similarly, $\kappa^{-1}_{M,N,L}(e_{M,N}\Box_D L)$ induces the inverse of $\omega_{M,N,L}$.

Suppose $M \in {}^E\mathcal{C}^C$ and $L \in {}^D\mathcal{C}^F$. Due to Remark 4.4, $e_{M,N\Box_D L}$ is E-F-bicolinear and $e_{M,N}$ is left E-colinear. Hence $e_{M,N}\Box_D L$ is E-F-bicolinear, as so is $\kappa_{M,N,L}$, by Remark 4.4. Then because of **2.4(i)**, $\omega_{M,N,L}$ is E-F-bicolinear. □

Corollary 4.8 *Let M, N and L be right C-comodules where C is a flat and cocommutative coalgebra. If M and L are flat, then $M\Box_C(N\Box_C L) \cong (M\Box_C N)\Box_C L$ as equalizers and as C-bicomodules.*

4.2. The group of Galois and biGalois objects

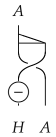
We start by recalling the construction of Schauenburg's group of biGalois objects. We will assume that the antipode is an isomorphism which allows to describe easier the inverse element.

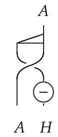
Definition 4.9 *Let $H \in \mathcal{C}$ be a Hopf algebra. An H-bicomodule algebra $A \in \mathcal{C}$ is said to be an H-biGalois object if it is a right H-Galois object and a left H-Galois object.*

Let A be a right H-comodule algebra and B a left H-comodule algebra. Then $A\Box_H B$ may be seen as the equalizer of two algebra morphisms and by **2.2(i)** it admits an algebra structure so that $e_{A,B} : A\Box_H B \to A \otimes B$ is an algebra morphism ([31, Lemma 2.3(2)]). If, in addition, A, B are H-bicomodule algebras and H is flat, by **2.2(ii)** then $A\Box_H B$ becomes an H-bicomodule algebra, with left structure inherited from A and right one inherited from B, such that $e_{A,B}$ is an H-bicomodule algebra morphism. The next result is [31, Theorems 5.2 and 6.6].

Theorem 4.10 *Let H be a flat Hopf algebra whose antipode is an isomorphism.*

(i) For an H-biGalois object A in $\mathcal{C}$, let $\overline{A}$ denote the opposite algebra of A ($\nabla_{\overline{A}} = \nabla_A \Phi_{A,A}$) with H-bicomodule structure given by

$\lambda_{\overline{A}} =$ 

and $\rho_{\overline{A}} =$ 

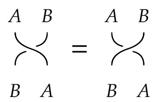
where the sign minus stands for S^{-1}. Then $\overline{A}$ is an H-biGalois object and there are H-bicomodule algebra isomorphisms $A\Box_H\overline{A} \cong H$ and $\overline{A}\Box_H A \cong H$.

(ii) The set $\text{BiGal}(\mathcal{C}; H)$ of isomorphism classes of H-biGalois objects is a group with multiplication $[A][B] = [A\Box_H B]$, identity element $[H]$ and the inverse of $[A]$ is $[\overline{A}]$.

For further purposes we must describe how the isomorphisms $H \cong A\Box_H\overline{A}$ and $H \cong \overline{A}\Box_H A$ are induced. This is explained in detail in [31, Remark 3.5]. Consider the morphism $\gamma_r := can_r^{-1}(\eta_A \otimes H) : H \longrightarrow A \otimes A$, where can_r denotes the canonical isomorphism of A as a right H-Galois object. Then $\gamma_r : H \to \overline{A} \otimes A$ is an algebra morphism. Endowing $A \otimes A$ with the codiagonal H-comodule structure, γ_r factors through ${}^{coH}(A \otimes A)$. On the other hand, ${}^{coH}(A \otimes A)$ is isomorphic, as an equalizer, to $\overline{A}\Box_H A$ (left version of [31, Lemma 2.4]). Then, there is a unique morphism $\tilde{\gamma}_r : H \longrightarrow \overline{A}\Box_H A$ such that $e_{\overline{A},A}\tilde{\gamma}_r = \gamma_r$. In [31, Lemma 4.2] it is proved that $\tilde{\gamma}_r$ is an isomorphism of H-bicomodule algebras. Symmetrically, there is an isomorphism of H-bicomodule algebras $\tilde{\gamma}_l : H \to A\Box_H\overline{A}$ such that $e_{A,\overline{A}}\tilde{\gamma}_l = \gamma_l$ where now $\gamma_l = can_l^{-1}(H \otimes \eta_A) : H \longrightarrow A \otimes A$.

We proceed to prove the final necessary results to construct the group $\text{Gal}(\mathcal{C}; H)$ of H-Galois objects for a cocommutative Hopf algebra H. We will construct it as a subgroup of $\text{BiGal}(\mathcal{C}; H)$. Since H is cocommutative, we can consider every right H-comodule as a left H-comodule and as an H-bicomodule via (4.9). This fact introduces a restriction to construct $\text{Gal}(\mathcal{C}; H)$, as we will show next, and we are forced to make the following assumption, satisfied when $\mathcal{C}$ is symmetric. In Section 6 a non-symmetric case where this holds is discussed.

Assumption 4.11 *For any two H-Galois objects A and B the equality*

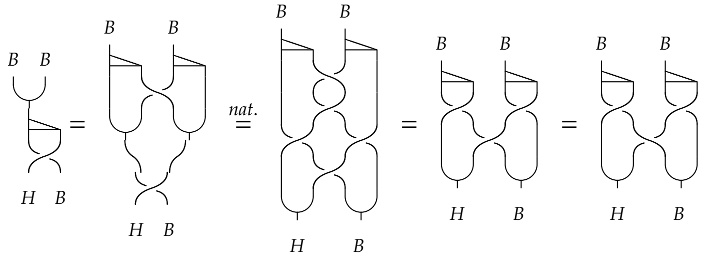


holds, i.e., $\Phi_{A,B} = \Phi_{B,A}^{-1}$. We say that the braiding acting between two H-Galois objects is symmetric.

Dealing with bicomodules (where the left and right comodule structures are not necessarily related) gives a freedom when manipulating with biGalois objects, which we do not have when only handling one sided comodule structure. In the latter case one is conditioned in order that the compatibility conditions be fulfilled. The following two lemmas are examples of this.

Lemma 4.12 *Let H be a cocommutative Hopf algebra, B a right H-comodule algebra and suppose that $\Phi_{B,H} = \Phi_{H,B}^{-1}$. Then B is a left H-comodule algebra.*

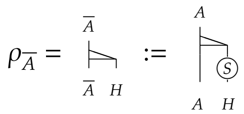
Proof. Recall that B is a left H-comodule with structure morphism $\lambda_B = \Phi^{-1}_{H,B}\rho_B$ by (4.9). We prove that λ_B is an algebra morphism. We compute:



where in the last equation we used the assumption $\Phi_{B,H} = \Phi^{-1}_{H,B}$. The compatibility with unit is obvious. □

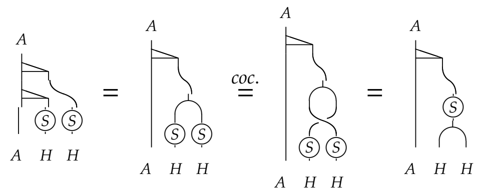
Lemma 4.13 *Let H be a cocommutative and flat Hopf algebra and A a right H-comodule algebra. Under Assumption 4.11:*

(i) $\overline{A}$ is a right H-comodule algebra with right H-comodule structure given by

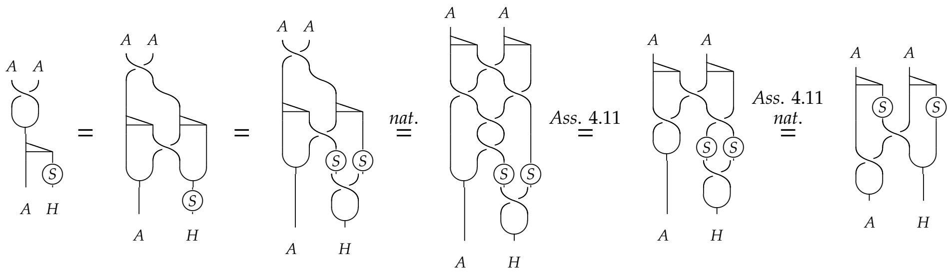


(ii) If A is an H-Galois object, then so is $\overline{A}$ with the above H-comodule structure.

Proof. (i) First let us prove that $\rho_{\overline{A}}$ endows $\overline{A}$ with a right H-comodule structure. We compute

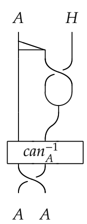


For the compatibility of the multiplication and the right H-comodule structure of $\overline{A}$ we compute

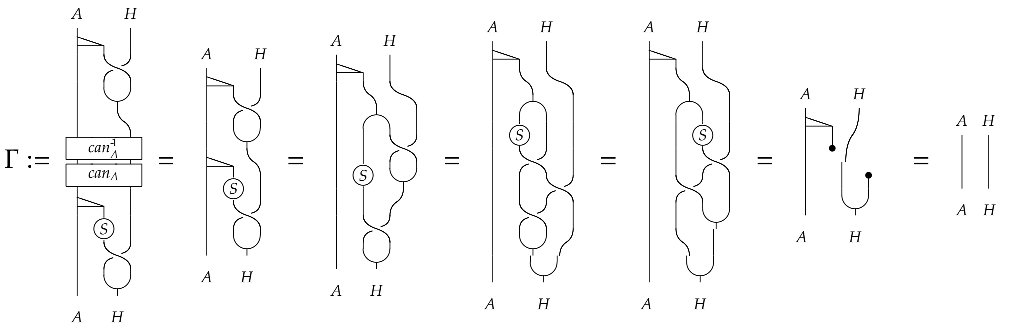


Notice that H is an H-Galois object because H is flat (Corollary 3.13) and we may use Assumption 4.11. The compatibility of the unit and the right H-comodule structure of $\overline{A}$ is clear.

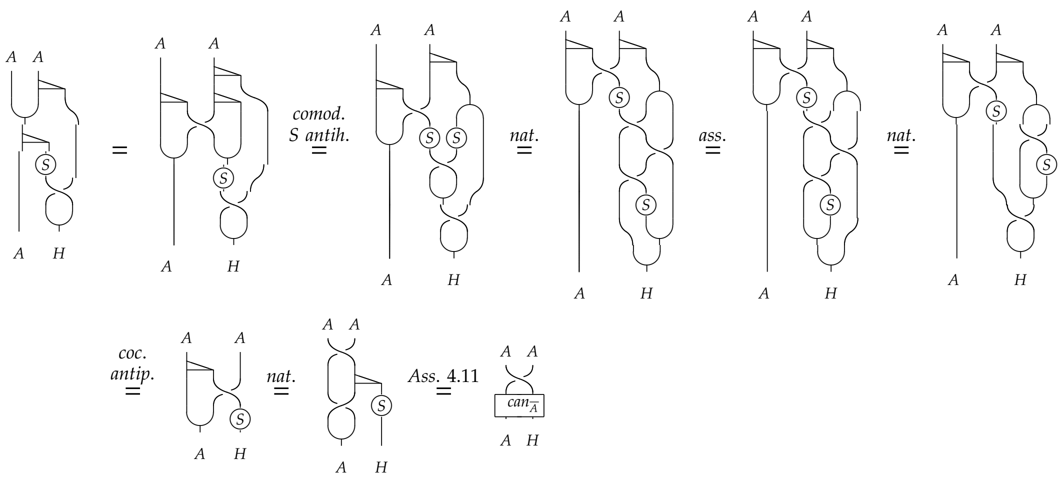
(ii) Obviously $\overline{A}$ is faithfully flat. Let us prove that $can_{\overline{A}}$ is an isomorphism. We will show that $\vartheta : \overline{A} \otimes H \to \overline{A} \otimes \overline{A}$ given by



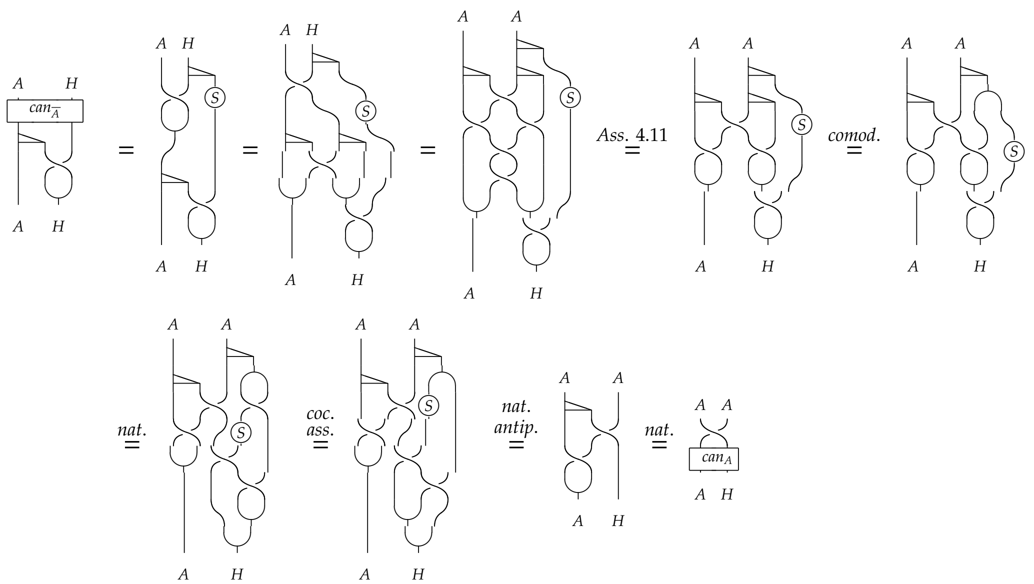
is the inverse of $can_{\overline{A}}$. Compute the following composition:



On the other hand, for the lower half of the diagram Γ we have



This implies that ϑ is a right inverse for $can_{\overline{A}}$. On the other hand:

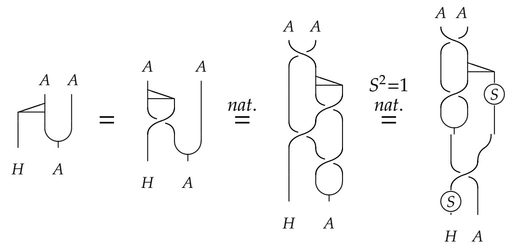


From this it follows that ϑ is a left inverse of $can_{\overline{A}}$ (note that Assumption 4.11 is being used). □

We are now in a position to construct the group of Galois objects.

Theorem 4.14 *If H is a flat and cocommutative Hopf algebra in $\mathcal{C}$ and the Assumption 4.11 is fulfilled, then the set* $\mathrm{Gal}(\mathcal{C};H)$ *of isomorphism classes of (right) H-Galois objects is an abelian subgroup of* $\mathrm{BiGal}(\mathcal{C};H)$.

Proof. Notice first that, since H is cocommutative, the square of the antipode is the identity. Let A be a right H-Galois object. Make it into a left H-comodule and an H-bicomodule via $\lambda_A = \Phi^{-1}_{H,A}\rho_A$. Under the Assumption 4.11, A is a left H-comodule algebra with the above structure by Lemma 4.12. We claim that A is a left H-Galois object and consequently an H-biGalois object. We will denote it by ${}^\bullet A$. The morphism $can^l_A : A \otimes A \longrightarrow H \otimes A$ is an isomorphism because it may be written as a composition of the following isomorphisms:

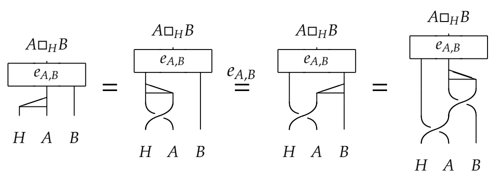
$can^l_A =$  $= (S \otimes A)\Phi^{-1}_{H,A} can^r_{\overline{A}} \Phi_{A,A}.$

Bear in mind that in Lemma 4.13(ii) we showed that $can^r_{\overline{A}}$ is an isomorphism.

The map $j : \mathrm{Gal}(\mathcal{C};H) \to \mathrm{BiGal}(\mathcal{C};H), [A] \to [{}^\bullet A]$, is clearly injective. Hence we can see $\mathrm{Gal}(\mathcal{C};H)$ as a subset of $\mathrm{BiGal}(\mathcal{C};H)$. We next prove that it is indeed a subgroup.

