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Numerical Solution of Quadratic SDE with Measurable Drift

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Abstract. In this paper we are interested in solving numerically quadratic SDEs with non-necessary continuous drift of the from

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t f(X_s)\sigma^2(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where, x is the initial data b and σ are given coefficients that are assumed to be Lipschitz and bounded and f is a measurable bounded and integrable function on the whole space $\mathbb{R}$.

Numerical simulations for this class of SDE of quadratic growth and measurable drift, induced by the singular term $f(x)\sigma^2(x)$, is implemented and illustrated by some examples. The main idea is to use a phase space transformation to transform our initial SDEs to a standard SDE without the discontinuous and quadratic term. The Euler–Maruyama scheme will be used to discretize the new equation, thus numerical approximation of the original equation is given by taking the inverse of the space transformation. The rate of convergence are shown to be of order $\frac{1}{2}$.

1. Introduction

Let $[0, T]$ be a bounded time interval and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P})$ a filtered probability space, on which is defined $W = \{W_t\}_{t\in[0,T]}$ a one–dimensional standard Brownian motion. We denote by $\{\mathcal{F}_t\}_{t\in[0,T]}$ the filtration generated by W, completed with P–null sets and made right continuous.

Our goal in this paper is to investigate numerical solutions of a class of SDE that are quadratic in space and may have discontinuous integrable drift. Concretely we are interested in studying numerical simulations of $\mathbb{R}$–valued SDEs of quadratic type of the form

$$X_t = x + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \mathrm{f}(X_s)\sigma^2(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \tag{1}$$

where f is bounded measurable and integrable function over the whole space $\mathbb{R}$.
We shall refer to this equation as $\mathrm{Eq}(x, b + \mathrm{f}\sigma^2, \sigma)$.

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The main idea is to use the following transformation $\mathrm{F_f}$ defined, for every $x \in \mathbb{R}$, by

$$\mathrm{F_f}(x) = \int_0^x \exp\left(2\int_0^y \mathrm{f}(t)\mathrm{d}t\right)\mathrm{d}y. \tag{2}$$

We shall construct a numerical scheme for the equation (1) and prove a rate of convergence to the exact solution which in turn is unique in the strong sense.

Euler–Maruyama numerical scheme is considered to obtain approximate solutions to (1). The convergence of the Euler–Maruyama scheme has been investigated for different modes of convergence by many authors among them, Kloeden and Platen (1999) [10], Higham (2001) [8], Bokil *et al.* (2020) [3] and the references therein. We refer to Mao *et al.* (2006) [12] and Higham and Kloeden (2021) [9] for applications in finance. A survey of progress numerical methods of solutions to stochastic differential equations can be found in Burrage *et al.* (2004) [4] in which a number of application in different areas were discussed with a particular focus on computational biology applications. They also have presented various classes of explicit and implicit methods for strong solutions.

Most of known results on numerical approximations of SDEs assume enough regularity of the coefficients such as Lipschitz continuity property and some times bundedness of the derivatives of the coefficients up to some order. Our goal in this paper is to go beyond the continuity by considering measurable integrable drifts. The main idea is to use the transformation $\mathrm{F_f}$, which is bilipschitz: that $\mathrm{F_f}$ and its inverse $\mathrm{F_f^{-1}}$ are Lipschitz continuous functions, to eliminate the singular part in the SDE (1) and obtain and ordinary SDEs with nice coefficients. This idea has been used by [7, 11, 14] for singular SDEs and by [1, 5, 6] for backward stochastic differential equations with measurable drifts and related partial differential equations.

Numerical Euler–Maruyama approximations $(x_t^n)_{0\le t\le T}$ is given for $x_t = \mathrm{F_f}(X_t)$ and the rate of convergence is found to be of order 0.5, then numerical approximations of the original process $(X_t)_{0\le t\le T}$ is defined by $X_t^n = \mathrm{F_f^{-1}}(x_t^n)$. Moreover, a comparison between the numerical approximations and the exact solutions are illustrated.

The paper is organized as follows: In section 2, we give some useful notations and state some preliminary results. The section 3 is devoted to some examples and numerical Euler–Maruyama scheme for the solution of the new SDE obtained after the transformation $\mathrm{F_f}$. Estimation of the error is given, and then the estimation of the error of the original equation is deduced and is shown to be the same as the rate of convergence of the SDE without the singular and quadratic term. Moreover comparison between the exact solution and the approximation of the original equation is illustrated numerically. This paper will be closed with the section 4 as an appendix in which technical results on the existence and uniqueness for this class of singular SDE are proved.

2. Notations and preliminary results.

Definition 2.1. *We say that an $\mathbb{R}$–valued stochastic process $X_\cdot$ is a solution of the Itô stochastic differential equation starting from X_0: $\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$ for $0 \le t \le T$,*
if the following properties hold true:

(i) *$X_\cdot$ is progressively measurable with respect to $\mathcal{F}_\cdot^W$ the canonical filtration of the Brownian motion,*

(ii) $b(\cdot, X_\cdot) \in L^1([0, T] \times \Omega, \mathrm{d}s \otimes \mathrm{d}\mathbb{P})$,

(iii) $\sigma(\cdot, X_\cdot) \in L^2([0, T] \times \Omega, \mathrm{d}s \otimes \mathrm{d}\mathbb{P})$,

(iv) $X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$ *a.s. for all* $0 \le t \le T$.

The next lemma is very useful, more precisely, it plays a crucial role in the proof of the Proposition 4.3 and the Theorem 4.5 in the appendix. In fact, the transformation $\mathrm{F_f}$ allows us to eliminate the singular part $\int_0^t \mathrm{f}(X_s)\sigma^2(X)\,\mathrm{d}s$ in the equation (1).

