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Nonoscillatory Solutions of Neutral Dynamic Equations with Positive and Negative Coefficients on Time Scales

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Abstract. In this paper, we establish the existence of nonoscillatory solutions to neutral dynamic equations with positive and negative coefficients on time scales of the form

$$(x(t) - \int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau)^{\Delta} + \sum_{i=1}^{l} r_i(t)x(\delta_i(t)) - \sum_{j=1}^{m} s_j(t)x(\eta_j(t)) = 0.$$

Two examples are included to illustrate our presented results. In our approaches, a key role is played by the fixed point technique.

1. Introduction

The theory of time scales, which has received a lot of attention, was introduced by Hilger [1] in order to unify continuous and discrete analysis. The study of dynamic equations on time scales reveals such discrepancies, and helps to avoid proving results twice, once for differential equations and once for difference equations. In recent years, there has been much significant research activity concerning the oscillation and nonoscillation of dynamic equations on time scales, we refer readers to the references [2-13]. We also refer to the papers [14-16] for oscillatory and nonoscillatory solutions to models from mathematical biology and physics formulated by partial differential equations and such that their long time behavior is connected to the external source, idealized by nonlocal and/or taxis-driven terms. For example, in 2007, Zhu and Wang [2] established some necessary and sufficient conditions for the existence of nonoscillatory solutions to the neutral functional dynamic equation

$$(x(t) + p(t)x(g(t)))^{\Delta} + f(t, x(h(t))) = 0.$$

In 2019, Zhou, Alsaedi and Ahmad [3] studied the existence of oscillatory and nonoscillatory solutions for the delay dynamic equation

$$(y(t) - C(t)y(t-\xi))^{\Delta} + P(t)y(t-\eta) - Q(t)y(t-\delta) = 0.$$

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Chen, Lv, He and Li [4] considered the existence of nonoscillatory solutions to neutral dynamic equation

$$(x(t)-\int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau)^\Delta+\int_c^d \omega(t,\nu)x(h(t,\nu))\Delta\nu=0.$$

Motivated by the above works, we are concerned with first-order neutral functional dynamic equations of the following form

$$(x(t)-\int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau)^\Delta+\sum_{i=1}^{l} r_i(t)x(\delta_i(t))-\sum_{j=1}^{m} s_j(t)x(\eta_j(t))=0, \tag{1}$$

where $t\in\mathbb{T},\mathbb{T}=[t_0,\infty)_\mathbb{T}:=\{t\in\mathbb{T}: t\geq t_0\}$ is a time scale, $t_0\in\mathbb{R}$. Eq. (1) is often used for mathematical modelling of various physical, chemical and biological systems. The main feature of Eq. (1) is that the positive and negative perturbations are separated. However, there have been few studies in present papers.

In this paper, we obtain some new sufficient conditions for the existence of nonoscillatory solutions to Eq. (1) by the fixed point theory in Banach space and the theory of time scales. The results of this paper enrich the research of nonoscillatory solutions of dynamic equations on time scales.

As it is customary, a solution is called oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory.

This paper is organized as follows. In Section 2, we recall some preliminaries and lemmas. In Section 3, we will establish the existence of nonoscillatory solutions for Eq. (1). Finally, some applications are presented in Section 4.

2. Preliminaries

A time scale is an nonempty closed subset of the real numbers $\mathbb{R}$, such as $\mathbb{R}$, natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, cantor set, etc.. Let $\mathbb{T}$ be a time scale with $\sup\mathbb{T}=\infty$. We denote the closed interval in $\mathbb{T}$ by $[a,b]:=\{t\in\mathbb{T}: a\leq t\leq b\}$. Open intervals, half-open intervals and others are defined accordingly.

According to [17], we recall some concepts related to time scales.

Definition 2.1. *For $t\in\mathbb{T}$, we define a forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$. If $t<\sigma(t)(t<\sup\mathbb{T})$ as well as $t=\sigma(t)$, then t is right-scatterd and right-dense, respectively. A backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t):=\sup\{s\in\mathbb{T}: s<t\}$. If $t>\rho(t)(t>\inf\mathbb{T})$ as well as $t=\rho(t)$, then t is left-scatterd and left-dense, respectively. The graininess operator $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$.*

Definition 2.2. *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k:=\mathbb{T}\setminus\{m\}$; otherwise $\mathbb{T}^k=\mathbb{T}$.*

For a function $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, we define f^Δ to be number (provided it exists) with property that given any $\varepsilon>0$, there is a neighborhood U of t (i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that

$$\Big|[f(\sigma(t))-f(s)]-f^\Delta(t)[\sigma(t)-s]\Big|\leq\varepsilon|\sigma(t)-s|, \text{ for all } s\in U.$$

We call f^Δ the delta or Hilger derivative of f at t.