Since the square of the antipode is the identity, the inverse of a biGalois object is described in Theorem 4.10(i). Let A be a right H-Galois object and consider the H-biGalois object ${}^\bullet A$. Consider $\overline{{}^\bullet A}$ with structures as in Theorem 4.10(i). Notice that the right H-comodule structure coincides with the structure of $\overline{A}$ in Proposition 4.13 due to Assumption 4.11 and that the left structure stems from this one as in (4.9). This means that $[\overline{{}^\bullet A}] \in \mathrm{Gal}(\mathcal{C};H)$.

Pick two right H-Galois objects A and B and make them into H-biGalois objects. Consider the cotensor product ${}^\bullet A \Box_H {}^\bullet B$. Its right (resp. left) H-comodule structure is inherited from the one of B (resp. A). The left H-comodule structure comes from the right one as in (4.9) in view of the following computation:

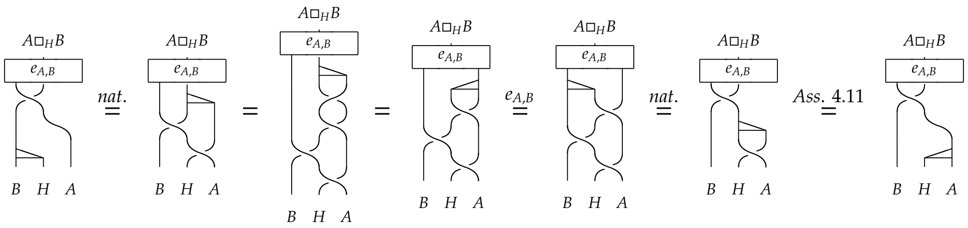


Therefore $[{}^\bullet A][{}^\bullet B] \in \mathrm{Gal}(\mathcal{C};H)$.

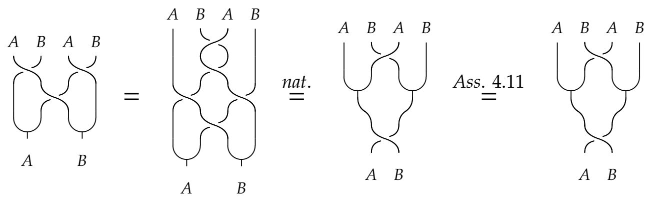
We finally show that $\mathrm{Gal}(\mathcal{C};H)$ is abelian. We check that the braiding induces a morphism $\Psi : A\Box_H B \longrightarrow B\Box_H A$ that will be an H-comodule algebra morphism. It will be well defined if we prove that in the diagram

$$\begin{array}{ccccc} A\Box_H B & \xrightarrow{e_{A,B}} & A \otimes B & & \\ \Psi\downarrow & & \downarrow \Phi_{A,B} & & \\ B\Box_H A & \xrightarrow{e_{B,A}} & B \otimes A & \overset{\rho_B \otimes A}{\underset{B \otimes \lambda_A}{\rightrightarrows}} & B \otimes H \otimes A \end{array} \tag{4.12}$$

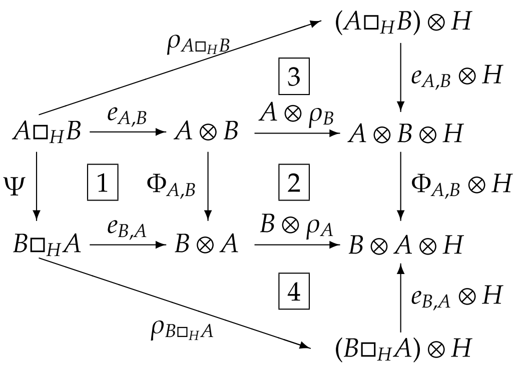
it is $(\rho_B \otimes A)\Phi_{A,B}e_{A,B} = (B \otimes \lambda_A)\Phi_{A,B}e_{A,B}$. We have:



Note that $\Phi^{-1}_{B,A}$ induces a morphism in the other direction – changing the sign of the braiding in the upper diagrams gives analogous computation. This induced morphism will be the inverse of Ψ. We will now prove that Ψ is an algebra morphism. We know that $e_{A,B}$ and $e_{B,A}$ are algebra morphisms. Then by **2.4(iii)**, Ψ will be an algebra morphism if we show that so is $\Phi_{A,B}$. This is true since we have



We finally check that Ψ is right H-colinear. Consider the diagram



We have that Diagram ⟨2⟩ commutes when composed with $e_{A,B}$:

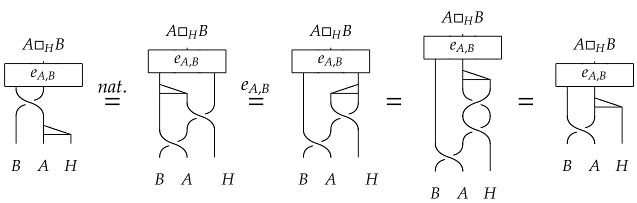


Diagram ⟨1⟩ commutes by the definition of Ψ. Diagrams ⟨3⟩ and ⟨4⟩ commute by the definitions of $\rho_{A\Box_H B}$ and $\rho_{B\Box_H A}$ respectively. Then the outer diagram in the above picture commutes, yielding

$$(\Phi_{A,B}\otimes H)(e_{A,B}\otimes H)\rho_{A\Box_H B} = (e_{B,A}\otimes H)\rho_{B\Box_H A}\Psi.$$

On the other hand, tensoring ⟨1⟩ from the right with H, one obtains

$$(\Phi_{A,B}\otimes H)(e_{A,B}\otimes H) = (e_{B,A}\otimes H)(\Psi\otimes H).$$

Substituting this in the preceding equation, one gets

$$(e_{B,A}\otimes H)(\Psi\otimes H)\rho_{A\Box_H B} = (e_{B,A}\otimes H)\rho_{B\Box_H A}\Psi.$$

Since H is flat, $e_{B,A}\otimes H$ is a monomorphism, yielding that Ψ is right H-colinear. □

As a consequence of the above theorem we get:

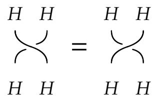
Corollary 4.15 *Let H be a flat and cocommutative Hopf algebra in a symmetric monoidal category $\mathcal{C}$. Then the set* $\mathrm{Gal}(\mathcal{C}; H)$ *is an abelian group.*

In a braided non-symmetric category Assumption 4.11 will be satisfied on an important subclass of H-Galois objects that we next define:

Definition 4.16 *An H-Galois object which is isomorphic to H as a right H-comodule is called an H-Galois object with a normal basis.*

There is a smart observation of Schauenburg in [30, Corollary 5] that will allow us to consider the group of Galois objects with a normal basis. We quote it here:

Theorem 4.17 *If a Hopf algebra $H \in \mathcal{C}$ is cocommutative (or commutative), then*



i.e., $\Phi^2_{H,H} = id_{H\otimes H}$.

Proposition 4.18 *For a cocommutative or commutative Hopf algebra H the Assumption 4.11 is fulfilled on H-Galois objects with a normal basis.*

Proof. We have that an H-Galois object with normal basis is isomorphic to H (as an object). Then the statement follows from Theorem 4.17 and naturality of the braiding. □

Corollary 4.19 *Let H be a flat and cocommutative Hopf algebra in $\mathcal{C}$. Then the set* $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ *of isomorphism classes of H-Galois objects with a normal basis is an abelian group.*

Proof. Go through the proof of Theorem 4.14 bearing in mind the above proposition. Take into account that if A, B are two right H-Galois objects with a normal basis, then $A\Box_H B$ is too because $A\Box_H B \cong H\Box_H H \cong H$ as H-comodules. Using the antipode, we have that $\overline{A}$ is isomorphic to H as H-comodules if A is so. □

For H flat and cocommutative the group $\mathrm{BiGal}(\mathcal{C}; H)$ may be recovered from $\mathrm{Gal}(\mathcal{C}, H)$ and the group $\mathrm{Aut}_{Hopf}(H)$ of Hopf automorphisms of H, since it is isomorphic to the semidirect product of the latter two. This is proved by Schauenburg in [32, Lemma 4.7] for a flat and cocommutative Hopf algebra over a commutative ring. The same arguments used in his proof are valid in our categorical setting as we show. We recall the following universal property of biGalois objects.

Proposition 4.20 *Let A be an H-biGalois object with left structure morphism λ_A.*

(i) If K is an algebra in $\mathcal{C}$ and $\phi : A \to K \otimes A$ is an algebra morphism, then there is a unique algebra morphism $f : H \to K$ such that $\phi = (f \otimes A)\lambda_A$.
(ii) If $\lambda'_A : A \to H \otimes A$ is another left H-comodule structure on A making it into an H-biGalois object, then there is a unique $f \in \mathrm{Aut}_{Hopf}(H)$ such that $\lambda'_A = (f \otimes A)\lambda_A$.

Proof. (i) Combine the following results in [31]: Lemma 4.2, Theorem 4.3 and Proposition 4.4.

(ii) Applying (i) to (A, λ_A), there is a unique algebra morphism $f : H \to H$ such that $\lambda'_A = (f \otimes A)\lambda_A$. The existence of its inverse is assured by applying (i) to (A, λ'_A). We now check that f is a bialgebra morphism.

That $\Delta f = (f \otimes f)\Delta$ and $\varepsilon f = \varepsilon$ follows from (i) and the following computations:

$$\begin{aligned}((\Delta f) \otimes A)\lambda_A &= (\Delta \otimes A)\lambda'_A \\ &= (H \otimes \lambda'_A)\lambda'_A \\ &= (H \otimes f \otimes A)(H \otimes \lambda_A)(f \otimes A)\lambda_A \\ &= (H \otimes f \otimes A)(f \otimes H \otimes A)(H \otimes \lambda_A)\lambda_A \\ &= (H \otimes f \otimes A)(f \otimes H \otimes A)(\Delta \otimes A)\lambda_A \\ &= (((f \otimes f)\Delta) \otimes A)\lambda_A.\end{aligned}$$

$$((\varepsilon f) \otimes A)\lambda_A = (\varepsilon \otimes A)\lambda'_A = Id_A = (\varepsilon \otimes A)\lambda_A.$$

Being f a bialgebra morphism, it is a Hopf algebra morphism. □

Let A be a left H-comodule algebra and $f \in \mathrm{Aut}_{Hopf}(H)$. The morphism $\lambda_A^f = (f \otimes A)\lambda_A$ endows A with a structure of left H-comodule algebra. We will denote this new object by ${}^fA$. Moreover, if A is an H-bicomodule algebra, then ${}^fA$ is such, too. Symmetrically, we write A^f for the right version of the above fact.

Theorem 4.21 *Let $H \in \mathcal{C}$ be a flat and cocommutative Hopf algebra. Under Assumption 4.11 (e.g. $\mathcal{C}$ symmetric) $\mathrm{Aut}_{Hopf}(H)$ acts on $\mathrm{Gal}(\mathcal{C}, H)$ by $A \triangleleft f = A^{f^{-1}}$ and there is a group isomorphism*

$$\Psi : \mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Gal}(\mathcal{C}, H) \to \mathrm{BiGal}(\mathcal{C}; H), (f, [A]) \mapsto [{}^f(\dot{A})].$$

Proof. If A is right H-Galois object and $f \in \mathrm{Aut}_{Hopf}(H)$, then $A^{f^{-1}}$ is a right H-Galois object too because $can_{A^{f^{-1}}} = (A \otimes f^{-1})can_A$ and clearly $A^{f^{-1}}$ is faithfully flat. This gives a right action of $\mathrm{Aut}_{Hopf}(H)$ on $\mathrm{Gal}(\mathcal{C}, H)$ and we may consider the semidirect product $\mathrm{Aut}_{Hopf}(H) \ltimes \mathrm{Gal}(\mathcal{C}, H)$.

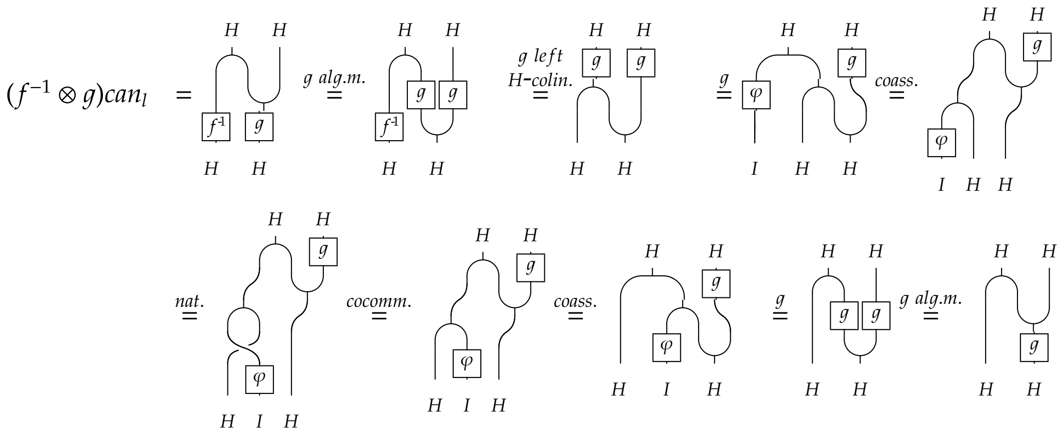
We check that Ψ is a group morphism. Let $[A], [B] \in \mathrm{Gal}(\mathcal{C}, H)$ and $f, g \in \mathrm{Aut}_{Hopf}(H)$. We have the following isomorphism of H-bicomodule algebras:

$$\begin{aligned}{}^f(\dot{A})\Box_H {}^g(\dot{B}) &= {}^f(\dot{A})^{gg^{-1}}\Box_H {}^g(\dot{B}) \\ &= ({}^f(\dot{A})^{g^{-1}})^g\Box_H {}^g(\dot{B}) \\ &\cong {}^f(\dot{A})^{g^{-1}}\Box_H \dot{B} \\ &= {}^{fgg^{-1}}(\dot{A})^{g^{-1}}\Box_H \dot{B} \\ &= {}^{fg}({}^{g^{-1}}(\dot{A})^{g^{-1}}\Box_H \dot{B}) \\ &= {}^{fg}(\dot{}(A^{g^{-1}})\Box_H \dot{B}). \\ &\cong {}^{fg}(\dot{}(A^{g^{-1}}\Box_H B)).\end{aligned}$$

This shows that $\Psi(f, [A])\Psi(g, [B]) = \Psi(fg, (A \triangleleft g)\Box_H B) = \Psi((f, [A])(g, [B]))$.

We show the injectivity of Ψ. Assume that ${}^f(\dot{A}) \cong H$ as H-bicomodule algebras. Then $A \cong H$ as right H-comodule algebras and hence $\dot{A} \cong H$ as H-bicomodule algebras. Hence we have an isomorphism $g : H \to {}^fH$ of H-bicomodule algebras. By (i) of the preceding proposition, there is an algebra morphism

$\varphi : H \to I$ such that $g = (\varphi \otimes H)\Delta_H$. Now we calculate:



$= (H \otimes g)can_l.$

Using this,

$$\begin{aligned} f^{-1} \otimes \eta_H &= (f^{-1} \otimes g)(H \otimes \eta_H) \\ &= (f^{-1} \otimes g)can_l can_l^{-1}(H \otimes \eta_H) \\ &= (H \otimes g)can_l can_l^{-1}(H \otimes \eta_H) \\ &= (H \otimes g)(H \otimes \eta_H) \\ &= (H \otimes \eta_H). \end{aligned}$$

Applying $H \otimes \varepsilon$ to this equality, we get $f^{-1} = Id_H$.

We finally show the surjectivity of Ψ. Let B be an H-biGalois object. Then it is a right H-Galois object. Turn it into a left H-comodule algebra and consider the H-biGalois object ${}^{\bullet}B$. By (ii) of the above proposition applied to $({}^{\bullet}B, \lambda_{{}^{\bullet}B})$, there is $f \in \mathrm{Aut}_{Hopf}(H)$ such that $\lambda_B = (f \otimes B)\lambda_{{}^{\bullet}B}$. Hence $B \cong {}^{f}({}^{\bullet}B)$ as H-bicomodule algebras. □

In Section 6 we will compute the group of biGalois objects for some examples of cocommutative braided Hopf algebras.