Let us denote by $\mathcal{W}_1^2(\mathbb{R})$ the space of continuous functions g defined on $\mathbb{R}$ such that both g and its generalized derivatives g'_ℓ and g''_ℓ are locally integrable on $\mathbb{R}$.

Lemma 2.2. *The function* $\mathrm{F_f}$ *defined (2) satisfies,*

$$\mathrm{F}_{\mathrm{f}}''(x) - 2\mathrm{f}(x)\mathrm{F}_{\mathrm{f}}'(x) = 0, \text{ for a.e. } x \tag{3}$$

and has the following properties:
(i) $\mathrm{F_f}$ *and* $\mathrm{F}_{\mathrm{f}}^{-1}$ *are quasi–isometry, that is for any* $x,\ y \in \mathbb{R}$ *and* $|\mathrm{f}|_1 = \int_{\mathbb{R}} |\mathrm{f}(x)|\,\mathrm{d}x$

$$\begin{aligned} e^{-2|\mathrm{f}|_1}\left|x-y\right| &\le \left|\mathrm{F_f}(x)-\mathrm{F_f}(y)\right| \le e^{2|\mathrm{f}|_1}\left|x-y\right|, \\ e^{-2|\mathrm{f}|_1}\left|x-y\right| &\le \left|\mathrm{F}_{\mathrm{f}}^{-1}(x)-\mathrm{F}_{\mathrm{f}}^{-1}(y)\right| \le e^{2|\mathrm{f}|_1}\left|x-y\right|. \end{aligned} \tag{4}$$

(ii) $\mathrm{F_f}$ *is a one to one function. Both* $\mathrm{F_f}$ *and its inverse function* $\mathrm{F}_{\mathrm{f}}^{-1}$ *belong to belongs to* $\mathcal{W}_1^2(\mathbb{R})$.

Proof of (i). By definition the functions $\mathrm{F_f}$ and its inverse $\mathrm{F}_{\mathrm{f}}^{-1}$ are continuous, one to one, strictly increasing functions, moreover $\mathrm{F}_{\mathrm{f}}''(x) - 2\mathrm{f}(x)\mathrm{F}_{\mathrm{f}}'(x) = 0$ for a.e. $x \in \mathbb{R}$.
In addition $\mathrm{F}_{\mathrm{f}}'(x) = \exp(2\int_0^x \mathrm{f}(t)\mathrm{d}t)$, hence for every $x \in \mathbb{R}$

$$e^{-2|\mathrm{f}|_1} \le \mathrm{F}_{\mathrm{f}}'(x) \le e^{2|\mathrm{f}|_1} \text{ and } e^{-2|\mathrm{f}|_1} \le (\mathrm{F}_{\mathrm{f}}^{-1})'(x) \le e^{2|\mathrm{f}|_1}. \tag{5}$$

Proof of (ii). Using the inequality (5), one can show that both $\mathrm{F_f}$ and $\mathrm{F}_{\mathrm{f}}^{-1}$ are $C^1(\mathbb{R})$. Since the second generalized derivative $\mathrm{F}_{\mathrm{f}}''$ satisfies (3) for almost all x, we get that $\mathrm{F}_{\mathrm{f}}''$ belongs to $L^1(\mathbb{R})$. Therefore, $\mathrm{F_f}$ belongs to the space $\mathcal{W}_1^2(\mathbb{R})$. Using again assertion (i), one can check easily that $\mathrm{F}_{\mathrm{f}}^{-1}$ also belongs to $\mathcal{W}_1^2(\mathbb{R})$.
Assumptions:
Let $b : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $\quad \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following assumptions:

($\mathbf{A}_{1.1}$) X_0 is $\mathcal{F}_0$–measurable with $\mathbb{E}|X_0|^p < \infty$, for $p \ge 1$.
($\mathbf{A}_{1.2}$) There exists a $L > 0$ such that, for all $r \in [0,T]$, $x,\ \acute{x} \in \mathbb{R}$

$$|b(r,x) - b(r,\acute{x})| + |\sigma(x) - \sigma(\acute{x})| \le L|x - \acute{x}|.$$

($\mathbf{A}_{1.3}$) The functions b, σ and f are bounded functions.
($\mathbf{A}_{1.4}$) f is integrable over the whole space $\mathbb{R}$.
($\mathbf{A}_{1.5}$) There exists a $L_1 > 0$ such that, for all $r, s \in [0,T]$, $x,\ \acute{x} \in \mathbb{R}$

$$|b(r,x) - b(s,x)| \le L_1(1+|x|)|r-s|^{\frac{1}{2}}.$$

It is clear the above conditions imply that b and σ satisfy the global linear growth condition: that is there exists some $C > 0$ such that, for all $0 \le r \le T$, $x \in \mathbb{R}$: $|b(r,x)| + |\sigma(x)| \le C(1+|x|)$.
For $s \in [0,\ T]$ and $x \in \mathbb{R}$ we set $\overline{b}(s,x) = \mathrm{F}_{\mathrm{f}}'(\mathrm{F}_{\mathrm{f}}^{-1}(x))b(s,\mathrm{F}_{\mathrm{f}}^{-1}(x))$ and $\overline{\sigma}(x) = \mathrm{F}_{\mathrm{f}}'(\mathrm{F}_{\mathrm{f}}^{-1}(x))\sigma(\mathrm{F}_{\mathrm{f}}^{-1}(x))$.

Theorem 2.3. *Let* b, σ *and* f *satisfy* $(\mathbf{A}_{1.2}) - (\mathbf{A}_{1.4})$ *then* $\mathrm{Eq}(x, b + \mathrm{f}\sigma^2, \sigma)$ *has a unique solution.*

Proof. From Theorem 4.5, in the appendix (the section 4 below), $(X_t)_{0\le t\le T}$ is a solution to $\mathrm{Eq}(x, b+\mathrm{f}\sigma^2, \sigma)$ with $X_0 = x$, if and only if $(x_t = \mathrm{F_f}(X_t))_{0\le t\le T}$ is a solution to $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$. It remains to prove that $\overline{b}$ and $\overline{\sigma}$ satisfy $(\mathbf{A}_{1.1}) - (\mathbf{A}_{1.3})$ which are easy to check since F_{f}' is bounded and Lipschitz and also $\mathrm{F}_{\mathrm{f}}^{-1}$ is a Lipschitz function thanks to the Lemma 2.2.