Definition 2.3. *A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all such functions is denoted by $C_{rd}=C_{rd}(\mathbb{T})=C_{rd}(\mathbb{T},\mathbb{R})$.*

Definition 2.4. *Every rd-continuous function has an antiderivative. In particular if $t_0\in\mathbb{T}$, then an antiderivative of f is defined by $F(t):=\int_{t_0}^t f(\tau)\Delta\tau (t\in\mathbb{T})$. And we define the Cauchy integral by*

$$\int_r^s f(t)\Delta t=F(s)-F(r) \text{ for all } r,s\in\mathbb{T}.$$

The following theorems will be used to prove our main results in the next section.

Lemma 2.5. ([2]) *Suppose that* $X \subseteq BC[T_0, \infty)_{\mathbb{T}}$ *is bounded and uniformly Cauchy. Further, suppose that* X *is equi-continuous on* $[T_0, T_1]_{\mathbb{T}}$ *for any* $T_1 \in [T_0, \infty)_{\mathbb{T}}$. *Then* X *is relatively compact.*

Lemma 2.6. ([18] *Krasnoselskii's fixed point theorem) Suppose that* Ω *is a Banach space and* X *is bounded, convex and closed subset of* Ω. *Suppose further that there exist two operators* $U, S : X \to \Omega$ *such that*
(1) $Ux + Sy \in X$ *for all* $x, y \in X$;
(2) U *is a contraction mapping;*
(3) S *is completely continuous.*
Then $U + S$ *has a fixed point in* X.

Lemma 2.7. ([2]) *Suppose that* Ω *is a Banach space and* X *is bounded, convex and closed subset of* Ω. *Suppose further that there exist an operator* $F : X \to \Omega$ *such that*
(1) $Fx \in X$ *for all* $x \in X$;
(2) F *is completely continuous.*
Then F *has a fixed point in* X.

3. Main results

Throughout this section, we will assume in Eq. (1) that

(H_1) $r_i(t), s_j(t) \in C_{rd}(\mathbb{T}, [0, \infty))$, where $i = 1, 2, ..., l$, $j = 1, 2, ..., m$, and $l + m = q$;

(H_2) $\delta_i(t), \eta_j(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$, $g(t, \tau) \in C_{rd}(\mathbb{T} \times [a, b], \mathbb{T})$, and $\max_{\tau\in[a,b]} g(t, \tau) \le t$, $\delta_i(t) \le t$, $\eta_j(t) \le t$, for $t \ge t_0$, $\lim_{t\to\infty} \max_{\tau\in[a,b]} g(t, \tau) = \infty$, $\lim_{t\to\infty} \delta_i(t) = \infty$, $\lim_{t\to\infty} \eta_j(t) = \infty$;

(H_3) $p(t, \tau) \in C_{rd}(\mathbb{T} \times [a, b]_{\mathbb{T}}, [0, \infty))$, $0 < P(t) = \int_a^b p(t, \tau)\Delta\tau \le \alpha < 1$ and $0 \le \lim_{t\to\infty} P(t) = \beta \le \alpha$ for all $t \in \mathbb{T}$;

(H_4) There exists $T_0 \in \mathbb{T}$ large enough such that

$$\int_{T_0}^{\infty} r_i(\xi)\Delta\xi \le \frac{1-\alpha}{q} < 1 \;\text{ and }\; \int_{T_0}^{\infty} s_j(\xi)\Delta\xi \le \frac{1-\alpha}{q} < 1 \;\text{ for any } i \text{ or } j.$$

In the sequel, we use the notation

$$z(t) = x(t) - \int_a^b p(t, \tau)x(g(t, \tau))\Delta\tau - \int_{T_0}^{t} \sum_{j=1}^{m} s_j(\tau)x(\eta_j(\tau))\Delta\tau. \tag{2}$$

Lemma 3.1. *If* $x(t)$ *is an eventually positive solution of Eq. (1), then eventually* $z^{\Delta}(t) < 0$ *and* $z(t) > 0$.