5. A short exact sequence for the group of Galois objects

In this section we present the main result of this paper, the construction of the short exact sequence

$$1 \longrightarrow \mathrm{H}^2(\mathcal{C}; H, I) \xrightarrow{\ \zeta\ } \mathrm{Gal}(\mathcal{C}; H) \xrightarrow{\ \xi\ } \mathrm{Pic}^{co}(\mathcal{C}; H),$$

connecting second Sweedler cohomology group, the group of Galois objects and the Picard group of a flat and cocommutative Hopf algebra $H \in \mathcal{C}$. The construction is done in several steps: the first subsection is devoted to Sweedler cohomology; second subsection deals with the definition and properties of the map ζ; the results needed for the exactness are established in the third subsection; fourth subsection treats the Picard group and the definition of ξ and in the fifth subsection all the pieces are put together to prove the result.

5.1. *Sweedler cohomology*

The construction of Sweedler cohomology done in [33] passes *mutatis mutandis* to the categorical setting. There is only one point about the braiding to care about.

Let C be a coalgebra and A an algebra in $\mathcal{C}$. The set $\mathcal{S} := \mathcal{C}(C, A)$ of morphisms from C to A is a monoid with the *convolution product* $f * g = \nabla_A(f \otimes g)\Delta_C$ for $f, g \in \mathcal{S}$, and unit $1_{\mathcal{S}} = \eta_A\varepsilon_H$. If C is cocommutative and A commutative, then $\mathcal{C}(C, A)$ is abelian. We denote by $\mathrm{Reg}(C, A)$ the group of morphisms from C to A invertible with respect to the convolution product.

Let H be a cocommutative Hopf algebra and A a commutative H-module algebra in $\mathcal{C}$. We denote by $H^{\otimes n}$ the n-fold tensor product of H. For $n = 0$ it is $H^{\otimes 0} = I$. Since H is cocommutative, $\Phi_{H,H} = \Phi^{-1}_{H,H}$ (Theorem 4.17) and hence $H^{\otimes n}$ is again a cocommutative Hopf algebra. Then $\mathrm{Reg}(H^{\otimes n}, A)$ is abelian. For $i = 0, \dots, n+1$ and $f \in \mathrm{Reg}(H^{\otimes n}, A)$ we define morphisms $\partial_i : \mathrm{Reg}(H^{\otimes n}, A) \longrightarrow \mathrm{Reg}(H^{\otimes (n+1)}, A)$ by:

$$\begin{aligned}
\partial_0(f) &= \mu(H \otimes f),\\
\partial_i(f) &= f(\underbrace{H \otimes \cdots \otimes H}_{i-1} \otimes \nabla_H \otimes H \cdots \otimes H), \quad i = 1, \dots, n,\\
\partial_{n+1}(f) &= f \otimes \varepsilon_H,
\end{aligned}$$

where $\mu : H \otimes A \longrightarrow A$ denotes the H-module structure morphism of A. The maps ∂_i's are group morphisms. We set $\partial_i^{-1}(f) = \partial_i(f^{-1})$ and further define

$$d_n := \partial_0 * \partial_1^{-1} * \cdots * \partial_{n+1}^{(-1)^{n+1}} : \mathrm{Reg}(H^{\otimes n}, A) \longrightarrow \mathrm{Reg}(H^{\otimes (n+1)}, A),$$

where $(\partial_i * \partial_j)(f) := \partial_i(f) * \partial_j(f)$ in $\mathrm{Reg}(H^{\otimes (n+1)}, A)$. One then has that d_i's are group morphisms (commutativity of $\mathrm{Reg}(H^{\otimes (n+1)}, A)$ is needed for this) and $d_i d_{i-1} = 1_{\mathrm{Reg}(H^{\otimes (i+1)}, A)}$, for $i \geq 1$, which makes

$$\mathrm{Reg}(I, A) \xrightarrow{d_0} \mathrm{Reg}(H, A) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathrm{Reg}(H^{\otimes n}, A) \xrightarrow{d_n} \mathrm{Reg}(H^{\otimes n+1}, A) \xrightarrow{d_{n+1}} \cdots$$

into a complex.

Definition 5.1 *Morphisms from* $\mathrm{Z^n}(\mathcal{C}; H, A) := \mathrm{Ker}(d_n)$ *are called n-cocycles and those from* $\mathrm{B^n}(\mathcal{C}; H, A) := \mathrm{Im}(d_{n-1})$ *n-coboundaries. The quotient group*

$$\mathrm{H^n}(\mathcal{C}; H, A) = \mathrm{Z^n}(\mathcal{C}; H, A) / \mathrm{B^n}(\mathcal{C}; H, A)$$

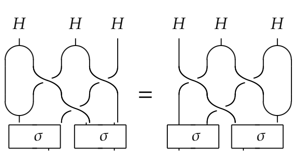
is called n-th Sweedler cohomology group with values in A.

Two n-cocycles f and g are called *cohomologous*, denoted by $f \sim g$, if they are in the same class in $\mathrm{H^n}(\mathcal{C}; H, A)$. That is, $f * g^{-1} \in \mathrm{B^n}(\mathcal{C}; H, A)$, or equivalently, $f = d_{n-1}h * g$, for some $h \in \mathrm{Reg}(H^{\otimes (n-1)}, A)$.

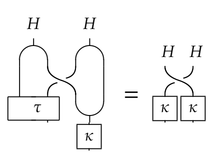
Let us consider second Sweedler cohomology group for $A = I$. The left H-action on I is then given by $\varepsilon : H \cong H \otimes I \longrightarrow I$. A *2-cocycle* is then a morphism $\sigma \in \mathrm{Reg}(H \otimes H, I)$ for which it holds

$$d_2(\sigma) = (\partial_0 * \partial_1^{-1} * \partial_2 * \partial_3^{-1})(\sigma) = 1_{\mathrm{Reg}(H^{\otimes 3}, I)}.$$

A *2-coboundary* is a morphism $\tau \in \mathrm{Reg}(H \otimes H, I)$ for which there exists $\kappa \in \mathrm{Reg}(H, I)$ so that $\tau = d_1(\kappa) = (\partial_0 * \partial_1^{-1} * \partial_2)(\kappa)$. In braided diagrams the 2-cocycle and the 2-coboundary conditions, rewritten as $(\partial_1 * \partial_3)(\sigma) = (\partial_2 * \partial_4)$ and $\tau * \partial_1(\kappa) = (\partial_0 * \partial_2)(\kappa)$ respectively, take the form:

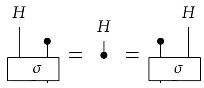


(5.13)



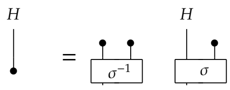
(5.14)

Definition 5.2 *A 2-cocycle σ that satisfies*

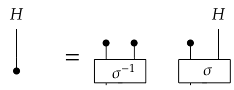


is called normalized.

Composing $d_2(\sigma^{-1}) = 1_{\mathrm{Reg}(H^{\otimes 3},I)}$ with $H \otimes \eta_H \otimes \eta_H$ and $d_2(\sigma) = 1_{\mathrm{Reg}(H^{\otimes 3},I)}$ with $\eta_H \otimes \eta_H \otimes H$, one obtains respectively:

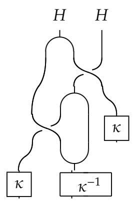


(5.15)



(5.16)

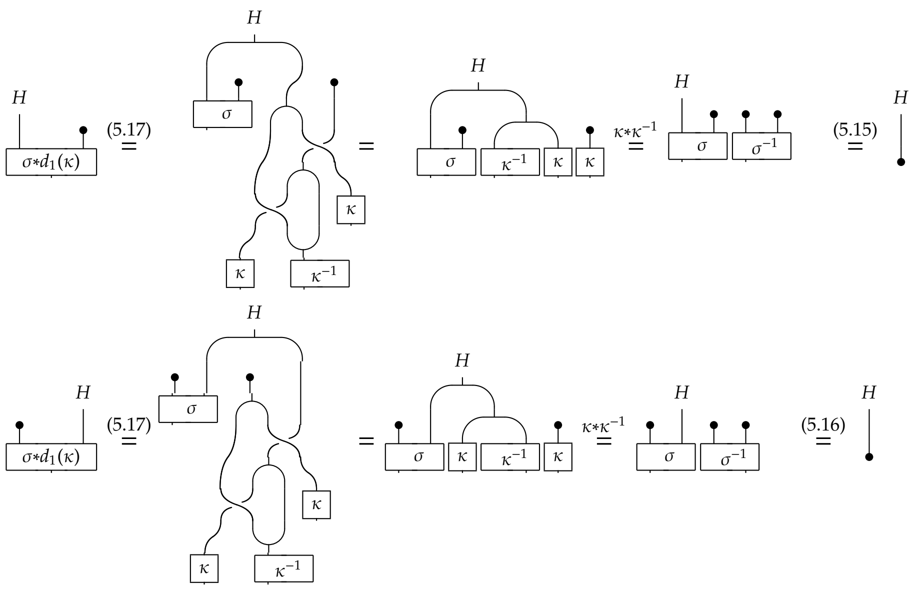
In future computations we will need the explicit form of the 2-coboundary condition $d_1(\kappa) = (\partial_0 * \partial_1^{-1} * \partial_2)(\kappa)$,



(5.17)

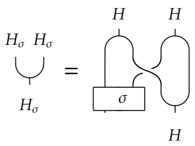
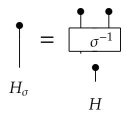
Lemma 5.3 *Every 2-cocycle σ is cohomologous to a normalized one.*

Proof. Clearly $\kappa = \sigma^{-1}(H \otimes \eta_H) \in \mathrm{Reg}(H, I)$ and $\sigma \sim \sigma * d_1(\kappa)$. We prove that the latter 2-cocycle is normalized. Note that $\kappa^{-1} = \sigma(H \otimes \eta_H)$. We have:



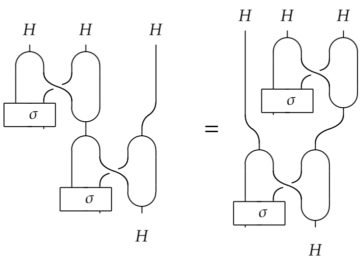
□

Proposition 5.4 *Let $H \in \mathcal{C}$ be a cocommutative Hopf algebra and $\sigma \in \mathrm{Reg}(H \otimes H, I)$. We define $H_\sigma := H$ as H-comodule and consider the following morphisms:*

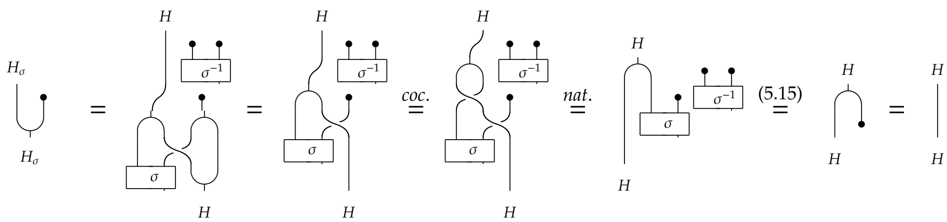

and

(5.18)

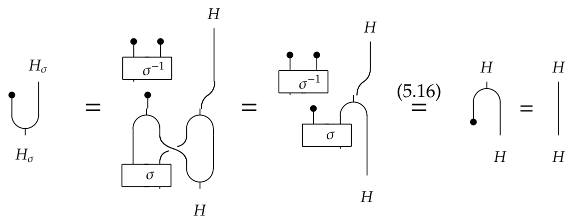
If σ is a 2-cocycle, then H_σ is a right H-comodule algebra. Moreover, if σ is normalized, then the unit on H_σ coincides with η_H.

Proof. Using that H is cocommutative and applying Theorem 4.17 we will prove that the 2-cocycle condition expressed in Diagram (5.13) is equivalent to

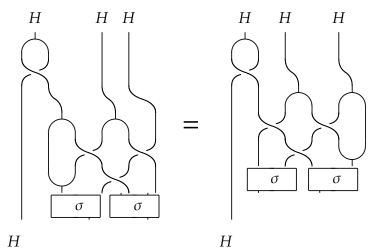

(5.19)

This just means that the multiplication of H_σ is associative. For the unit property we find

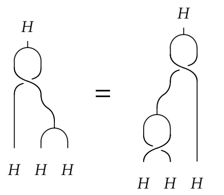




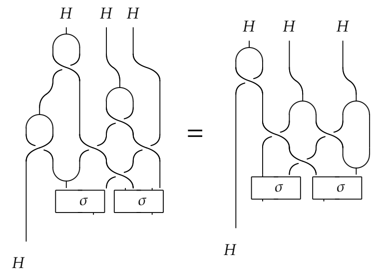
Thus H_σ is an algebra. We proceed to prove what the 2-cocycle condition in Diagram (5.13) is equivalent to (5.19). Starting from Diagram (5.13) one obtains:



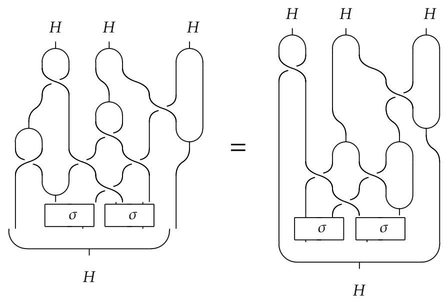
By cocommutativity of H, coassociativity and naturality we have:



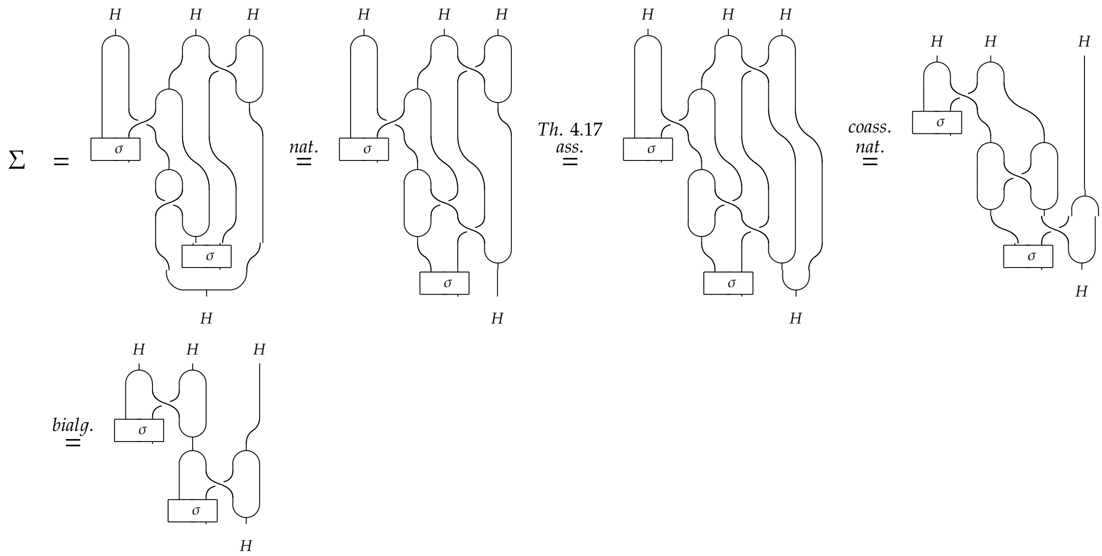
We apply this to the first tensor factor in the left hand-side diagram and simultaneously cocommutativity of H to the second tensor factor and we get:



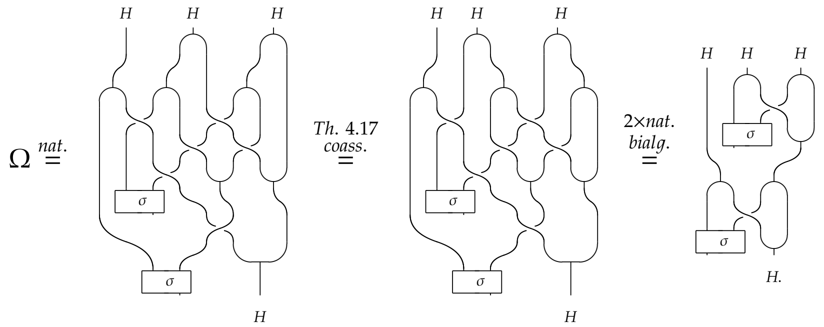
Then it is also true that:



Denote the left hand-side by Σ and the right one by Ω. By naturality (and left and right unity constraints) we have that Σ further equals to

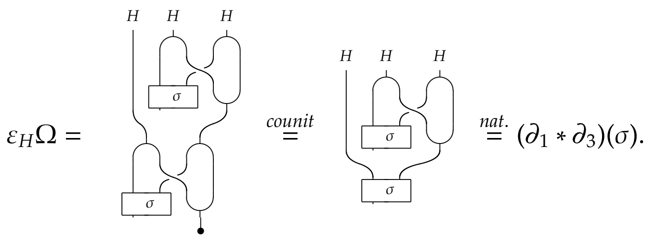


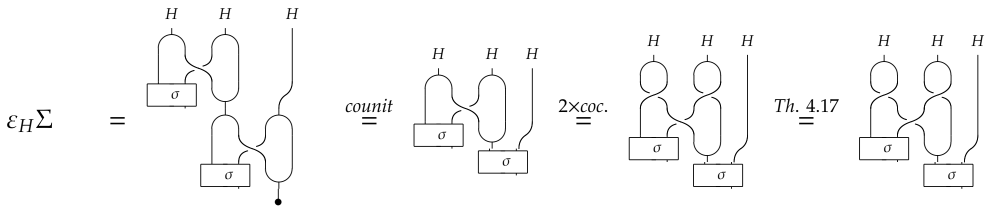
Similarly, Ω equals to



Thus the equation $\Sigma = \Omega$ finally yields (5.19).