3. Numerical approximations

3.1. *Euler–Maruyama scheme for* $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$

Let $x_\cdot$ be the unique solution to $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$: that is

$$x_t = x_0 + \int_0^t \overline{b}(s, x_s)\,\mathrm{d}s + \int_0^t \overline{\sigma}(x_s)\,\mathrm{d}W_s,$$

where $x_0 = \mathrm{F_f}(x)$.
In what follows we shall give some useful estimates for the solutions $x_\cdot$.

Lemma 3.1. *Assume that* $(\mathbf{A}_{1.1})-(\mathbf{A}_{1.2})$ *hold. Then, for any* $p \geq 1$, *there exists a constant* $C>0$, *which depends on* T *and* p, *such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|x_s|^p\right] \leq C\left(1+e^{2p|\mathrm{f}|_1}|x|^p\right).$$

Proof: The proof is similar to that of the Lemma 3.5 in [2], therefore we omit it here.

Lemma 3.2. *Assume that* $(\mathbf{A}_{1.1})-(\mathbf{A}_{1.2})$ *hold. Thus, for any* $p \geq 1$, $0 \leq s \leq t \leq T$, *there exists a constant* $C_1>0$, *which depends on* T *and* p, *such that*

$$\mathbb{E}\left[|x_t-x_s|^p\right] \leq C_1\left(1+e^{2p|\mathrm{f}|_1}|x|^p\right)(t-s)^{\frac{p}{2}}.$$

Proof: The proof is similar to that of the Lemma 3.6 in [2], therefore we omit it here.

Let us now construction the Euler–Maruyama scheme numerical scheme for the equation $\mathrm{Eq}(\mathrm{F}_{\mathrm{f}}(x),\overline{b},\overline{\sigma})$ as follows

$$x^n_t=\mathrm{F}_{\mathrm{f}}(x)+\int_0^t \overline{b}\left(\eta_n(s),x^n_{\eta_n(s)}\right)\mathrm{d}s+\int_0^t\overline{\sigma}\left(x^n_{\eta_n(s)}\right)\mathrm{d}W_s,$$

where $\eta_n(s)=\frac{kT}{n}$, $k=0,\ldots,n-1$, and $s\in[\frac{k}{n}T,\frac{k+1}{n}T[$ and n is large enough to be tending towards infinity. Since the coefficients $\overline{b}$ and $\overline{\sigma}$ are relatively nice say that are Lipschitz continuous and satisfy the linear growth condition, we shall prove in the following proposition that $(x^n_\cdot)$ converges uniformly in $L^p(\Omega)$ to the unique solution $x_\cdot=(x_s)_{0\leq s\leq T}$ of $\mathrm{Eq}(\mathrm{F}_{\mathrm{f}}(x),\overline{b},\overline{\sigma})$. Moreover the rate of convergence is shown to be of order 0.5.

Proposition 3.3. *The sequence* $x^n_\cdot$ *converges to* $x_\cdot$ *uniformly in* $L^p(\Omega)$ *with* 0.5 *as a rate of convergence*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^n_t-x_t\right|^p\right]\leq\frac{c}{(\sqrt{n})^p}\ \textit{for}\ \ p\geq 2.$$

Proof: Since $x^n_\cdot$ and $x_\cdot$ start from the same point, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^n_t-x_t\right|^p\right]=\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\left(\overline{b}\left(s,x_s\right)-\overline{b}\left(\eta_n(s),x^n_{\eta_n(s)}\right)\right)\mathrm{d}s+\int_0^t\left(\overline{\sigma}\left(x_s\right)-\overline{\sigma}\left(x^n_{\eta_n(s)}\right)\right)\mathrm{d}W_s\right|^p\right].$$

Applying the inequality $\left(\sum_{i=1}^m a_i\right)^p \leq m^{(p-1)\vee 0}\sum_{i=1}^m a_i^p$ for any $m\in\mathbb{N}$, $a_i\geq 0$, $p>0$, Jensen's inequality, Burkholder-Davis-Gundy's inequality, assumptions $(\mathbf{A}_{1.1})-(\mathbf{A}_{1.5})$, Lemmas 3.1 -3.2 and $0\leq s-\eta_n(s)\leq\frac{T}{n}$, we obtain for $t\in[0,T]$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^n_t-x_t\right|^p\right]\leq\frac{c}{\sqrt{n}^p}+c\int_0^T\mathbb{E}\left[\sup_{0\leq u\leq s}\left|x^n_u-x_u\right|^p\right]\mathrm{d}s.$$

Using the Gronwall's inequality, we get for $c_1=ce^{cT}$:

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^n_t-x_t\right|^p\right]\leq\frac{c}{n^{\frac{p}{2}}}e^{cT}=\frac{c_1}{(\sqrt{n})^p}.$$

This completes the proof.

Now, let us set $X^n_t:=\mathrm{F}_{\mathrm{f}}^{-1}(x^n_t)$, the goal is to show that the sequence $X^n_\cdot$ converges to $X_\cdot$ uniformly in $L^p(\Omega)$ with 0.5 as a rate of convergence.