Proof. Since $x(t)$ is an eventually positive solution of Eq. (1), there exists $T_1 \ge T_0$ such that

$$x(g(t, \tau)) > 0, x(\delta_i(t)) > 0, x(\eta_j(t)) > 0 (i = 1, 2, ..., l, j = 1, 2, ..., m) \;\text{ for }\; t \ge T_1.$$

In view of (H_1) and (H_2), we get

$$z^{\Delta}(t) = (x(t) - \int_a^b p(t, \tau)x(g(t, \tau))\Delta\tau)^{\Delta} - \sum_{j=1}^{m} s_j(t)x(\eta_j(t)) = -\sum_{i=1}^{l} r_i(t)x(\delta_i(t)) < 0,$$

which implies that $z(t)$ is decreasing for $t \ge T_1$.

Next, we will show that $z(t) > 0$. If $z(t) \to -\infty$ as $t \to \infty$, then $x(t)$ is unbounded by (2). Hence, there is the subsequence $\{t_n\}$ on $[T_1, \infty)_{\mathbb{T}}$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} x(t_n) = \infty$$

and $x(t_n) = \max_{T_1 \le t \le t_n} x(t)$ for each $n \in N$. Moreover, according to (H_3) and (H_4) we have

$$z(t_n) = x(t_n) - \int_a^b p(t_n, \tau)x(g(t_n, \tau))\Delta\tau - \int_{T_0}^{t_n} \sum_{j=1}^{m} s_j(\tau)x(\eta_j(\tau))\Delta\tau$$

$$\ge x(t_n)(1 - \int_a^b p(t_n, \tau)\Delta\tau - \int_{T_0}^{t_n} \sum_{j=1}^{m} s_j(\tau)\Delta\tau) \ge x(t_n)\frac{l(1-\alpha)}{q} > 0. \tag{3}$$

Hence, we get

$$\lim_{t\to\infty} z(t) = \lim_{n\to\infty} z(t_n) \ge 0,$$

which is in contradiction with $z(t) \to -\infty$. Therefore, $\lim_{t\to\infty} z(t) = A$ and is finite. As can be seen from the proof above, if $x(t)$ is unbounded, then $A \ge 0$.

If $x(t)$ is bounded, there is the subsequence $\{t'_n\}$ on $[T_1, \infty)_{\mathbb{T}}$ such that

$$\lim_{n\to\infty} x(t'_n) = \limsup_{t\to\infty} x(t) = \overline{B}.$$

Therefore, we have

$$x(t'_n) - z(t'_n) = \int_a^b p(t'_n, \tau)x(g(t'_n, \tau))\Delta\tau + \int_{T_0}^{t'_n} \sum_{j=1}^{m} s_j(\tau)x(\eta_j(\tau))\Delta\tau$$

$$\le y(t'_n)(\alpha + \frac{m(1-\alpha)}{q}) \le y(t'_n), \tag{4}$$

where $y(t'_n) = \max\{\max_{\tau\in[a,b]}\{x(g(t'_n, \tau))\}, \max_{1\le j\le m}\{x(\eta_j(s)) : T_1 \le s \le t'_n\}\}$. Hence, it follows that

$$\limsup_{n\to\infty} y(t'_n) \le \overline{B}.$$

Taking supremum limit of the two sides of (4) as $n \to \infty$, we get $\overline{B} - A \le \overline{B}$. Hence, $A \ge 0$. To sum up, we deduce $z(t) > 0$. □

Theorem 3.2. *If $x(t)$ is an eventually positive solution of Eq. (1), then* $\lim_{t\to\infty} x(t) = B > 0$ *or* $\lim_{t\to\infty} x(t) = 0$.

Proof. In view of Lemma 3.1, we have $\lim_{t\to\infty} z(t) = A \ge 0$ and A is finite.

We assert that $x(t)$ is bounded. In fact, if it is not true, it follows that $z(t) \to \infty$ from (3) which is in contradiction. Hence $x(t)$ is bounded. We suppose that $\limsup_{t\to\infty} x(t) = \overline{B}$ and $\liminf_{t\to\infty} x(t) = \underline{B}$.