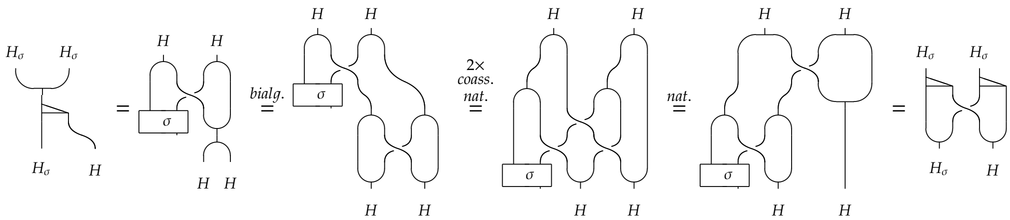
Conversely, we start now from Diagram (5.19) to obtain Diagram (5.13). Let Σ denote the left hand-side of (5.19) and Ω the right one. Then:





$\overset{nat.}{=} (\partial_2 * \partial_4)(\sigma).$

We continue with the proposition under proof. We should show that H_σ is an H-comodule algebra. For the compatibility of the H-comodule structure and multiplication of H_σ we have:



The compatibility with unit of H_σ is also satisfied, so H_σ is a right H-comodule algebra. The last statement is clear. □

In the above proof we have also established:

Corollary 5.5 *For a cocommutative Hopf algebra H a morphism $\sigma \in \mathrm{Reg}(H \otimes H, I)$ is a 2-cocycle if and only if the multiplication on H_σ defined in (5.18) is associative.*

5.2. Cocycles and Galois objects

We will now use the cocycle twisting of the multiplication presented in the previous section to define an injective group morphism from $\mathrm{H}^2(C; H, I)$ into the group of H-Galois objects with a normal basis.

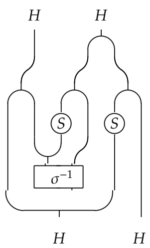
Proposition 5.6 *Let $H \in C$ be a flat and cocommutative Hopf algebra. The map*

$$\zeta : \mathrm{H}^2(C; H, I) \longrightarrow \mathrm{Gal}_{nb}(C; H), [\sigma] \mapsto [H_\sigma]$$

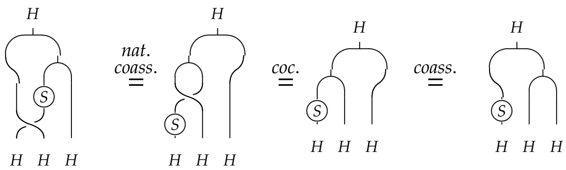
is a group monomorphism.

Proof. We have that $\mathrm{Gal}_{nb}(C; H)$ is a group due to Corollary 4.19. By Proposition 5.4 we know that H_σ is a right H-comodule algebra. We are going to prove here that H_σ is an H-Galois object. Clearly it will have a normal basis. Then we will prove that ζ does not depend on the choice of a representative of the class of σ, that it is compatible with the product and that it is injective.

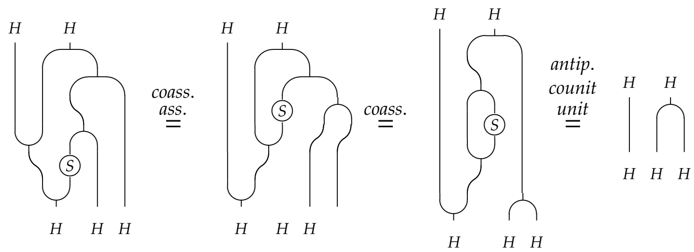
1) Let us prove that H_σ *is an H-Galois object*. In Corollary 3.13 we have seen that $H = H_\sigma$ is faithfully flat. We claim that the inverse of can_{H_σ} is given by

$\gamma :=$ 

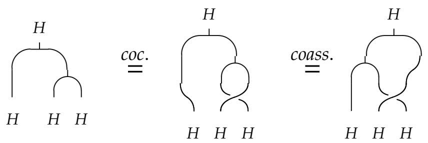
We first record several identities to be used in the proof of $\gamma can_{H_\sigma} = id_{H_\sigma \otimes H_\sigma}$. They are:



(5.20)

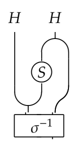


(5.21)

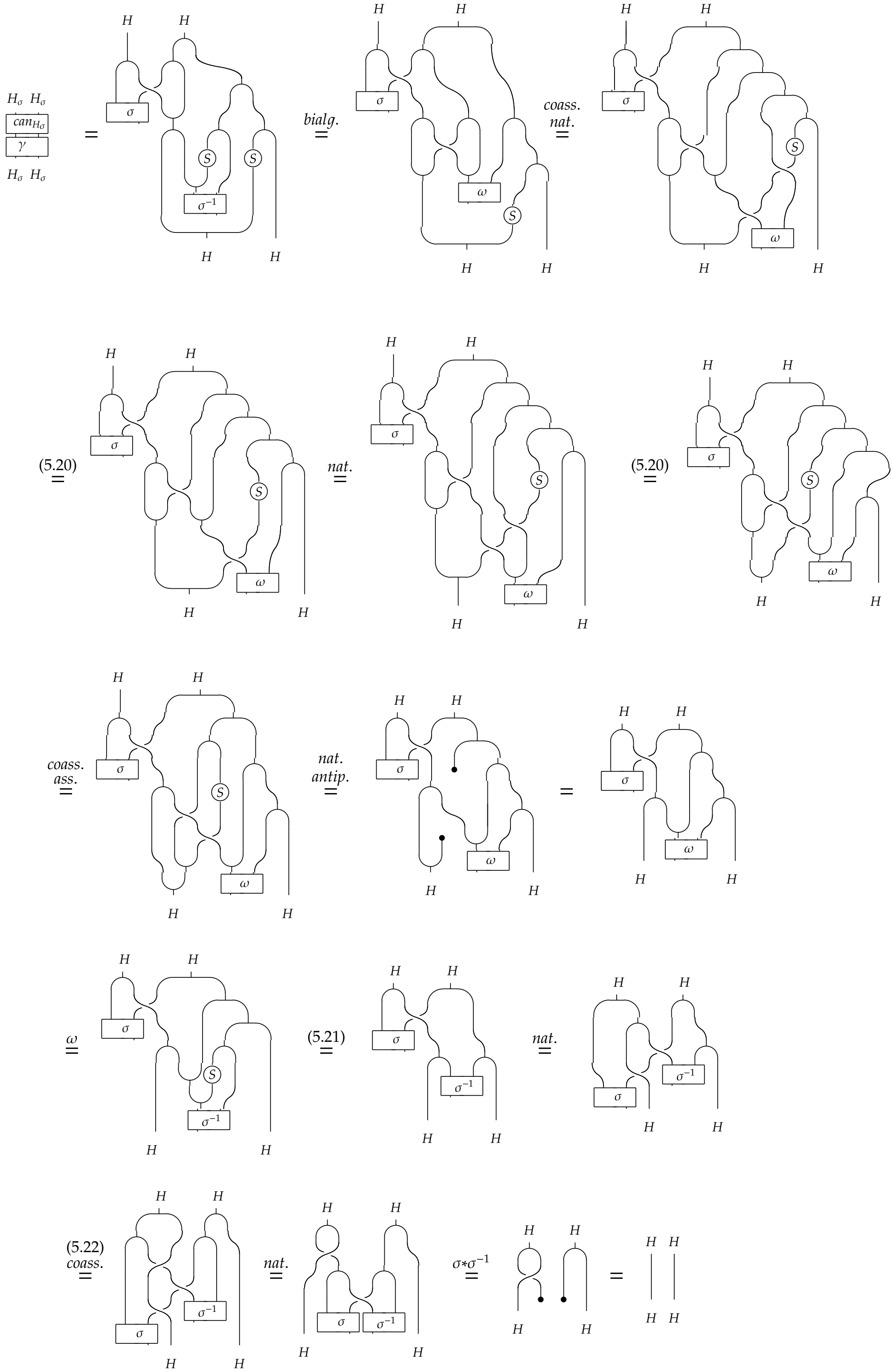


(5.22)

We set

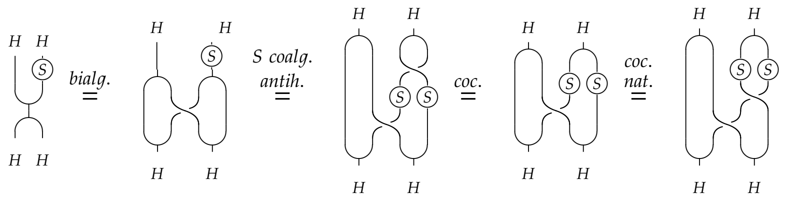
$\omega :=$ 

Then we have



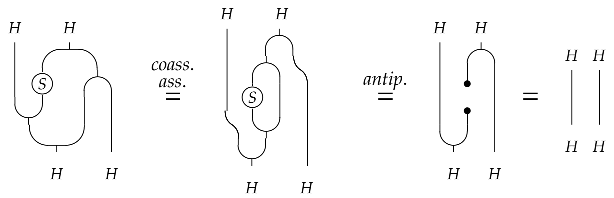
can_{H_σ}
γ
bialg.
coass. nat.
(5.20)
nat.
(5.20)
coass. ass.
nat. antip.
ω
(5.21)
nat.
(5.22) coass.
nat.
σ∗σ^{-1}

As before we next record a few identities needed in the proof of $can_{H_\sigma}\gamma = id_{H_\sigma \otimes H}$. We have



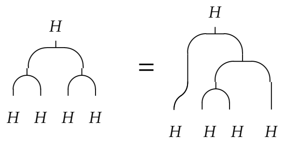
(5.23)

On the other hand, we also have

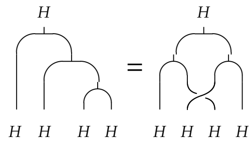


(5.24)

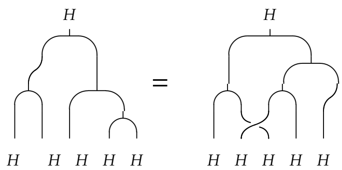
Applying coassociativity (and cocommutativity) in appropriate way one obtains the following equalities:



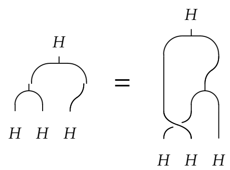
(5.25)



(5.26)

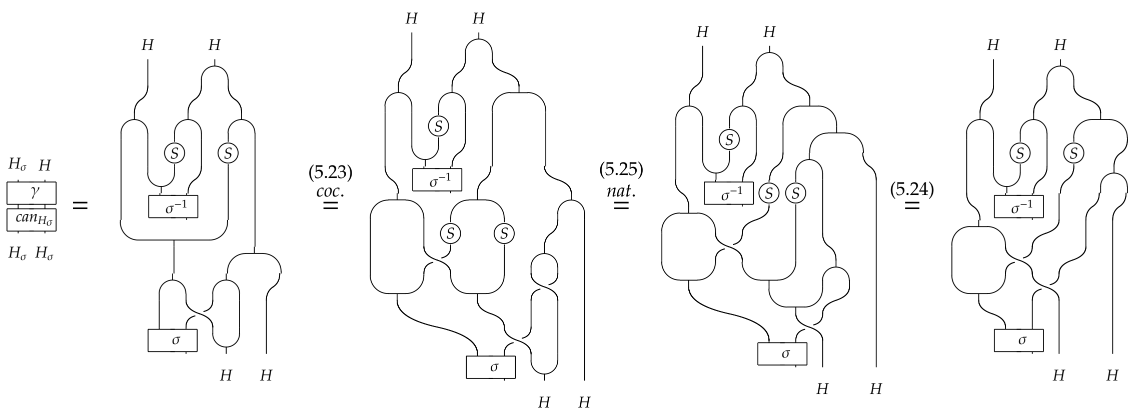


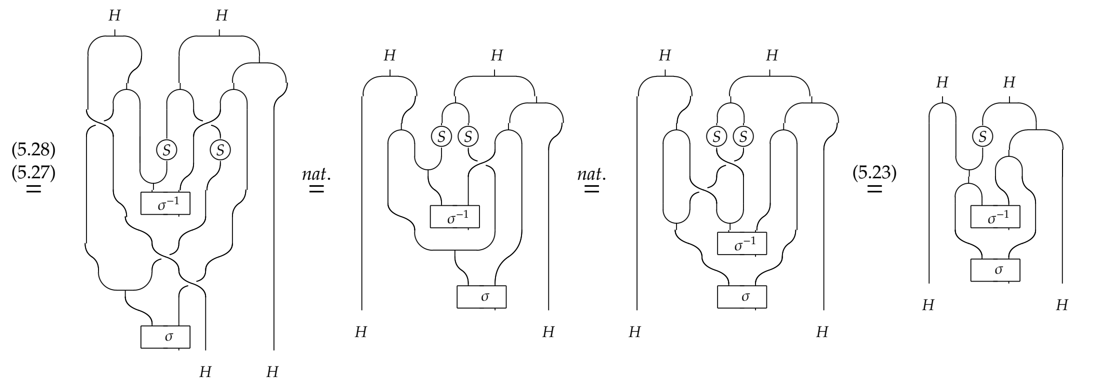
(5.27)

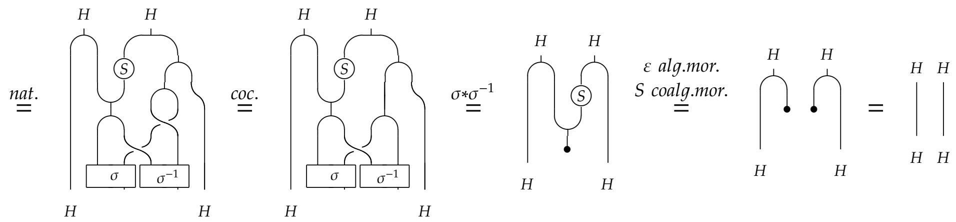


(5.28)

We now compute

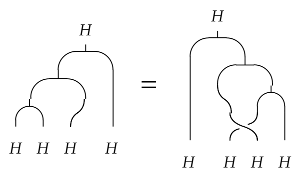


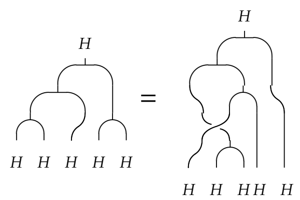




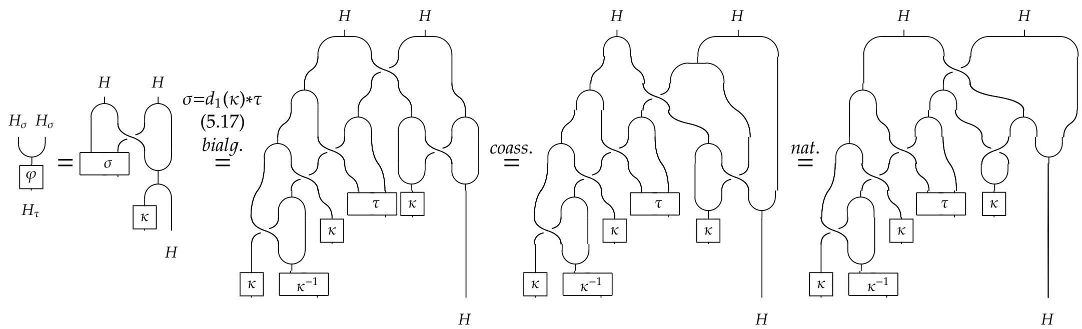
This proves that can_{H_σ} is an isomorphism and that H_σ is an H-Galois object.