Proposition 3.4. *For* $p\geq 2$ *there exist a constant* C *depending only on* T, p *and* $|\mathrm{f}|_1$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X^n_t-X_t\right|^p\right]\leq\frac{ce^{2p|\mathrm{f}|_1}}{(\sqrt{n})^p}.$$

Proof: Thanks to (4) we have

$$\left|X_t^n - X_t\right| = \left|\mathrm{F}_{\mathrm{f}}^{-1}\left(x_t^n\right) - \mathrm{F}_{\mathrm{f}}^{-1}(x_t)\right| \leq e^{2|\mathrm{f}|_1}\left|x_t^n - x_t\right|,$$

we have

$$\sup_{0\leq t\leq T}\left|X_t^n - X_t\right|^p \leq e^{2p|\mathrm{f}|_1}\sup_{0\leq t\leq T}\left|x_t^n - x_t\right|^p,$$

hence the estimation follows by taking the expectation in both sides of the above inequality and using the estimation of Proposition 3.3.

3.2. *Euler–Maruyama scheme for* $\mathrm{Eq}(x, b + f\sigma^2, \sigma)$

In this section we will give some examples to illustrate the applicability and interest of our abstract results. Let a, α, $\gamma \in \mathbb{R}$, and $\beta > 2$. We consider the following SDE

$$\begin{cases} \mathrm{d}x_t = \overline{b}(x_t)\,\mathrm{d}t + \overline{\sigma}(x_t)\,\mathrm{d}W_t, \\ x_0 = a^\beta, \end{cases} \tag{6}$$

where $\overline{b}$ and $\overline{\sigma}$ are given by

$$\overline{b}(x) = \begin{cases} \beta\left(\alpha x^{\frac{\beta-1}{\beta}} + \frac{\gamma^2(\beta-1)}{2}x^{\frac{\beta-2}{\beta}}\right) & \text{for} \quad 1 < x < 2 \\ \beta\left(\alpha + \frac{\gamma^2(\beta-1)}{2}\right) & \text{for} \quad x \leq 1 \\ \beta\left(\alpha 2^{\frac{\beta-1}{\beta}} + \frac{\gamma^2(\beta-1)}{2}2^{\frac{\beta-2}{\beta}}\right) & \text{for} \quad x \geq 2, \end{cases}$$

and

$$\overline{\sigma}(x) = \begin{cases} \gamma\beta x^{\frac{\beta-1}{\beta}} & \text{for} \quad 1 < x < 2 \\ \gamma\beta & \text{for} \quad x \leq 1 \\ \gamma\beta 2^{\frac{\beta-1}{\beta}} & \text{for} \quad x \geq 2. \end{cases}$$

The exact solution with $x_0 = a^\beta$ is given, $x^+ = \max(x, 0)$, by

$$x_t = ((a + \alpha t + \gamma W_t)^+)^\beta.$$

Indeed, the Itô's formula leads to

$$\begin{aligned} x_t &= a^\beta + \int_0^t \beta\left((a + \alpha s + \gamma W_s)^+\right)^{\beta-1}(\alpha \mathrm{d}s + \gamma \mathrm{d}W_s) \\ &\quad + \frac{1}{2}\int_0^t \int_0^t \beta(\beta - 1)\left((a + \alpha s + \gamma W_s)^+\right)^{\beta-2}\gamma^2 \mathrm{d}s \\ &= 1 + \int_0^t \beta x_s^{\frac{\beta-1}{\beta}}(\alpha \mathrm{d}s + \gamma \mathrm{d}W_s) + \frac{\gamma^2\beta(\beta-1)}{2}\int_0^t x_s^{\frac{\beta-2}{\beta}}\mathrm{d}s, \end{aligned}$$

therefore $x_\cdot$ is the unique solution of the non–Linear SDE

$$y_t = a^\beta + \beta\int_0^t \left(\alpha y_s^{\frac{\beta-1}{\beta}} + \frac{\gamma^2(\beta-1)}{2}y_s^{\frac{\beta-2}{\beta}}\right)\mathrm{d}s + \int_0^t \gamma\beta y_s^{\frac{\beta-1}{\beta}}\,\mathrm{d}W_s,$$

since the coefficients $\overline{b}$ and $\overline{\sigma}$ are Lipschitz and bounded on the interval $[1,\ 2]$.

3.3. Numerical simulations

Example 1: Let $\alpha = 3$, $\gamma = 0.4$, $\beta = 4$ and $a = \frac{1}{2}$. We consider the function f defined by

$$f(x) = \mathbb{1}_{[0,2]}(x) - \mathbb{1}_{[1,4]}(x).$$

Therefore, the exact solution of equation (6) is found

$$x_t = \left(\frac{1}{2} + 3t + 0.4W_t\right)^4.$$

Now we will apply the Euler-Maruyama (EM) method to the quadratic SDE (6) with step size 2^{-26} on the interval $[0, 1]$. In Figure 1, the trajectory of the exact solution x_{exact} is given in the left which is plotted as red and the trajectory of the EM approximation x_{app} is given in the right which is plotted as blue.

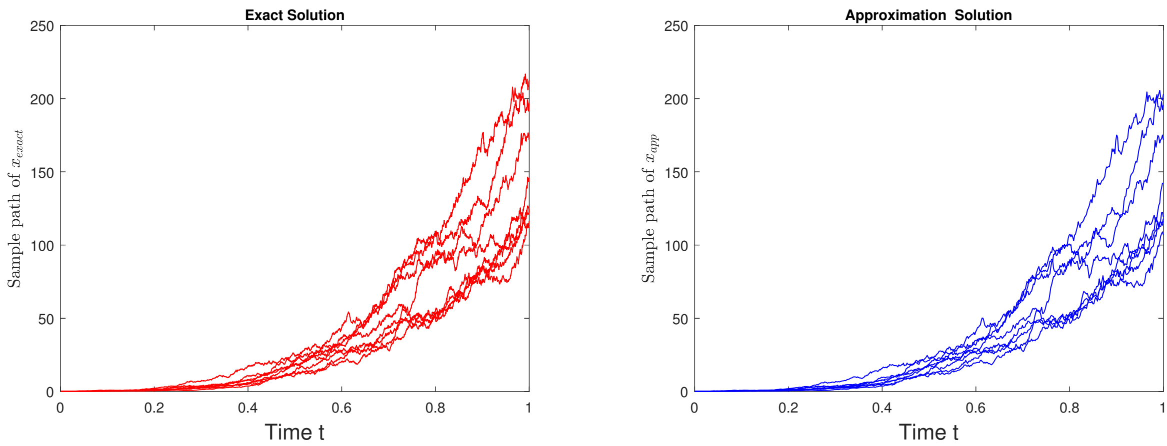

Figure 1: Exact solution (in red) and EM approximation (in blue) of 8 individual paths for system (6) on the interval $[0, 1]$ with parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$.