We define the function

$$S(t, m, T_0) = \int_{T_0}^{t} \sum_{j=1}^{m} s_j(\xi)\Delta\xi,$$

which is increasing and upper bounded for t, and there exists the limit λ as $t \to \infty$. It's not hard to get $\lambda \le \frac{m(1-\alpha)}{q}$. From (2), we have

$$A \ge \overline{B} - \beta\overline{B} - \lambda\overline{B}, A \le \underline{B} - \beta\underline{B} - \lambda\underline{B}.$$

Thus, we have $\overline{B} = \underline{B}$. To sum up, we get that $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = \frac{A}{1-\beta-\lambda}$. The proof is complete. □

Theorem 3.3. *Assume that*

$$\frac{(1-\alpha)lr_i(t)}{q} > \sum_{j=1}^{m} s_j(t) \quad for \quad each \quad t\in\mathbb{T} \quad and \quad all \quad i=1,2,...,l. \tag{5}$$

Then Eq. (1) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} x(t) = B > 0$.

Proof. Define the set of all continuous bounded functions

$$BC[T_0,\infty) := \{x : x\in C([T_0,\infty)_{\mathbb{T}},\mathbb{R}) \text{ and } \sup_{t\in[T_0,\infty)_{\mathbb{T}}} |x(t)| < \infty\},$$

with the norm $\|x\| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}}|x(t)|$. Then $BC[T_0,\infty)$ is a Banach space. Let

$$X = \{x : x\in BC[T_0,\infty)_{\mathbb{T}} \text{ and } \frac{(1-\alpha)}{q}K \le x(t) \le K, K>0\}.$$

It is easy to check that X is a bounded, convex and closed subset of $BC[T_0,\infty)$.

Now we define two operators U and $S : X \to BC[T_0,\infty)$ as follow

$$(Ux)(t) = \int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau,$$
$$(Sx)(t) = \frac{m}{q}(1-\alpha)K + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi - \int_t^\infty \sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi.$$

Next, we will show that U and S satisfy the conditions in Lemma 2.6.

(i) We will show that $Ux+Sy\in X$ is true for any $x,y\in X$ and $t\in[T_0,\infty)_{\mathbb{T}}$.

In view of (H_3), (H_4) and (5), we have

$$\begin{aligned}(Ux)(t)+(Sy)(t) \quad &\ge \frac{m}{q}(1-\alpha)K + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)y(\delta_i(\xi))\Delta\xi - \int_t^\infty \sum_{j=1}^{m} s_j(\xi)y(\eta_j(\xi))\Delta\xi \\ &\ge \frac{m}{q}(1-\alpha)K + \frac{1-\alpha}{q}K\int_t^\infty \sum_{i=1}^{l} r_i(\xi)\Delta\xi - K\int_t^\infty \sum_{j=1}^{m} s_j(\xi)\Delta\xi \\ &\ge \frac{m}{q}(1-\alpha)K + K\int_t^\infty (\frac{1-\alpha}{q}\sum_{i=1}^{l} r_i(\xi) - \sum_{j=1}^{m} s_j(\xi))\Delta\xi \\ &\ge \frac{m}{q}(1-\alpha)K \ge \frac{1-\alpha}{q}K\end{aligned}$$

and

$$\begin{aligned}(Ux)(t)+(Sy)(t) \quad &\le \frac{m}{q}(1-\alpha)K + \int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)y(\delta_i(\xi))\Delta\xi \\ &\le \frac{m}{q}(1-\alpha)K + \alpha K + lK\frac{1-\alpha}{q} = K.\end{aligned}$$

So $Ux+Sy\in X$ for any $x,y\in X$ and $t\in[T_0,\infty)_{\mathbb{T}}$.

(ii) We will show that U is a contraction operator.

For any $x, y \in X$ and $t \in [T_0, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} |(Ux)(t) - (Uy)(t)| &= |\int_a^b p(t,\tau)(x(g(t,\tau)) - y(g(t,\tau)))\Delta\tau| \\ &\leq \int_a^b p(t,\tau)\Delta\tau \parallel x - y \parallel \leq \alpha \parallel x - y \parallel . \end{aligned}$$

This implies that

$$\parallel Ux - Uy \parallel \leq \alpha \parallel x - y \parallel .$$

Hence, we conclude that U is a contraction operator on X.