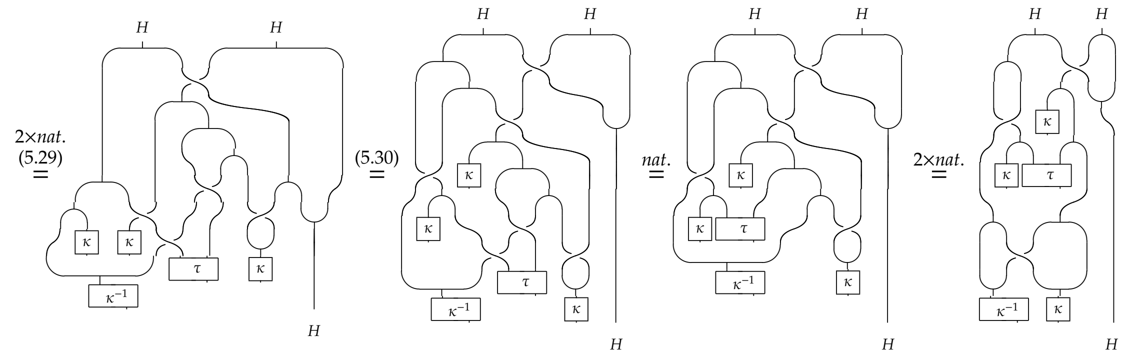
2) We next show that $\zeta : \mathrm{H}^2(\mathcal{C}; H, I) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ *does not depend on the choice of the representative.* Assume that $\sigma \sim \tau$ and let $\kappa \in \mathrm{Reg}(H, I)$ be such that $\sigma = d_1(\kappa) * \tau$. We will prove that $\varphi := (\kappa \otimes H)\Delta_H : H_\sigma \longrightarrow H_\tau$ is an isomorphism of right H-comodule algebras. Its inverse is $\varphi^{-1} := (\kappa^{-1} \otimes H)\Delta_H$. Colinearity is immediate. To check the compatibility with the algebra structure we record some necessary equalities. Applying three times coassociativity and once cocommutativity in an appropriate way we obtain

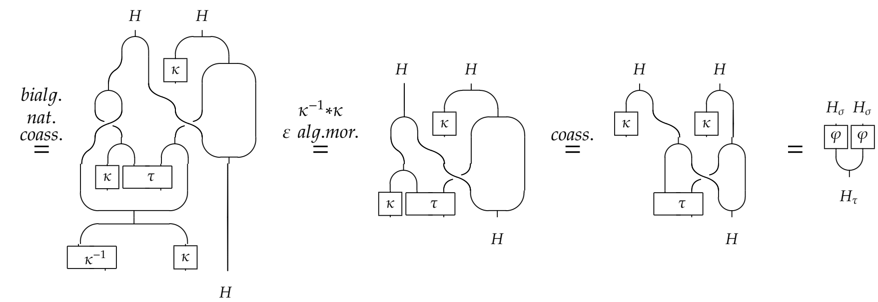
 (5.29)

 (5.30)

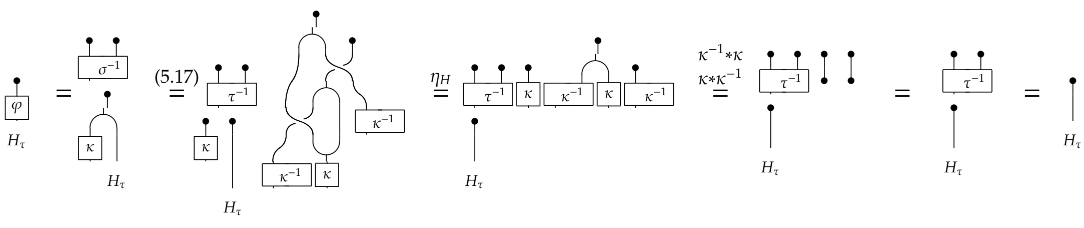
We now compute:







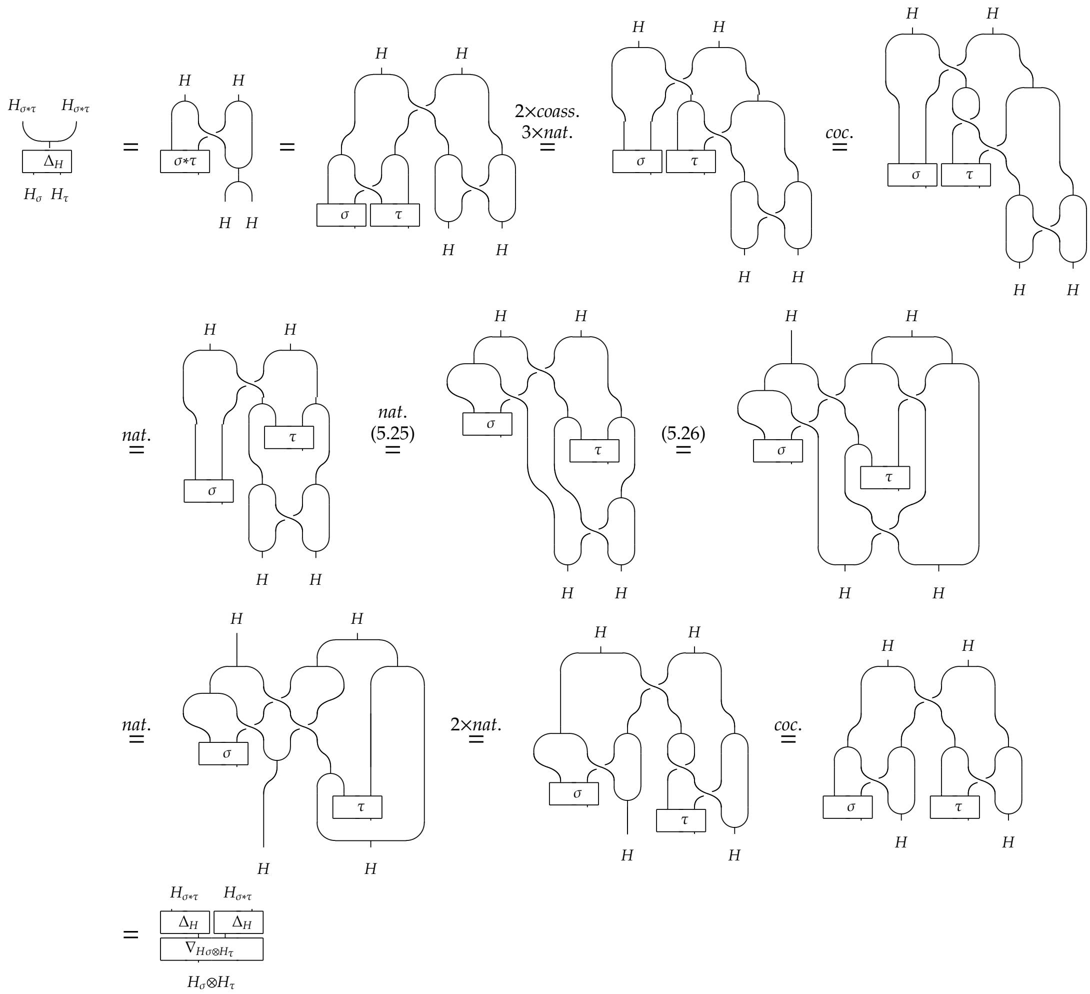
This proves that $\varphi : H_\sigma \longrightarrow H_\tau$ is compatible with multiplication. From the condition $\sigma = d_1(\kappa) * \tau$ we have $\sigma^{-1} = \tau^{-1} * d_1(\kappa^{-1})$. Then φ is also compatible with unit, since



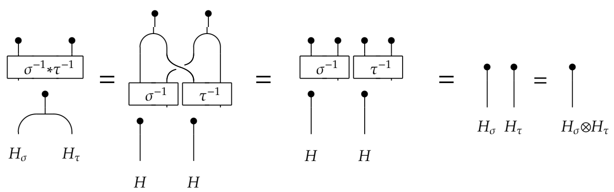
This finishes the proof that $\varphi : H_\sigma \longrightarrow H_\tau$ is an isomorphism of right H-Galois objects.

3) We next prove that ζ *is a group morphism* by showing that $H_{\sigma*\tau} \cong H_\sigma \Box_H H_\tau$ as right H-comodule algebras. We know from Lemma 4.6 that the morphism $\Delta_H : H \longrightarrow H \otimes H$ factors through the (bi)comodule isomorphism $\overline{\Delta}_H : H \longrightarrow H \Box_H H \cong H$ such that $e_{H,H}\overline{\Delta}_H = \Delta_H$. Furthermore, $H_{\sigma*\tau} = H$ and $H_\sigma \Box_H H_\tau \cong H \Box_H H$ as right H-comodules. It remains to prove that $\overline{\Delta}_H : H_{\sigma*\tau} \longrightarrow H_\sigma \Box_H H_\tau$ is an algebra morphism. By **2.4(a)** it suffices to prove that $\Delta_H : H_{\sigma*\tau} \longrightarrow H_\sigma \otimes H_\tau$ is an algebra morphism ($e : H_\sigma \Box_H H_\tau \longrightarrow H_\sigma \otimes H_\tau$ is such, since H

is a bialgebra). We have

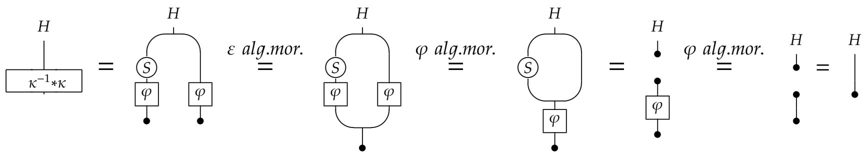


showing that $\Delta_H : H_{\sigma*\tau} \longrightarrow H_\sigma \otimes H_\tau$ is multiplicative. It also preserves the unit. We have $(\sigma * \tau)^{-1} = \sigma^{-1} * \tau^{-1}$ and thus

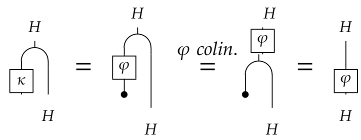


Thus $\overline{\Delta}_H : H_{\sigma*\tau} \longrightarrow H_\sigma \square_H H_\tau$ is an isomorphism of H-comodule algebras.

4) We finally prove that ζ *is injective,* that is, if $H_\sigma \cong H_\tau$ as H-comodule algebras, then $\sigma \sim \tau$. Let $\varphi : H_\sigma \longrightarrow H_\tau$ be the given H-comodule algebra isomorphism. We define $\kappa := \varepsilon_H \varphi : H \longrightarrow I$. It is convolution invertible with inverse $\kappa^{-1} = \kappa S = \varepsilon_H \varphi S$. Indeed,



We claim that $\sigma = d_1(\kappa) * \tau$. Note that $\varphi = (\kappa \otimes H)\Delta_H : H \longrightarrow H$, since



Then $\varphi : H_\sigma \longrightarrow H_\tau$ is the same one we have dealt with in 2). When we proved that $\varphi : H_\sigma \longrightarrow H_\tau$ is an algebra morphism in the diagram computation after (5.30), we have proved $\nabla_{H_\tau}(\varphi \otimes \varphi) = \varphi \nabla_{d_1(\kappa)*\tau}$ – taking into account the bialgebra compatibility written out there (see the thitd diagram from the start and the penultimate diagram in that computation). Observe that in the mentioned computation we have used only the definition of φ. Using that φ is an algebra morphism, $\nabla_{H_\tau}(\varphi \otimes \varphi) = \varphi \nabla_{H_\sigma}$, since φ is an isomorphism, from the former equality we get $\nabla_{H_\sigma} = \nabla_{d_1(\kappa)*\tau}$. Applying ε_H to this we obtain $\sigma = d_1(\kappa) * \tau$, as desired. □

Notice that in the course of the above proof we have shown that $H_\sigma \cong H_\tau$ as H-comodule algebras if and only if $\sigma \sim \tau$.

5.3. *Galois objects with a normal basis*

The aim of this subsection is to prove that the group monomorphism ζ from Proposition 5.6 is surjective, yielding $\mathrm{Gal}_{nb}(C; H) \cong \mathrm{H}^2(C; H, I)$. The proof for the following claim can be found in the first part of the proof of [1, Proposition 9], where it is formulated for symmetric monoidal categories. We show that the symmetricity hypothesis is not necessary.

Lemma 5.7 *Let A be an H-Galois object with a normal basis, where H is cocommutative. There is a convolution invertible right H-colinear morphism $\nu : H \longrightarrow A$ satisfying $\nu\eta_H = \eta_A$.*

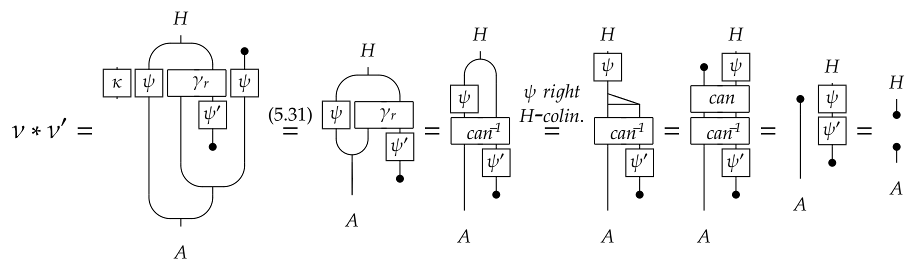
Proof. Denote by $\psi : H \longrightarrow A$ the right H-comodule isomorphism. We define the morphism $\nu : H \longrightarrow A$ as $\nu := \kappa \otimes \psi$, where $\kappa := \varepsilon_H \psi^{-1} \eta_A : I \longrightarrow I$ is a commuting factor (because of the left and right unity constraints). We first prove that $\nu\eta_H = \eta_A$. Indeed,

$$\begin{aligned} \nu\eta_H = (\psi\eta_H) \otimes \kappa &= (\psi S \eta_H) \otimes (\varepsilon_H \psi^{-1} \eta_A) \\ &= (\psi S)(\nabla_H (S \otimes H)\Delta_H)(\psi^{-1}\eta_A) \\ &= \psi S \nabla_H (S \otimes H)(\psi^{-1} \otimes H)\rho_A \eta_A \\ &= \psi S \nabla_H (S\psi^{-1} \otimes H)(\eta_A \otimes \eta_H) \\ &= \psi S S \psi^{-1} \eta_A \\ &= \psi\psi^{-1}\eta_A \\ &= \eta_A. \end{aligned}$$

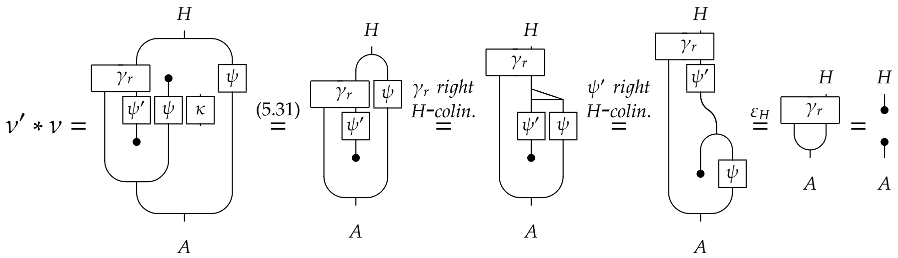
Then

$$\kappa \otimes (\psi\eta_H) = \eta_A. \tag{5.31}$$

We have that ν is right H-colinear, as so are κ and ψ. Let us prove that the convolution inverse of ν is given by $\nu' := \nabla_A(A \otimes (\varepsilon_H \psi^{-1}) \otimes A)(\gamma_r \otimes (\psi\eta_H))$. Recall that $\gamma_r = can^{-1}(\eta_A \otimes H)$. Since can is left A-linear we have $(\nabla_A \otimes A)(A \otimes \gamma_r) = can^{-1}$. In the following diagrams ψ' will stand for ψ^{-1}. We have:



On the other hand:



□

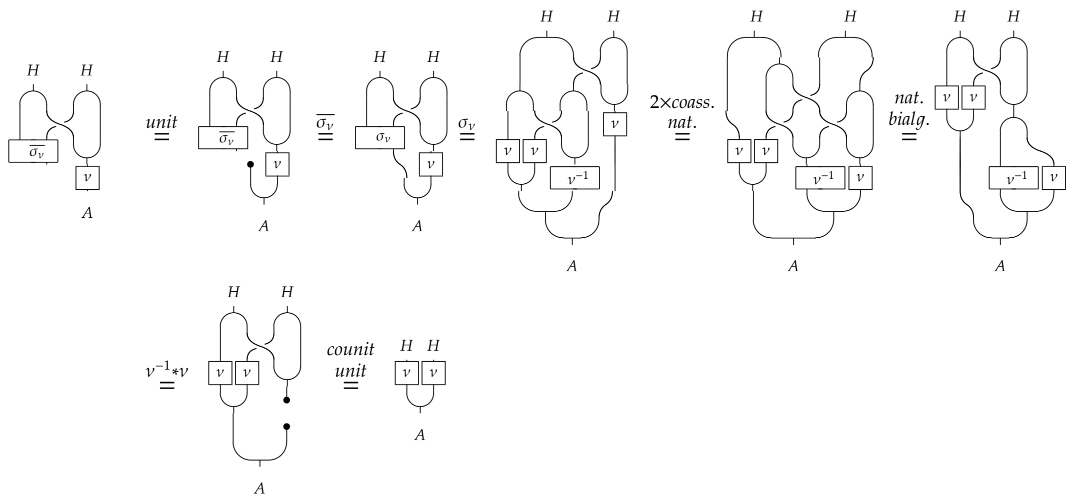
A right H-comodule algebra A for which there exists an H-colinear convolution invertible morphism $\nu : H \longrightarrow A$ is called *H-cleft*, and the morphism ν is called a *cleaving morphism*. One may always assume that $\nu\eta_H = \eta_A$, because otherwise one may take $\nu' := \nabla_A[(\nu^{-1}\eta_H) \otimes \nu]$. Thus together with the above lemma we have proved:

Corollary 5.8 *An H-Galois object with a normal basis is H-cleft.*

Due to [3, Proposition 1.2c)], for an H-Galois object A with a normal basis one has $A \cong A^{coH}\#_{\overline{\sigma_\nu}}H$ as H-comodule algebras, where $\overline{\sigma_\nu} \in \mathrm{H}^2(C; H, A^{coH})$ is obtained as a factorization through A^{coH} of the morphism $\sigma_\nu = (\nabla_A(\nu \otimes \nu)) * ((\nu^{-1}\nabla_H)) : H \otimes H \longrightarrow A$. Here ν is a cleaving morphism for A. The cocycle twisted smash product turns out to be isomorphic to $H_{\overline{\sigma_\nu}}$ from (5.18), since $A^{coH} \cong I$.

Theorem 5.9 *The monomorphism ζ from Proposition 5.6 is an isomorphism. Therefore* $\mathrm{Gal}_{nb}(C; H) \cong \mathrm{H}^2(C; H, I)$.

Proof. Take an H-Galois object A with a normal basis and let ν be like in Lemma 5.7. With $\overline{\sigma_\nu}$ the 2-cocycle as in the above paragraph, we know from Proposition 5.6 that $H_{\overline{\sigma_\nu}}$ is an H-Galois object. Let us prove that $\nu : H_{\overline{\sigma_\nu}} \longrightarrow A$ is an isomorphism of H-comodule algebras (Proposition 3.14). By Lemma 5.7 we know that ν is right H-colinear. That ν is compatible with multiplication follows from the following calculation:

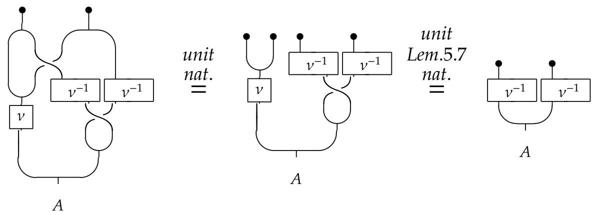


Note that in the second equality we identified the equalizers (A^{coH}, i_A) and (I, η_A) according to Proposition 3.9. We next check that ν is compatible with the unit. The convolution inverse of $\overline{\sigma_\nu}$ is induced by the convolution inverse of σ_ν, that is, $\sigma_\nu^{-1} = (\nu\nabla_H) * (\nabla_A\Phi_{A,A}(\nu^{-1} \otimes \nu^{-1}))$. Then:

$$\nu\eta_{H_{\overline{\sigma_\nu}}} = \nu\eta_H(\overline{\sigma_\nu})^{-1}(\eta_H \otimes \eta_H) \overset{Lem.5.7}{=} \eta_A(\overline{\sigma_\nu})^{-1}(\eta_H \otimes \eta_H) = \sigma_\nu^{-1}(\eta_H \otimes \eta_H).$$

On the other hand, $\nu\eta_H = \eta_A$ implies $\nu^{-1}\eta_H = \eta_A$. Finally,

$\sigma_\nu^{-1}(\eta_H \otimes \eta_H) =$



$= \eta_A$

□

5.4. The Picard group

Dually to the Morita theory which characterizes equivalences between categories of modules, Takeuchi proposed in [34] a theory that describes equivalences of categories of comodules for coalgebras over a field. This theory is called *Morita-Takeuchi theory*. Torrecillas and Zhang defined in [35] the Picard group of a coalgebra over a field as the set of isomorphism classes of bicomodules giving a Morita-Takeuchi equivalence. In this section we define the Picard group of a cocommutative coalgebra in a braided monoidal category.

Module categories and module functors over a monoidal category $\mathcal{C}$ were treated already in [26], under the names $\mathcal{C}$-categories and $\mathcal{C}$-functors, respectively. For newer reference see [12, Definitions 7.1. and 7.2.1]. It is immediate to check and well-known that the categories of the form ${}^C\mathcal{C}$ are right $\mathcal{C}$-module categories. For the purpose of our next result we recollect that a functor $\mathcal{F} : {}^D\mathcal{C} \longrightarrow {}^C\mathcal{C}$ is right $\mathcal{C}$-linear if for all $M \in {}^D\mathcal{C}$ and $X \in \mathcal{C}$ there are natural isomorphisms $\mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$ satisfying a pentagonal and a triangular diagram. Here $M \otimes X$ inherits the structure of a left D-comodule from that of M. Pareigis established in [27, Theorems 5.1 and 5.3] Morita Theorems for categories of modules in a monoidal category. Using his theorems, read in the opposite category, we will prove the following result.

Theorem 5.1 *Let C and D be flat coalgebras in $\mathcal{C}$. The following assertions are equivalent:*

(i) The functors $\mathcal{F} : {}^D\mathcal{C} \rightleftarrows {}^C\mathcal{C} : \mathcal{G}$ *establish a $\mathcal{C}$-module equivalence;*

(ii) There is $P \in {}^C\mathcal{C}^D$ flat and coflat in $\mathcal{C}^D$ such that $\mathcal{F}(-) \cong P\Box_D-$ and $Q \in {}^D\mathcal{C}^C$ flat and coflat in $\mathcal{C}^C$ such that $\mathcal{G}(-) \cong Q\Box_C-$, together with bicomodule isomorphisms $f : C \longrightarrow P\Box_D Q$ and $g : D \to Q\Box_C P$ so that the following diagrams

$$\begin{array}{ccc} P & \xrightarrow{\quad \overline{\lambda}_P \quad} & C\Box_C P \\ {\scriptstyle \overline{\rho}_P}\downarrow & & \downarrow{\scriptstyle f\Box_C P} \\ P\Box_D D & \xrightarrow[\quad P\Box_D g \quad]{} & P\Box_D(Q\Box_C P) \cong (P\Box_D Q)\Box_C P \end{array} \tag{5.32}$$

$$\begin{array}{ccc} Q & \xrightarrow{\quad \overline{\lambda}_Q \quad} & D\Box_D Q \\ {\scriptstyle \overline{\rho}_Q}\downarrow & & \downarrow{\scriptstyle g\Box_D Q} \\ Q\Box_C C & \xrightarrow[\quad Q\Box_C f \quad]{} & Q\Box_C(P\Box_D Q) \cong (Q\Box_C P)\Box_D Q \end{array} \tag{5.33}$$

commute. The isomorphisms $\overline{\lambda}$'s and $\overline{\rho}$'s are those from Lemma 4.6.

Proof. (i) ⇒ (ii) The coalgebras C and D become algebras in the opposite category $\mathcal{C}^{op}$, that we denote by C^o and D^o respectively. Taking into account that by duality, ${}_{C^o}(\mathcal{C}^{op}) \cong ({}^C\mathcal{C})^{op}$ and similarly for D, condition (i) read in $\mathcal{C}^{op}$ means that there are $\mathcal{C}^{op}$-module equivalence functors $\mathcal{F}^o : {}_{D^o}(\mathcal{C}^{op}) \longrightarrow {}_{C^o}(\mathcal{C}^{op})$ and $\mathcal{G}^o : {}_{C^o}(\mathcal{C}^{op}) \longrightarrow {}_{D^o}(\mathcal{C}^{op})$. By Morita Theorem [27, Theorem 5.1], $\mathcal{F}^o(-) \cong P^o \otimes_{D^o} -$ for some $P^o \in {}_{C^o}(\mathcal{C}^{op})_{D^o}$ and $\mathcal{G}^o(-) \cong Q^o \otimes_{C^o} -$ for some $Q^o \in {}_{D^o}(\mathcal{C}^{op})_{C^o}$ (here $\otimes_{C^o}$ denotes the tensor product over the algebra C^o, a construction dual to that of the cotensor product). Moreover, we have that P^o is coflat (in Pareigis' sense) in $(\mathcal{C}^{op})_{D^o}$ and Q^o is coflat in $(\mathcal{C}^{op})_{C^o}$, as well as that there are isomorphisms $f^o : P^o \otimes_{D^o} Q^o \longrightarrow C^o$ in ${}_{C^o}(\mathcal{C}^{op})_{C^o}$ and $g^o : Q^o \otimes_{C^o} P^o \longrightarrow D^o$ in ${}_{D^o}(\mathcal{C}^{op})_{D^o}$ so that there are two commutative diagrams, which read in $\mathcal{C}$ are Diagrams (5.32) and (5.33). Back in $\mathcal{C}$ we have thus objects $P \in {}^C\mathcal{C}^D$ coflat in $\mathcal{C}^D$ and $Q \in {}^D\mathcal{C}^C$ coflat in $\mathcal{C}^C$ such that $\mathcal{F}(-) \cong P\Box_D -$ and $\mathcal{G}(-) \cong Q\Box_C -$, and isomorphisms $f : C \longrightarrow P\Box_D Q$ in ${}^C\mathcal{C}^C$ and $g : D \to Q\Box_C P$ in ${}^D\mathcal{C}^D$ so that the desired two diagrams commute.

Let us prove that P is flat, then similarly Q will be flat, too. Consider the following equalizer in $\mathcal{C}$:

$$E \xrightarrow{e} X \underset{l}{\overset{k}{\rightrightarrows}} Y$$

Since D is flat,

$$D \otimes E \xrightarrow{D\otimes e} D \otimes X \underset{D\otimes l}{\overset{D\otimes k}{\rightrightarrows}} D \otimes Y$$

is an equalizer in $\mathcal{C}$, which by **2.3** becomes an equalizer in ${}^D\mathcal{C}$. We now apply that $P\Box_D - : {}^D\mathcal{C} \longrightarrow {}^C\mathcal{C}$ is a $\mathcal{C}$-module equivalence to conclude that

$$P\Box_D(D\otimes E) \xrightarrow{P\Box_D(D\otimes e)} P\Box_D(D\otimes X) \underset{P\Box_D(D\otimes l)}{\overset{P\Box_D(D\otimes k)}{\rightrightarrows}} P\Box_D(D\otimes B)$$

i.e.

$$(P\Box_D D)\otimes E \xrightarrow{(P\Box_D D)\otimes e} (P\Box_D D)\otimes X \underset{(P\Box_D D)\otimes l}{\overset{(P\Box_D D)\otimes k}{\rightrightarrows}} (P\Box_D D)\otimes Y$$

is an equalizer in ${}^C\mathcal{C}$ (we are using coflatness of P in $\mathcal{C}^D$). Applying the isomorphism $P\Box_D D \cong P$ and the fact that the forgetful functor $\mathcal{U} : {}^C\mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers since C is flat (**2.3**) , we finally obtain that

$$P\otimes E \xrightarrow{P\otimes e} P\otimes X \underset{P\otimes l}{\overset{P\otimes k}{\rightrightarrows}} P\otimes Y$$

is an equalizer in $\mathcal{C}$.

(ii) ⇒ (i) Note that we have the associativity laws in the two diagrams by Lemma Lemma 4.7. Likewise, by part (ii) of the mentioned lemma, we have $P\Box_D(Q\Box_C M) \cong (P\Box_D Q)\Box_C M \cong C\Box_C M \cong M$ for every $M \in {}^C\mathcal{C}$. Similarly, $Q\Box_C(P\Box_D N) \cong (Q\Box_C P)\Box_D N$
$\cong D\Box_D N \cong N$ for every $N \in {}^D\mathcal{C}$. In other words, putting $\mathcal{F} := P\Box_D - : {}^D\mathcal{C} \longrightarrow {}^C\mathcal{C}$ and $\mathcal{G} := Q\Box_C - : {}^C\mathcal{C} \longrightarrow {}^D\mathcal{C}$, we have $\mathcal{F}\mathcal{G} \cong \mathrm{Id}_{{}^C\mathcal{C}}$ and $\mathcal{G}\mathcal{F} \cong \mathrm{Id}_{{}^D\mathcal{C}}$. The functors $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}$-module equivalences because P and Q are coflat in $\mathcal{C}^D$ and $\mathcal{C}^C$, respectively. That the pentagon and the triangle axiom work, one sees by applying diagram (4.10) to the pentagon and the triangle axiom for the associativity and unity constraints of $\mathcal{C}$.
□

Definition 5.10 *Let $C \in \mathcal{C}$ be a flat and cocommutative coalgebra. An invertible C-comodule is a flat right C-comodule P coflat in $\mathcal{C}^C$ for which there exists a further flat right C-comodule Q coflat in $\mathcal{C}^C$ so that $P\Box_C Q \cong C$ and $Q\Box_C P \cong C$ by C-bicomodule isomorphisms f and g, respectively, so that the Diagrams (5.32) and (5.33) commute. We turned right C-comodules into left ones as in (4.9).*

From Morita theory it is clear that:

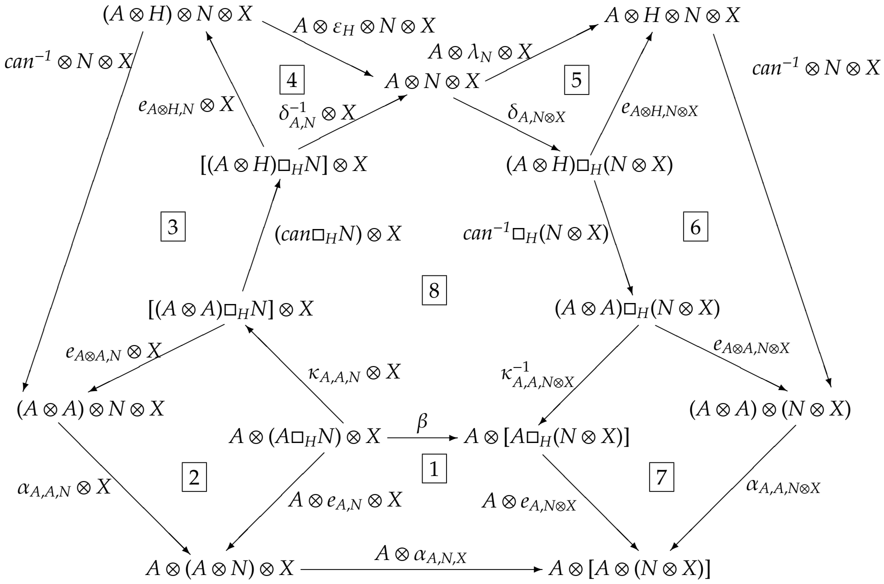
Proposition 5.11 *The set* $\mathrm{Pic}^{co}(C;C)$ *of isomorphism classes of invertible C-comodules in* $\mathcal{C}$ *is a group under the product induced by the cotensor product over C, with unit the class of C and the inverse for the class of P is the class of Q for which* $P\Box_C Q \cong C \cong Q\Box_C P$*. This group is called the Picard group of C.*

Recall that in the proof of Theorem 4.14 we constructed a morphism $\Psi : A\Box_H B \longrightarrow B\Box_H A$, induced by the braiding. As we have seen, it is a well defined isomorphism of H-bicomodules if $\mathcal{C}$ is symmetric, or if A and B are H-Galois objects and Assumption 4.11 holds, or A and B have a normal basis. If one of these conditions is satisfied, then in the definition of an invertible H-comodule the second isomorphism (that we denoted as g) is superfluous. Moreover, in this case the group $\mathrm{Pic}^{co}(\mathcal{C};H)$ is abelian.