In Figure 2, we give both the trajectories of x_{exact} and x_{app} with the time steps 2^{-18}. The discrepancy between the exact solution and the EM solution at the endpoint $t = 1$.

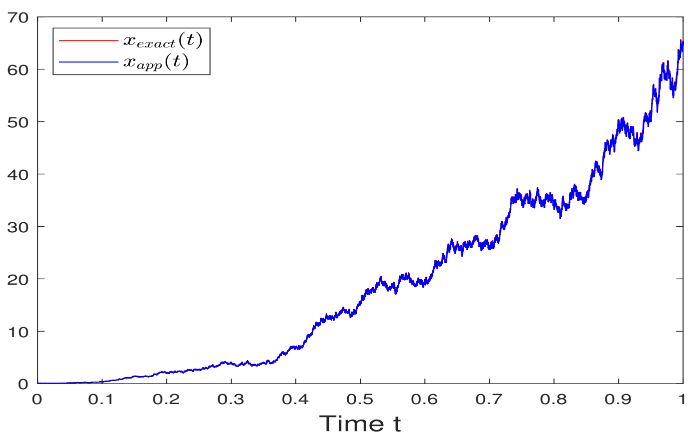

Figure 2: Exact solution and EM approximation for system (6) on the interval $[0, 1]$ with parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$.

Now, we shall find the expression of $\mathrm{F_f}(x) = \int_0^x \exp\left(2\int_0^y \mathrm{f}(t)\mathrm{d}t\right)\mathrm{d}y$ for $\mathrm{f}(x) = \mathbb{1}_{[0,2]}(x) - \mathbb{1}_{[1,4]}(x)$. By simple calculations, one gets

$$\mathrm{F_f}(x) = \begin{cases} x, & \text{if} \quad x < 0 \\ \frac{e^{2x}-1}{2}, & \text{if} \quad 0 \leq x \leq 1 \\ \frac{e^2-1}{2} + e^2\,(x-1) & \text{if} \quad 1 < x \leq 2 \\ \frac{3e^2-1}{2} + \dfrac{e^2 - e^{6-2x}}{2} & \text{if} \quad 2 < x \leq 4 \\ \frac{4e^2+e^{-2}-1}{2} + e^{-2}\,(x-4) & \text{if} \quad x > 4. \end{cases}$$

Based to the Euler–Maruyama scheme with step size 2^{-26} and the parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$, the trajectory of the exact solution $X_{exact} = \mathrm{F_f^{-1}}(x_{exact})$ is given in the left hand side of the figure 3 which is plotted in red. Moreover, we give the trajectory of the EM approximation $X_{app} = \mathrm{F_f^{-1}}(x_{app})$ in the right hand side which is plotted in blue.

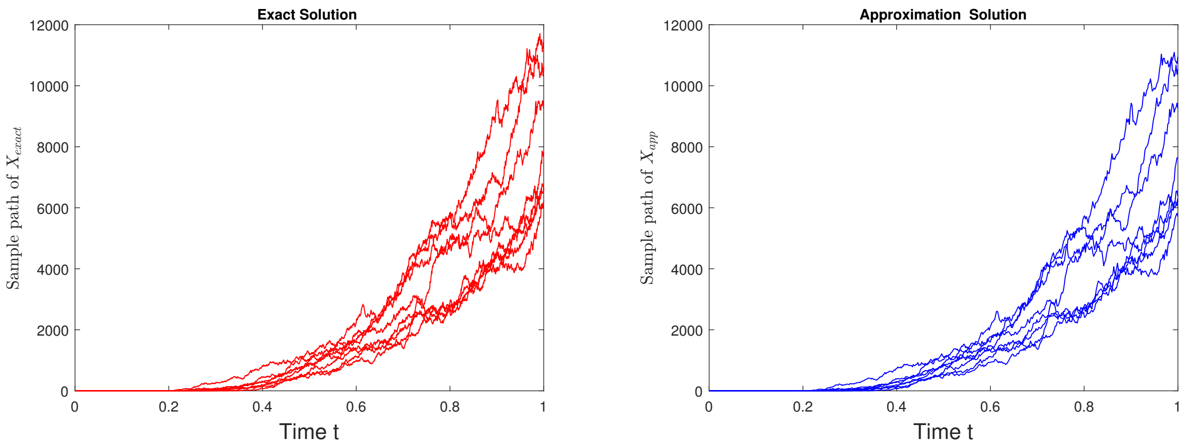

Figure 3: Exact solution (in red) and EM approximation (in blue) of 8 individual paths for system (6) on the interval $[0, 1]$ with parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$.

Moreover, by plotting the trajectories of X_{exact} and X_{app} with the time steps 2^{-18} (showing in Figure 4), we give the discrepancy between the exact solution $X_{exact} = \mathrm{F_f^{-1}}(x_{exact})$ and EM approximation $X_{app} = F^{-1}(x_{app})$ at the endpoint $t = 1$.