(iii) We now prove that S is completely continuous operator on X.

First, we show that S maps X into X. For any $x \in X$, according to the proof of (i) we have

$$(Sx)(t) \geq \frac{1-\alpha}{q}K$$

and

$$\begin{aligned} (Sx)(t) &\leq \frac{m}{q}(1-\alpha)K + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi \\ &\leq \frac{m}{q}(1-\alpha)K + lK\frac{1-\alpha}{q} = (1-\alpha)K < K. \end{aligned}$$

That is, S maps X into X.

Second, we show that S is continuous. Let $\{x_n\} \subset X$ and $\|x_n - x\| \to 0$ as $n \to \infty$. Since X is closed, we have $x \in X$. Then, $|x_n(t) - x(t)| \to 0$ is obvious for any $t \in [T_0, \infty)_{\mathbb{T}}$, and

$$\begin{aligned} |(Sx_n)(t) - (Sx)(t)| &\leq |\int_t^\infty \sum_{i=1}^{l} r_i(\xi)(x_n(\delta_i(\xi)) - x(\delta_i(\xi)))\Delta\xi| \\ &\quad + |\int_t^\infty \sum_{j=1}^{m} s_j(\xi)(x_n(\eta_j(\xi)) - x(\eta_j(\xi)))\Delta\xi| \\ &\leq (|\int_t^\infty \sum_{i=1}^{l} r_i(\xi)\Delta\xi| + |\int_t^\infty \sum_{j=1}^{m} s_j(\xi)\Delta\xi|) \parallel x_n - x \parallel \\ &\leq (l\frac{1-\alpha}{q} + m\frac{1-\alpha}{q}) \parallel x_n - x \parallel = (1-\alpha) \parallel x_n - x \parallel . \end{aligned}$$

By applying Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \|(Sx_n)(t) - (Sx)(t)\| = 0,$$

which proves that S is continuous on X.

Finally, we prove SX is relatively compact. Clearly, SX is bounded. Let $T_1 \in [T_0, \infty)_{\mathbb{T}}$ be large enough, and we take any $t_1, t_2 \in [T_1, \infty)$. Without loss of generality, we set $t_1 < t_2$. In view of (H_4), for any $\varepsilon > 0$, we have

$$\int_{t_1}^{t_2} r_i(\xi)\Delta\xi \leq \frac{\varepsilon}{2lK} \quad \text{and} \quad \int_{t_1}^{t_2} s_j(\xi)\Delta\xi \leq \frac{\varepsilon}{2mK}.$$

Hence, we infer

$$\begin{aligned}|(Sx)(t_2)-(Sx)(t_1)| &\leq |\int_{t_1}^{t_2}\sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi| + |\int_{t_1}^{t_2}\sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi| \\ &\leq (\frac{\varepsilon}{2lK}l + \frac{\varepsilon}{2mK}m)K = \varepsilon.\end{aligned}$$

This means that SX is uniformly Cauchy on $[T_1,\infty)_{\mathbb{T}}$.

Because $r_i(t)$ and $s_j(t)$ are rd-continuous on the interval $[T_0,T_1]$, they are bounded([5]). We take any $t_1,t_2 \in [T_0,T_1]$, and let

$$M = \max{}_{t\in[T_0,T_1]}\{\sup{}_{1\leq i\leq l} r_i(t),\ \sup{}_{1\leq j\leq m} s_j(t)\}.$$

Without loss of generality, we suppose $t_1 < t_2$. For any $\varepsilon > 0$, we have

$$\begin{aligned}|(Sx)(t_2)-(Sx)(t_1)| &\leq |\int_{t_1}^{t_2}\sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi| + |\int_{t_1}^{t_2}\sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi| \\ &\leq (lMK + mMK)|t_2-t_1| = qMK|t_2-t_1|.\end{aligned}$$

Hence, when $\delta = \frac{\varepsilon}{qMK}$ and $|t_2-t_1| < \delta$, $|(Sx)(t_2)-(Sx)(t_1)| < \varepsilon$ is true. Thus, SX is equi-continuous on $[T_0,T_1]$.

By Lemma 2.5, SX is relatively compact. So, S is completely continuous operator on X.