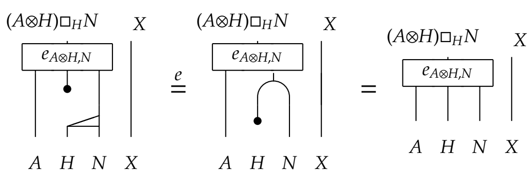
Proposition 5.12 *Let H be a flat and cocommutative Hopf algebra. A right H-Galois object A is an invertible H-comodule.*

Proof. Clearly, A and $\overline{A}$ are flat. From the paragraph after Theorem 4.10, we know that there are H-bicomodule isomorphisms $\tilde{\gamma}_r : H \longrightarrow \overline{A}\Box_H A$ and $\tilde{\gamma}_l : H \longrightarrow A\Box_H\overline{A}$. We are going to prove that A, and analogously $\overline{A}$, is coflat in $\mathcal{C}^H$ and that the Diagrams (5.32) and (5.33) commute for $P = A, Q = \overline{A}, f = \tilde{\gamma}_l$ and $g = \tilde{\gamma}_r$.

To prove coflatness of A we must show that the associativity constraint induces an isomorphism $(A\Box_H N)\otimes X \cong A\Box_H(N\otimes X)$ for arbitrary $N \in {}^H\mathcal{C}$ and $X \in \mathcal{C}$. For this purpose we consider the diagram



The morphisms κ are isomorphisms because A is flat (Lemma 4.3). Note that for $Z \in {}^H\mathcal{C}$ there is a natural isomorphism $\delta_{A,Z} : A\otimes Z \longrightarrow (A\otimes H)\Box_H Z$ induced by the structure morphism of Z. Its inverse is given by $\delta^{-1}_{A,Z} = (A\otimes\varepsilon_H\otimes Z)e_{A\otimes H,Z}$. Let β be equal to the composition of the six isomorphisms lying on the remaining edges of diagram $\langle 8\rangle$. Clearly, β is an isomorphism. The diagrams $\langle 2\rangle - \langle 7\rangle$ commute by definitions of the morphisms along diagram $\langle 8\rangle$. Using the equalizer property of $((A\otimes H)\Box_H N, e_{A\otimes H,N})$, which appears in diagram $\langle 4\rangle$, we find



(5.34)

We now have

$$\begin{aligned}
&(A \otimes e_{A,N\otimes X})\beta(\kappa^{-1}_{A,A,N} \otimes X)((can^{-1}\square_H N) \otimes X)\\
&\stackrel{\beta}{=} (A \otimes e_{A,N\otimes X})\kappa^{-1}_{A,A,N\otimes X}(can^{-1}\square_H(N \otimes X))\delta_{A,N\otimes X}(\delta^{-1}_{A,N} \otimes X)\\
&\stackrel{\langle 4\rangle-\langle 7\rangle}{=} \alpha_{A,A,N\otimes X}(can^{-1} \otimes N \otimes X)(A \otimes \lambda_N \otimes X)(A \otimes \varepsilon_H \otimes N \otimes X)(e_{A\otimes H,N} \otimes X)\\
&\stackrel{(5.34)}{=} (A \otimes \alpha_{A,N,X})(\alpha_{A,A,N} \otimes X)(can^{-1} \otimes N \otimes X)(e_{A\otimes H,N} \otimes X)\\
&\stackrel{\langle 2\rangle-\langle 3\rangle}{=} (A \otimes \alpha_{A,N,X})(A \otimes e_{A,N} \otimes X)(\kappa^{-1}_{A,A,N} \otimes X)((can^{-1}\square_H N) \otimes X).
\end{aligned}$$

From here

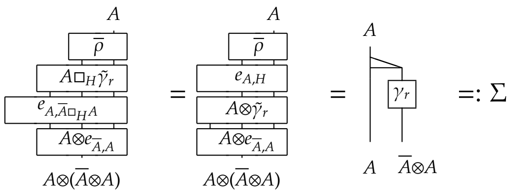
$$(A \otimes e_{A,N\otimes X})\beta = (A \otimes \alpha_{A,N,X})(A \otimes e_{A,N} \otimes X) \stackrel{(4.10)}{=} (A \otimes e_{A,N\otimes X})(A \otimes \theta_{A,N,X}).$$

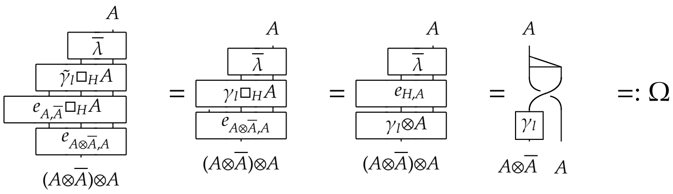
Since $A \otimes e_{A,N\otimes X}$ is a monomorphism, $\beta = A \otimes \theta_{A,N,X}$. Recalling that β is an isomorphism and that A is faithfully flat we conclude finally that $\theta_{A,N,X} : (A\square_H N) \otimes X \longrightarrow A\square_H(N \otimes X)$ is an isomorphism.

We next show that Diagram (5.32) commutes. The commutativity of the other one is similarly proved. Galois objects are faithfully flat, then Lemma Lemma 4.7 ensures the associativity laws in both diagrams. Let $\overline{\lambda} : A \longrightarrow H\square_H A$ and $\overline{\rho} : A \longrightarrow A\square_H H$ denote respectively the isomorphisms from Lemma 4.6 induced by the left and right H-comodule structure morphisms of A. In order to prove that

$$\begin{array}{ccc}
A & \xrightarrow{\quad\overline{\rho}\quad} & A\square_H H\\
{\scriptstyle\overline{\lambda}}\downarrow & & \downarrow{\scriptstyle A\square_H\tilde{\gamma}_r}\\
H\square_H A & \xrightarrow[\tilde{\gamma}_l\square_H A]{} & (A\square_H\overline{A})\square_H A \cong A\square_H(\overline{A}\square_H A)
\end{array} \tag{5.35}$$

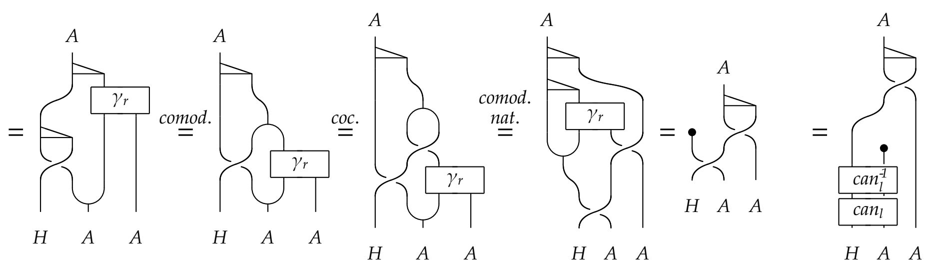
commutes we compute:





We compose Σ with $can_l \otimes A$ and obtain

$(can_l \otimes A)\Sigma$



$= (can_l \otimes A)\Omega$

Since $can_l \otimes A$ is an isomorphism, $\Sigma = \Omega$ up to associativity constraint, or

$$(A\otimes e_{\overline{A},A})e_{A,\overline{A}\square_H A}(A\square_H\tilde{\gamma}_r)\overline{\rho} = \alpha_{A,\overline{A},A}e_{A\otimes\overline{A},A}(e_{A,\overline{A}}\square_H A)(\tilde{\gamma}_l\square_H A)\overline{\lambda}. \tag{5.36}$$

The commutativity of the diagram

$$\begin{array}{ccccc}
(A\square_H\overline{A})\square_H A & \xrightarrow{e_{A,\overline{A}}\square_H A} & (A\otimes\overline{A})\square_H A & \xrightarrow{e_{A\otimes\overline{A},A}} & (A\otimes\overline{A})\otimes A\\
\downarrow{\scriptstyle \omega^{-1}_{A,\overline{A},A}} & & \downarrow{\scriptstyle \kappa^{-1}_{A,\overline{A},A}} & & \downarrow{\scriptstyle \alpha_{A,\overline{A},A}}\\
A\square_H(\overline{A}\square_H A) & \xrightarrow{e_{A,\overline{A}\square_H A}} & A\otimes(\overline{A}\square_H A) & \xrightarrow{A\otimes e_{\overline{A},A}} & A\otimes(\overline{A}\otimes A)
\end{array}$$

yields $\alpha_{A,\overline{A},A}e_{A\otimes\overline{A},A}(e_{A,\overline{A}}\square_H A) = (A\otimes e_{\overline{A},A})e_{A,\overline{A}\square_H A}\omega^{-1}_{A,\overline{A},A}$. With this equation (5.36) becomes

$$(A\otimes e_{\overline{A},A})e_{A,\overline{A}\square_H A}(A\square_H\tilde{\gamma}_r)\overline{\rho} = (A\otimes e_{\overline{A},A})e_{A,\overline{A}\square_H A}\omega^{-1}_{A,\overline{A},A}(\tilde{\gamma}_l\square_H A)\overline{\lambda}.$$

As $A\otimes e_{\overline{A},A}$ and $e_{A,\overline{A}\square_H A}$ are monomorphisms, we get $(A\square_H\tilde{\gamma}_r)\overline{\rho} = \omega^{-1}_{A,\overline{A},A}(\tilde{\gamma}_l\square_H A)\overline{\lambda}$, proving that Diagram (5.35) commutes. □

5.5. The exact sequence

After preparing all the necessary ingredients, we can at last construct the announced short exact sequence.

Theorem 5.13 *Let C be a braided monoidal category with equalizers and $H \in C$ a flat cocommutative Hopf algebra. Under Assumption 4.11, there is a short exact sequence of abelian groups*

$$1 \longrightarrow \mathrm{H}^2(C;H,I) \xrightarrow{\zeta} \mathrm{Gal}(C;H) \xrightarrow{\xi} \mathrm{Pic}^{co}(C;H),$$

where $\zeta([\sigma]) = [H_\sigma]$ and $\xi([A]) = [A]$ for $[\sigma] \in \mathrm{H}^2(C;H,I)$ and $[A] \in \mathrm{Gal}(C;H)$.

Proof. In Theorem 4.14 we proved that, under Assumption 4.11, $\mathrm{Gal}(C;H)$ is an abelian group. By Proposition 5.6, ζ is injective. From Proposition 5.12, ξ is well defined. Since the group structures in $\mathrm{Gal}(C;H)$ and $\mathrm{Pic}^{co}(C;H)$ are both induced by the cotensor product over H, ξ is clearly a group morphism. The kernel of ξ is precisely $\mathrm{Gal}_{nb}(C;H)$. Exactness is proved in Theorem 5.9. □

We will now examine the relationship between this sequence and a similar one of Alonso Álvarez and Fernández Vilaboa constructed in [1, Theorem 11] and [2, Proposition 0.3] for a finite and cocommutative Hopf algebra H in a closed symmetric monoidal category with equalizers and coequalizers. The authors proved that there is a short exact sequence

$$1 \longrightarrow \mathrm{H}^2(C;H,I) \longrightarrow \mathrm{Gal}(C;H) \xrightarrow{\overline{\xi}} \mathrm{Pic}(C;H^*), \tag{5.37}$$

where the group $\mathrm{Pic}(C;H^*)$ consists of isomorphism classes of invertible left H^*-modules. We present our findings in an abbreviated form here, the details can be found in [13, Section 4.5].

Let $\mathrm{Pic}^{co}_f(C;H) \subseteq \mathrm{Pic}^{co}(C;H)$ denote the subgroup of finite invertible H-comodules and let $\beta : \mathrm{Pic}^{co}_f(C;H) \longrightarrow \mathrm{Pic}(C;H^*)$ denote a morphism given by $\beta([M]) := [M^*]$, where M^* is a dual object of M whose right H^*-module structure is induced by the right H-comodule structure of M. In [13, Theorem 5.2.5], which is included in [11, Proposition 5.12], we proved that in a closed symmetric category C and finite H, H-Galois objects are faithfully projective, thus finite. Hence $\mathrm{Im}(\xi) \subseteq \mathrm{Pic}^{co}_f(C;H)$, and then clearly $\beta\xi = \overline{\xi}$. The following is [13, Lemma 4.5.2].

Lemma 5.2 *Assume that C is a closed symmetric monoidal category with equalizers and coequalizers and that H is a finite and cocommutative Hopf algebra. If $\xi : \mathrm{Gal}(C;H) \longrightarrow \mathrm{Pic}^{co}(C;H)$ is the morphism from Theorem 5.13, then $\beta\xi : \mathrm{Gal}(C;H) \longrightarrow \mathrm{Pic}(C;H^*)$ is the group morphism $\overline{\xi}$ from the sequence (5.37).*

In a closed category C the modules determining the group $\mathrm{Pic}(C;H^*)$ from [1] and [2] which are moreover H^*-coflat (in the sense of [25, Page 202]) can be viewed as C-autoequivalences of ${}_{H^*}C$, dually to Theorem 5.1. Let $\mathrm{Pic}^{cofl}(C;H^*) \subseteq \mathrm{Pic}(C;H^*)$ denote the corresponding subgroup. Note that the dual condition to the flatness appearing in Theorem 5.1, 2) is omitted, as it is automatically fulfilled. Namely, C is closed and the tensor functor, as a left adjoint, preserves coequalizers. The relationship between the groups $\mathrm{Pic}^{co}(C;H)$ of invertible Picard H-comodules and $\mathrm{Pic}^{cofl}(C;H^*)$ is revealed by the following proposition, which is [13, Proposition 4.5.3].

Proposition 5.3 *Assume that C is a closed braided monoidal category with equalizers and coequalizers and that H is a finite and cocommutative Hopf algebra. There is a group isomorphism $\mathrm{Pic}^{co}(C;H) \cong \mathrm{Pic}^{cofl}(C;H^*)$.*

6. Examples

This final section will be devoted to examples to illustrate the main results of this paper. We will take C the category of left H-modules over a quasi-triangular Hopf algebra $(H,\mathcal{R})$ (over a field K). We will compute the group of Galois objects for two examples of $B \in C$ and we will compare it with the second Sweedler cohomology group of B. The examples we will deal with are Radford Hopf algebra H_ν and Nichols Hopf algebra $E(n)$.

6.1. Radford biproducts and Majid's bosonization

It is well known (see [23, Theorem 10.4.2] and [22, Theorem 9.2.4]) that a Hopf algebra H with bijective antipode over a field K (or just a bialgebra H) has a quasi-triangular structure $\mathcal{R}$ if and only if ${}_H\mathcal{M}$ is a braided monoidal category with braiding $\Phi_{\mathcal{R}} := \tau\mathcal{R}$, where τ is the flip map. The quasi-triangular structure on H is recovered by setting $\mathcal{R} := \tau\Phi(1_H \otimes 1_H)$. The difference here between the bialgebra and the Hopf algebra case is that in the latter the bijectivity of the antipode enables one to have an explicit expression for the inverse of the quasi-triangular structure $\mathcal{R}$, namely, it is: $\mathcal{R}^{-1} = S(\mathcal{R}^{(1)}) \otimes \mathcal{R}^{(2)} = \mathcal{R}^{(1)} \otimes S^{-1}(\mathcal{R}^{(2)})$. Moreover, recall that triangular structure corresponds to a symmetric braiding.