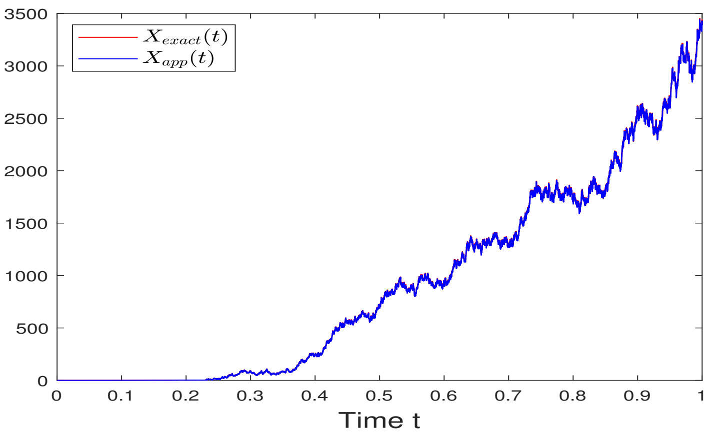

Figure 4: Exact solution and EM approximation (in blue) for system (6) on the interval $[0, 1]$ with parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$.

Strong convergence of Euler–Maruyama:
We used five different time steps: $\Delta t = 2^{-26}, 2^{-25}, 2^{-24}, 2^{-23}, 2^{-22}$ and 1000 realizations for each discretization. In Figure 5, the log-log plot of the experimental error at $T = 1$ between the exact solution x_{exact} and the EM approximation x_{app} is plotted in red in the left hand side and the log-log plot of the experimental error at $T = 1$ between the exact solution $X_{exact} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{exact})$ and the EM approximation $X_{app} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{app})$ is plotted in blue, in the right hand side, with respect to the 5 different time steps.

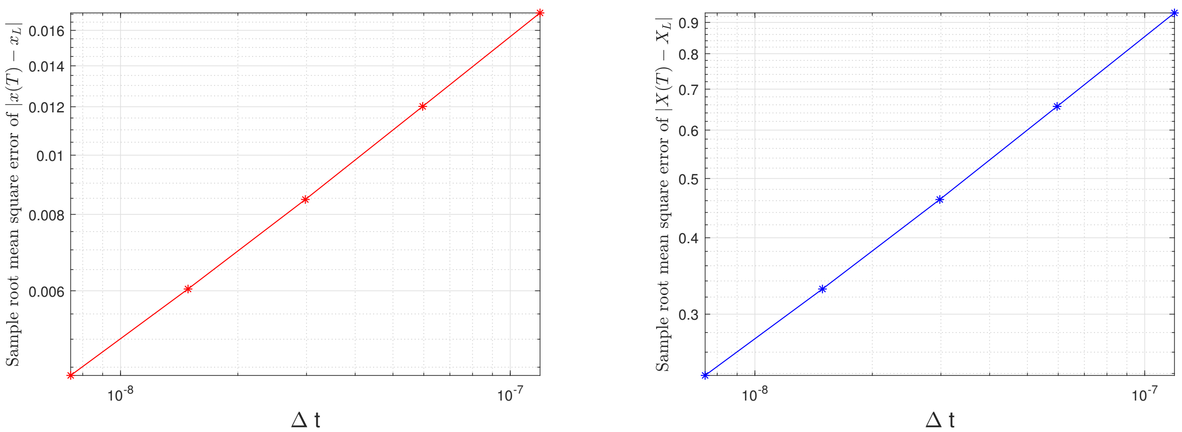

Figure 5: Log-log plot of the root mean square error from the numerical approximation for system (6) at $T = 1$ with parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$.

Example 2: Let $\alpha = 2$, $\gamma = 0.6$, $\beta = 3$ and $a = 1$. Set

$$\mathrm{f}(x) := \begin{cases} \sin(x), & \text{if} \quad x \in [-\pi, \frac{\pi}{2}] \\ 0, & \text{otherwise.} \end{cases}$$

Hence, in this case the exact solution of equation (6) is given by

$$x_t = (1 + 2t + 0.6W_t)^3 .$$

Now, we will apply the EM method to the quadratic SDE (6) with step size 2^{-24} on the interval $[0, 1]$. In

Figure 6, the trajectories of the exact solution x_{exact} are plotted in red in the left hand side and the trajectories of the EM approximation x_{app} are plotted in blue the right hand side.

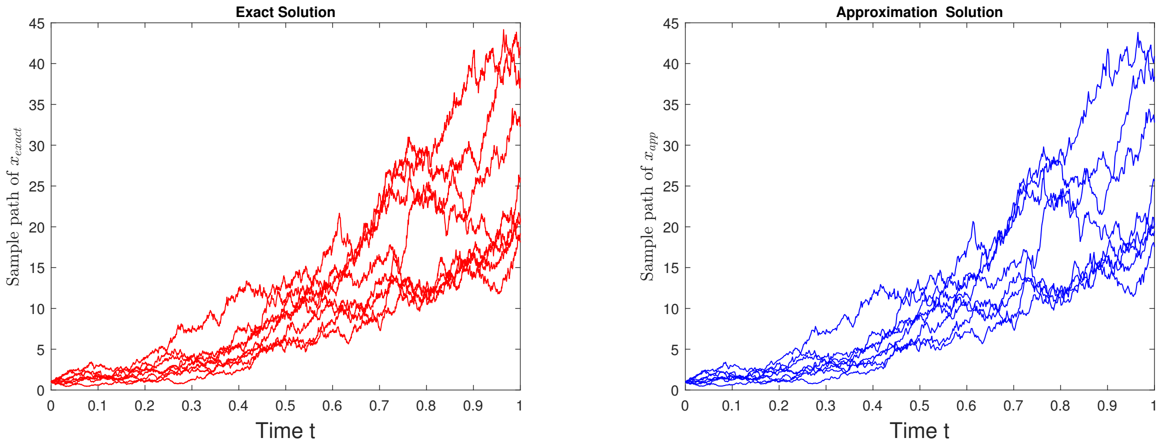

Figure 6: Exact solution and EM approximation of 8 individual paths for system (6) on the interval $[0, 1]$ with parameters $\alpha = 2$, $\gamma = 0.6$ and $\beta = 3$.