According to Lemma 2.6, there exists $x \in X$ such that$(U+S)(x) = x$, which is a bounded nonoscillatory solution of Eq. (1) with $\lim_{t\to\infty} x(t) = B > 0$. □

Theorem 3.4. *Let* $G_1(t) = \max_{\tau\in[a,b]}\{g(t,\tau)\}$ *and* $G_2(t) = \min_{\tau\in[a,b]}\{g(t,\tau)\}$. *If there exists* $T^* \in \mathbb{T}$ *with* $T^* > \max\{1,T_0\}$ *such that*

$$\int_t^\infty \sum_{j=1}^{m} \frac{s_j(\xi)}{\eta_j(\xi)}\Delta\xi \leq \frac{P(t)}{G_1^2(t)} - \frac{1}{t^2},\ \ t\in[T^*,\infty)_{\mathbb{T}} \tag{6}$$

and

$$\int_t^\infty \sum_{i=1}^{l} \frac{r_i(\xi)}{\delta_i(\xi)}\Delta\xi \leq \frac{1}{t} - \frac{P(t)}{G_2(t)},\ \ t\in[T^*,\infty)_{\mathbb{T}}, \tag{7}$$

then, Eq. (1) has a bounded nonoscillatory solution $x(t)$ *with* $\lim_{t\to\infty} x(t) = 0$.

Proof. Similar to Theorem 3.3, we define the Banach space $BC[T^*,\infty)$. Let

$$X = \{x : x \in BC[T^*,\infty)_{\mathbb{T}}\ \text{ and }\ t^{-2} \leq x(t) \leq t^{-1}\}.$$

Then, X is a bounded, convex and closed subset of $BC[T^*,\infty)$. Define the following operator F on X

$$(Fx)(t) = \int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi - \int_t^\infty \sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi.$$

First, we will show that $Fx \in X$ for any $x \in X$ and $t \in [T^*,\infty)_{\mathbb{T}}$. In view of (H_3), (H_4), (6) and (7), we have

$$\begin{aligned}(Fx)(t) &\geq P(t)\frac{1}{g^2(t,\tau)} - \int_t^\infty \sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi \\ &\geq P(x)\frac{1}{G_1^2(t)} - \int_t^\infty \sum_{j=1}^{m} \frac{s_j(\xi)}{\eta_j(\xi)}\Delta\xi \\ &\geq P(t)\frac{1}{G_1^2(t)} - P(t)\frac{1}{G_1^2(t)} + \frac{1}{t^2} > \frac{1}{t^2}\end{aligned}$$

and

$$\begin{aligned}(Fx)(t) \quad &\le P(t)\frac{1}{g(t,\tau)} + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi \\ &\le P(x)\frac{1}{G_1^2(t)} + \int_t^\infty \sum_{i=1}^{l} \frac{r_i(\xi)}{\delta_i(\xi)}\Delta\xi \\ &\le P(t)\frac{1}{G_2(t)} + \frac{1}{t} - P(t)\frac{1}{G_2(t)} \le \frac{1}{t}.\end{aligned}$$

So $Fx \in X$ for any $x \in X$ and $t \in [T^*, \infty)_{\mathbb{T}}$.

Similar to the proof of Theorem 3.3, we can show that F satisfies the rest conditions in Lemma 2.7. Hence, there exists $x \in X$ such that

$$\begin{aligned}x(t) = &\int_a^b p(t,\tau)x(g(t,\tau))\Delta\tau + \int_t^\infty \sum_{i=1}^{l} r_i(\xi)x(\delta_i(\xi))\Delta\xi \\ &- \int_t^\infty \sum_{j=1}^{m} s_j(\xi)x(\eta_j(\xi))\Delta\xi, \quad t \in [T^*, \infty)_{\mathbb{T}}.\end{aligned}$$

According to the definition of X, we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Remark 3.5. *Clearly, the discussed equation in [3] is a particular case of Eq. (1). Because Eq. (1) contains both positive and negative parameters, the application of the model (1) is more extensive. On the other hand, one can easily see that the results obtained in [2,3,4,18] cannot be applied to Eq. (1). So, our results are new and interesting.*

4. Examples

In the section, we would like to illustrate the results by means of the following examples.