Let H be a bialgebra and B an algebra in ${}_H\mathcal{M}$ and a coalgebra in ${}^H\mathcal{M}$. Radford's biproduct Theorem [28, Theorem 2.1 and Proposition 2], in the reformulation due to Majid's observation, characterizes that the space $B \times H = B \otimes H$ is a bialgebra (respectively, a Hopf algebra), with particularly introduced bialgebra (resp. Hopf algebra) structure which one calls *Radford biproduct*, if and only if B is a bialgebra (resp. a Hopf algebra) in the braided monoidal category ${}_H^H\mathcal{YD}$ of Yetter-Drinfel'd H-modules.

For a quasi-triangular bialgebra $(H,\mathcal{R})$ every left H-module M belongs to ${}_H^H\mathcal{YD}$ with coaction $\lambda : M \longrightarrow H \otimes M$ given by $\lambda(m) := \mathcal{R}^{(2)} \otimes (\mathcal{R}^{(1)} \triangleright_H m)$ for $m \in M$. Moreover, ${}_H\mathcal{M}$ is a braided monoidal subcategory of ${}_H^H\mathcal{YD}$. The process of obtaining an ordinary Hopf algebra $B \times H$ out of a Hopf algebra B in ${}_H\mathcal{M}$ as above is called *bosonization* by Majid. It is a particular case of Radford's Theorem.

6.2. Example: Radford Hopf algebra

A family of Hopf algebras generalizing Sweedler Hopf algebra was constructed by Radford in [29]. It is

$$H_\nu = K\langle g, x | g^{2\nu} = 1, x^2 = 0, gx = -xg\rangle$$

for ν an odd natural number and $char(K) \nmid 2\nu$. The element $g \in H_\nu$ is group-like whereas x is a $(g^\nu, 1)$-primitive element, i.e., $\Delta(x) = 1 \otimes x + x \otimes g^\nu$ and $\varepsilon(x) = 0$. The antipode is defined as $S(g) = g^{-1}$ and $S(x) = g^\nu x$. The quasi-triangular structures on H_ν have the form

$$\mathcal{R}_{s,\beta} = \frac{1}{2\nu}\Big(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i \otimes g^{sl}\Big) + \frac{\beta}{2\nu}\Big(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i x \otimes g^{sl+\nu} x\Big) \tag{6.38}$$

for $1 \leqslant s < 2\nu$ odd, $\beta \in K$ and ω a 2ν-th primitive root of unity.

The Hopf algebra H_ν is a Radford biproduct of $B = K[x]/(x^2)$ and the group algebra $L = K\mathbb{Z}_{2\nu}$. Since $K\mathbb{Z}_{2\nu}$ is a self-dual Hopf algebra (because K contains a primitive 2ν-th root of unity), we can identify the category $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ of $\mathbb{Z}_{2\nu}$-graded vector spaces with the category ${}_L\mathcal{M}$. The L-module structure on a $\mathbb{Z}_{2\nu}$-graded vector space M is given by $g \cdot m = \omega^{deg(m)s} m$ for homogeneous $m \in M$. Furthermore, we have that ((6.38)) for $\beta = 0$ determines a quasi-triangular structure on L. The braiding $\Phi_{\mathcal{R}_{s,0}}$ on ${}_L\mathcal{M}$ arising from $\mathcal{R}_{s,0}$ coincides with the one induced in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ by the corresponding bicharacter, that is, $\Phi_{\mathcal{R}_{s,0}}(m \otimes n) = \omega^{deg(m)deg(n)s} n \otimes m$ for $M, N \in \mathcal{G}r_{\mathbb{Z}_{2\nu}}$ and homogeneous $m \in M, n \in N$. *The braiding* $\Phi_{\mathcal{R}_{s,0}}$ *is not a symmetry if* $s \neq \nu$.

The algebra $B = K[x]/(x^2)$ is a $\mathbb{Z}_{2\nu}$-graded vector space by setting $B_0 = K1, B_\nu = Kx$, and $B_\sigma = 0$ for all $\sigma \neq 0, \nu$. It becomes a commutative and cocommutative Hopf algebra in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ with the coalgebra structure and antipode given as follows:

$$\Delta(1) = 1 \otimes 1,\ \varepsilon(1) = 1,\ S(1) = 1,\ \Delta(x) = 1 \otimes x + x \otimes 1,\ \varepsilon(x) = 0,\ S(x) = -x.$$

We now may consider the Hopf algebra $B \times L$ obtained by Majid's bosonization. It is not difficult to show that the map $\Psi : H_\nu \longrightarrow K[x]/(x^2) \times K\mathbb{Z}_{2\nu}, G \mapsto 1 \times g, X \mapsto x \times g^\nu$ is a Hopf algebra isomorphism. Here we denote the generators of H_ν by G and X instead of g and x. We will compute the group $\mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$ and check that it coincides with $\mathrm{H}^2(C; K[x]/(x^2), K)$. Although $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is not a symmetric category for $s \neq \nu$ and thus we do not know a priori if $\mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$ is a group, it will turn out that it coincides with the subgroup $\mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$, which we know is a group by Corollary 4.19.

For $\alpha \in K$ we define the algebra $C(\alpha) := K\langle y | y^2 = \alpha\rangle$. It is $\mathbb{Z}_{2\nu}$-graded by $C(\alpha)_0 = K, C(\alpha)_\nu = Ky$ and the rest of homogeneous components are zero. Furthermore, $C(\alpha)$ is a right B-comodule algebra with the comodule structure morphism $\rho : C(\alpha) \longrightarrow C(\alpha) \otimes B$ given by

$$\rho(1) = 1 \quad \text{and} \quad \rho(y) = 1 \otimes x + y \otimes 1.$$

It is easy to see that $can : C(\alpha) \otimes C(\alpha) \longrightarrow C(\alpha) \otimes B$ is a $\mathbb{Z}_{2\nu}$-graded isomorphism. Thus $C(\alpha)$ is indeed a B-Galois object in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$. Moreover, it has the normal basis property since the map $B \to C(\alpha), 1 \mapsto 1, x \mapsto y$ is a B-comodule isomorphism.

Conversely, we show that any B-Galois object A in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is of this form. From the isomorphism $can : A \otimes A \to A \otimes B$ we obtain $dim_K(A) = dim_K(B) = 2$. Since $K1 \subset A_0$, it follows that $dim_K(A_0) \geq 1$. If $dim_K(A_0) = 2$, then $A = A_0$ and since the structure morphism $\rho : A \to A \otimes B$ is $\mathbb{Z}_{2\nu}$-graded, $\rho(A) \subseteq A \otimes K1$, implying $A \subseteq A^{co(B)} = K$, a contradiction. As $dim_K(A_0) = 1$, there is a unique $1 \leq \epsilon < 2\nu$ such that $A_\epsilon \neq \{0\}$. This forces $\epsilon = \nu$. Thus we may write $A = A_0 \oplus A_\nu$, where $A_0 = K1$ and $A_\nu = Ku$ with $u \neq 0$ and $u^2 = \gamma \in K$. Knowing that ρ is a morphism in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$, we obtain $\rho(u) = a \otimes x + bu \otimes 1$, for some $a, b \in K$. We claim that $a \neq 0$. If $a = 0$, then $can(1 \otimes u) = bu \otimes 1 = can(bu \otimes 1)$, contradiction. Take $v = \frac{1}{a}u$. Then $\rho(v) = 1 \otimes x + bv \otimes 1$. Since $(A \otimes \varepsilon)\rho(v) = v$, we get $b = 1$. Note that $v^2 = \alpha$, for some $\alpha \in K$. It is easy to check that $\varphi : A \longrightarrow C(\alpha)$, defined by $\varphi(1) = 1$ and $\varphi(v) = y$, is an B-comodule algebra isomorphism. We have thus proved that any B-Galois object has the normal basis property and we can consider the group $\mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B)$.

Theorem 6.1 *The map* $\Omega : (K, +) \longrightarrow \mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B), \alpha \mapsto [C(\alpha)]$ *is a group isomorphism.*

Proof. For $\alpha, \beta \in K$ let $C(\alpha) = K\langle y | y^2 = \alpha\rangle$, $C(\beta) = K\langle z | z^2 = \beta\rangle$ and $C(\alpha + \beta) = K\langle w | w^2 = \alpha + \beta\rangle$. We prove that Ω is a group morphism by giving a right B-comodule algebra morphism $\theta : C(\alpha + \beta) \longrightarrow C(\alpha)\square_B C(\beta)$ (recall Proposition 3.14). Let ρ_α and ρ_β denote the right B-comodule structure morphisms of $C(\alpha)$ and $C(\beta)$, respectively. We turn $C(\beta)$ into a left B-comodule via the inverse braiding. If λ_β denotes the left B-comodule structure morphism, then $\lambda_\beta(1) = 1 \otimes 1$ and $\lambda_\beta(z) = 1 \otimes z + x \otimes 1$. It is easy to see that $\{1 \otimes 1, 1 \otimes z + y \otimes 1\}$ is a K-basis of $C(\alpha)\square_B C(\beta)$. Recall that we consider $C(\alpha) \otimes C(\beta)$ as a right B-comodule via $C(\alpha) \otimes \rho_\beta$. It is also easy to show that $\theta : C(\alpha + \beta) \longrightarrow C(\alpha)\square_B C(\beta)$, defined by $\theta(1) = 1 \otimes 1$ and $\theta(w) = 1 \otimes z + y \otimes 1$, is a right B-comodule algebra morphism in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$.

We prove that Ω is injective. Let $\vartheta : C(\alpha) \longrightarrow C(\beta)$ be a $\mathbb{Z}_{2\nu}$-graded right B-comodule algebra isomorphism. Then $\vartheta(1) = 1$ and $\vartheta(y) = \kappa z$, for some $\kappa \in K$. Since ϑ is right B-colinear, $(\vartheta \otimes B)\rho_\alpha(y) = \rho_\beta \vartheta(y)$. That is,

$1 \otimes x + \kappa z \otimes 1 = \kappa \otimes x + \kappa z \otimes 1$. Hence $\kappa = 1$. On the other hand, $\alpha = \vartheta(\alpha) = \vartheta(y^2) = \vartheta(y)^2 = \kappa^2 z^2 = \beta$. That Ω is surjective was previously proved. □

Observe that $C(\alpha)$ is just B_{σ_α} where the cocycle $\sigma_\alpha : B \otimes B \to K$ is given by $\sigma_\alpha(1 \otimes 1) = 1, \sigma_\alpha(1 \otimes x) = \sigma_\alpha(x \otimes 1) = 0$ and $\sigma_\alpha(x \otimes x) = \alpha$. On the other hand, taking into account the grading of B, any normalized 2-cocycle of B with values in K is of this form. Then $H^2(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B, K) \cong \mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B) = \mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B) \cong (K, +)$.

We next compute the group of biGalois objects of B. Any Hopf automorphism of B is of the form $f_\lambda(1) = 1$ and $f_\lambda(x) = \lambda x$ for $\lambda \in K^\times$ and $\mathrm{Aut}_{Hopf}(B) \cong K^\times$. It is easy to show that $B_{\sigma_\alpha} \triangleleft f_\lambda \cong B_{\sigma_{\alpha\lambda^2}}$ as right B-comodule algebras. Then $\mathrm{BiGal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; B) \cong K^\times \ltimes (K, +)$ with product $(\lambda, \alpha)(\gamma, \beta) = (\lambda\gamma, \gamma^2\alpha + \beta)$, due to Theorem 4.21.

6.3. Example: Nichols Hopf algebra

Nichols Hopf algebra $E(n)$ will provide us with an example where the morphism ζ in Theorem 5.13 is not surjective. This is explained in [11, Remark 6.10]. Nichols' Hopf algebra is

$$E(n) = K\langle g, x_i, i, j \in \{1, ..., n\} | g^2 = 1, x_i^2 = 0, gx_i = -x_i g, x_i x_j = -x_j x_i \rangle$$

and it has a triangular structure given by $\mathcal{R}_0 = \frac{1}{2}(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g)$, the same one as for Sweedler Hopf algebra $H_4 = E(1)$, which is $\mathcal{R}_{0,0}$ from (6.38) with $\nu = 1$. The element $g \in E(n)$ is grouplike, whereas $x_i \in E(n)$ for $i = 1, ..., n$ are $(g, 1)$-primitive elements, that is $\Delta(x_i) = 1 \otimes x_i + x_i \otimes g$ and $\varepsilon(x_i) = 0$. The antipode is given by $S(g) = g^{-1}$ and $S(x_i) = gx_i$.

Nichols' Hopf algebra $E(n)$ is isomorphic to the Radford biproduct $K[x_n]/(x_n^2) \times E(n-1)$. Set $L = E(n-1)$. The algebra $B = K[x_n]/(x_n^2)$ is an L-module via $g \cdot x_n = -x_n$ and $x_i \cdot x_n = 0, i = 1, ..., n-1$. The Hopf algebra structure of B in the category ${}_L\mathcal{M}$ is similar to that of $K[x]/(x^2)$: $\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1, \varepsilon(x_n) = 0$ and $S(x_n) = -x_n$.

Similarly as in the case of H_ν, it is proved that the algebra morphism $\Psi' : E(n) \longrightarrow K[x_n]/(x_n^2) \times E(n-1)$, defined on generators by $\Psi'(G) = 1 \otimes g, \Psi'(X_i) = 1 \otimes x_i$ and $\Psi'(X_n) = x_n \otimes g$, for $i = 1, ..., n-1$, is a Hopf algebra isomorphism. Here we denote the generators of $E(n)$ by G and $X_i, i = 1, ..., n$ instead of g and $x_i, i = 1, ..., n$.

Similarly as in Theorem 6.1 we have the group isomorphism

$$Gal({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong (K, +)^n \tag{6.39}$$

given by

$$(\alpha, \alpha_1, ..., \alpha_{n-1}) \mapsto M(\alpha, \alpha_1, ..., \alpha_{n-1}) = K\langle y | y^2 = \alpha \rangle$$

where $M(\alpha, \alpha_1, \cdots, \alpha_{n-1})$ has a structure of an $E(n-1)$-module by $g \cdot y = -y$ and $x_i \cdot y = \alpha_i$, for $i \in \{1, \cdots, n-1\}$, and of a right $K[x_n]/(x_n^2)$-comodule by $\rho(1) = 1 \otimes 1, \rho(y) = 1 \otimes x_n + y \otimes 1$.

Looking at the above given L-module structure on B, it is clear that $M(\alpha, \alpha_1, ..., \alpha_{n-1})$ is isomorphic to $K[x_n]/(x_n^2)$ as a left L-module only if all $\alpha_i = 0, i = 1, ..., n-1$. Consequently, in the case of Nichols' Hopf algebra the subgroup of Galois objects with normal basis is a proper subgroup in the whole group, as we announced. In other words, the morphism $\zeta : H^2({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2), K) \longrightarrow \mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$ from Theorem 5.13 is not surjective and the $K[x_n]/(x_n^2)$-comodules $M(\alpha, \alpha_1, ..., \alpha_{n-1})$ where some $\alpha_i \neq 0$ give non-trivial elements in the group $\mathrm{Pic}^{co}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$.

For the Galois objects with normal basis we thus have

$$\mathrm{Gal}_{nb}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong H^2({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2), K) \cong (K, +)$$

and its elements are represented by $M(\alpha, 0, ..., 0)$, which, similarly as in the case of H_ν, equal B_{σ_α}, where $\sigma_\alpha : B \otimes B \to K$ is the cocycle given by $\sigma_\alpha(1 \otimes 1) = 1, \sigma_\alpha(1 \otimes x_n) = \sigma_\alpha(x_n \otimes 1) = 0$ and $\sigma_\alpha(x_n \otimes x_n) = \alpha$.

From the above said we find: $\mathrm{BiGal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong K^\times \ltimes (K, +)^n$.

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