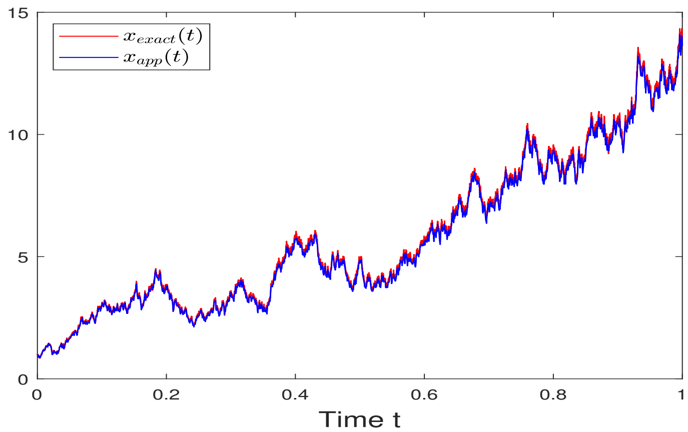

Figure 7: Exact solution and EM approximation for system (6) on the interval $[0, 1]$ with parameters $\alpha = 2$, $\gamma = 0.6$ and $\beta = 3$.

In Figure 7, we plot both trajectories of x_{exact} and x_{app} with the time steps 2^{-13}. The discrepancy between the exact solution and the EM solution at the endpoint $t = 1$.
Again, we need to find the expression of $\mathrm{F_f}(x) = \int_0^x \exp\left(2\int_0^y \mathrm{f}(t)\mathrm{d}t\right)\mathrm{d}y$ for $\mathrm{f}(x) = \sin(x)\mathbb{1}_{[-\pi, \frac{\pi}{2}]}(x)$, by simple calculations, the expression of F is given as follows

$$\mathrm{F_f}(x) = \begin{cases} -e^4 x, & \text{if} \quad x < -\pi \\ \int_0^x e^{2(1-\cos y)}\mathrm{d}y, & \text{if} \quad -\pi \le x \le \frac{\pi}{2} \\ 3.97125 + e^2\left(x - \dfrac{\pi}{2}\right) & \text{if} \quad x > \frac{\pi}{2}. \end{cases}$$

Based to the Euler–Maruyama scheme with step size 2^{-24} and the parameters $\alpha = 3$, $\gamma = 0.4$ and $\beta = 4$, the trajectories of the exact solution $X_{exact} = \mathrm{F_f^{-1}}(x_{exact})$ are plotted in red in the left hand side of the Figure 8.

Moreover, the trajectories of the EM approximation $X_{app} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{app})$ are plotted in blue in the right hand side.

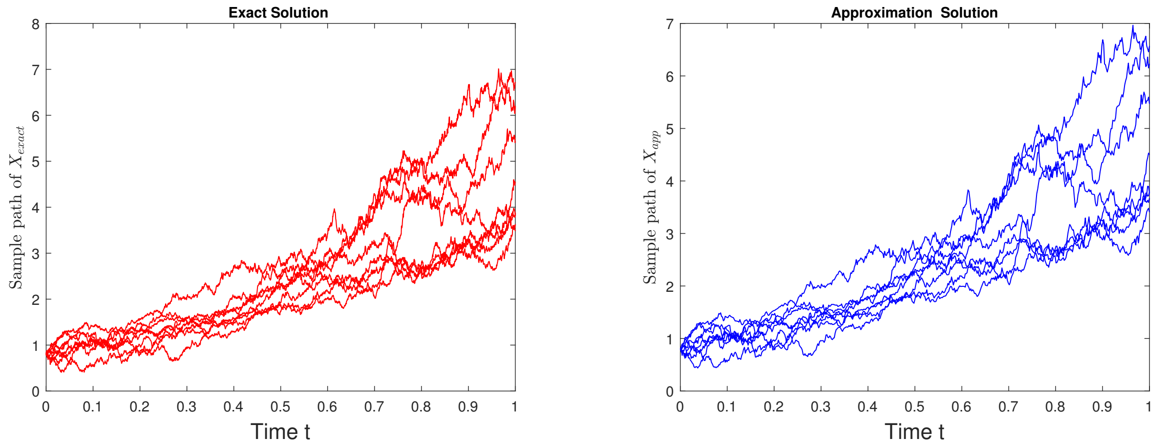

Figure 8: Exact solution and EM approximation of 8 individual paths for system (6) on the interval $[0, 1]$ with parameters $\alpha = 2$, $\gamma = 0.6$ and $\beta = 3$.

Moreover, by plotting the trajectories of X_{exact} and X_{app} with the time steps 2^{-13} (showing in Figure 9), we give the discrepancy between the exact solution $X_{exact} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{exact})$ and EM approximation $X_{app} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{app})$ at the endpoint $t = 1$.

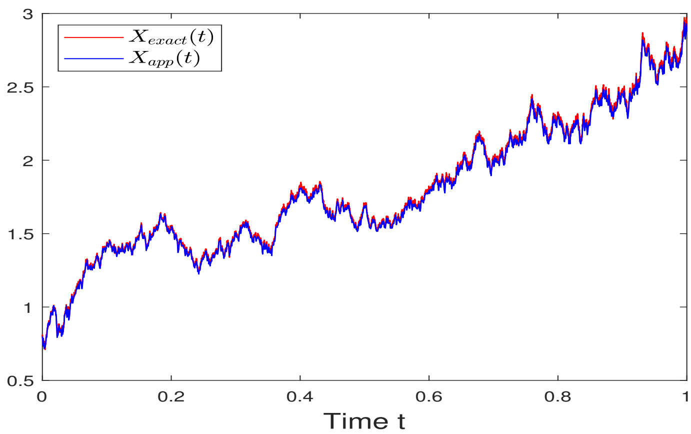

Figure 9: Exact solution and EM approximation for system (6) on the interval $[0, 1]$ with parameters $\alpha = 2$, $\gamma = 0.6$ and $\beta = 3$.