Example 4.1. Let $\mathbb{T} = 2^{N_0}$. Consider the following equation

$$(x(t) - \int_1^4 \frac{\sin\frac{\tau}{4}\sin\frac{3\tau}{4}}{2t} x(t-\tau+1)\Delta\tau)^{\Delta} + \frac{1}{t\sigma(t)}x(\frac{3}{4}t) - \frac{1}{5(t^2+\sigma^2(t))}x(\frac{2}{3}t) = 0, \tag{8}$$

where $\mathbb{T} = 2^{N_0} = \{1, 2, 4, 8...\}$, $\sigma(t) = 2t$, $\mu(t) = t$, $p(t,\tau) = \frac{\sin\frac{\tau}{4}\sin\frac{3\tau}{4}}{2t}$, $r(t) = \frac{1}{t\sigma(t)}$, $s(t) = \frac{1}{5(t^2+\sigma^2(t))}$, $q = 2$, $l = m = 1$. Then we have

$$\begin{aligned}P(t) = &\int_1^4 \frac{\sin\frac{\tau}{4}\sin\frac{3\tau}{4}}{2t}\Delta\tau = \frac{1}{2t}\int_1^4 \sin\frac{\tau}{4}\sin\frac{3\tau}{4}\Delta\tau \\ &= \frac{1}{2t}(\mu(1)\sin\frac{1}{4}\sin\frac{3}{4} + \mu(2)\sin\frac{1}{2}\sin\frac{3}{2}) \\ &= \frac{1}{2t}(\sin\frac{1}{4}\sin\frac{3}{4} + 2\sin\frac{1}{2}\sin\frac{3}{2}) \le \frac{2}{3t} \le \frac{2}{3} = \alpha < 1,\end{aligned}$$

$$\int_{T_0}^\infty \frac{1}{t\sigma(t)}\Delta t = \frac{1}{T_0} \le \frac{1}{6} = \frac{1-\alpha}{q} < 1,$$

$$\int_{T_0}^\infty \frac{1}{5(t^2+\sigma^2(t))}\Delta t \le \int_{T_0}^\infty \frac{1}{5t\sigma(t)}\Delta t = \frac{1}{5T_0} \le \frac{1-\alpha}{q} < 1,$$

and

$$\frac{1-\alpha}{q} r(t) \geq s(t),$$

where $T_0 \geq 8$. Hence, by Theorem 3.3 with $\alpha = \frac{2}{3} < 1$, Eq. (8) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} x(t) > 0$.

Example 4.2. Let $\mathbb{T} = \{t \geq 1 : t \in \mathbb{R}\}$. Consider the following equation

$$(x(t) - \int_1^2 \frac{2(\tau-1)}{t+1} x((t-\tau)^{\frac{1}{2}})\Delta\tau)^\Delta + \frac{1}{t^2+1} x(\frac{4}{5}t) - \frac{1}{t^3+1} x(\frac{5}{6}t) = 0, \tag{9}$$

where $p(t,\tau) = \frac{2(\tau-1)}{t+1}$, $r(t) = \frac{1}{t^2+1}$, $s(t) = \frac{1}{t^3+1}$, $q = 2$, $l = m = 1$. Taking $t \geq 3$, we have

$$P(t) = \int_1^2 \frac{2(\tau-1)}{t+1}\Delta\tau = \frac{1}{t+1} \leq \frac{1}{4} = \alpha < 1, G_1(t) = (t-1)^{\frac{1}{2}}, G_2(t) = (t-2)^{\frac{1}{2}},$$

$$\int_3^\infty \frac{1}{t^2+1}\Delta t \leq \frac{3}{8} \leq \frac{1-\alpha}{q} < 1, \int_3^\infty \frac{1}{t^3+1}\Delta t \leq \frac{3}{8} \leq \frac{1-\alpha}{q} < 1,$$

$$\int_t^\infty \frac{1}{s^2+1} \cdot \frac{1}{\frac{4}{5}s}\Delta s \leq \frac{1}{(t+1)(t-1)} - \frac{1}{t^2},$$

and

$$\int_t^\infty \frac{1}{s^3+1} \cdot \frac{1}{\frac{5}{6}s}\Delta s \leq \frac{1}{t} - \frac{1}{(t+1)(t-2)^{\frac{1}{2}}}.$$

Hence, by Theorem 3.4, Eq. (9) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$.

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