Strong convergence of Euler–Maruyama:
We used five different time steps: $\Delta t = 2^{-24}, 2^{-23}, 2^{-22}, 2^{-21}, 2^{-20}$ and 1000 realizations for each discretization. In Figure 10, the log-log plot of the experimental error at $T = 1$ between the exact solution x_{exact} and the EM approximation x_{app} is plotted in red in the left hand side and the log-log plot of the experimental error at $T = 1$ between the exact solution $X_{exact} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{exact})$ and the EM approximation $X_{app} = \mathrm{F}_{\mathrm{f}}^{-1}(x_{app})$ is plotted in blue, in the right hand side, with respect to 5 different time steps.

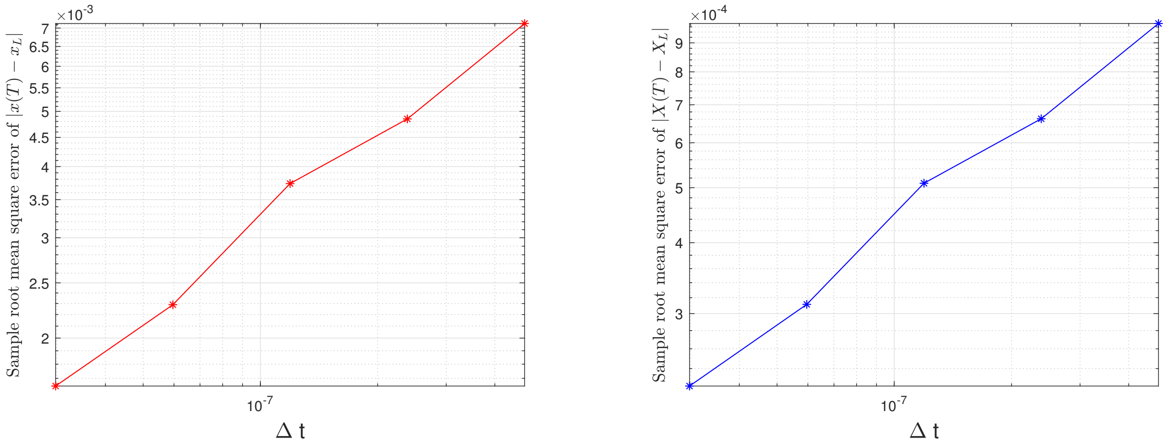

Figure 10: Log-log plot of the root mean square error from the numerical approximation for system (6) at $T = 1$ with parameters $\alpha = 2$, $\gamma = 0.6$ and $\beta = 3$.

4. Appendix

4.1. Short review on local time

Let us introduce some specific notation for this subsection. The sign function is defined to be

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

Note that our definition of sign is not symmetric. We further define

$$\varphi_0(x) = |x| \quad \text{and} \quad \varphi_a(x) = |x - a|.$$

Then $\operatorname{sign}(x)$ is the left derivative of $\varphi_0(x)$, and $\operatorname{sign}(x-a)$ is the left derivative of $\varphi_a(x)$. since $\varphi_a(x)$ is convex by the well–known Tanaka's formula we have for a continuous semimartingale X

$$\varphi_a(X_t) = |X_t - a| = |X_0 - a| + \int_0^t \operatorname{sign}(X_s - a)\, \mathrm{d}X_s + A_t^a, \tag{7}$$

where A_t^a is the increasing process of the continuous semimartingale $\varphi_a(X_.)$.

Definition 4.1. *The local time at a of X, denoted $L_t^a(X_.)$, is defined to be the process given by $L_t^a(X_.) = A_t^a$.*

It has been shown in Protter [13] that the stochastic integral $\int_0^t \operatorname{sign}(X_s - a)\mathrm{d}X_s$ in (7) has a version which is jointly measurable in (ω, t, a) and càdlàg in t. Therefore so does $(A_t^a)_{t\geq 0}$, and finally so too does the local time $L_t^a(X_.)$. We always choose this jointly measurable, càdlàg version of the local time, without any special mention. In fact the local time $(L_t^a(X_.))_{t\geq 0}$ of a continuous semimartingale X is continuous in t and right continuous with left limit with respect to a: that is

$$\lim_{x\to a+, s\to t} L_s^x(X_.) = L_t^{a+}(X_.) = L_t^a(X_.) \text{ and } \lim_{x\to a-, s\to t} L_s^x(X_.) = L_t^{a-}(X_.) \text{ exist.}$$

The next Proposition whose proof can be found in Protter [13] is quite simple yet crucial to proving the properties of $L_t^a(X_.)$ that justify its name. For each real number x we set $x^+ = x \vee 0 = \max(x, 0)$ and $x^- = -(x \wedge 0) = \min(-x, 0)$, thus $|x| = x^- + x^+$.

Proposition 4.2. *Let X be a continuous semimartingale and let $L_t^a(X.)$ be its local time at time t and level a. Then*

$$\begin{aligned}(X_t - a)^+ - (X_0 - a)^+ &= \int_0^t \mathbb{1}_{\{X_s > a\}} \mathrm{d}X_s + \frac{1}{2} L_t^a(X.),\\ (X_t - a)^- - (X_0 - a)^- &= -\int_0^t \mathbb{1}_{\{X_s \leq a\}} \mathrm{d}X_s + \frac{1}{2} L_t^a(X.).\end{aligned}$$

Moreover, for a.a. ω, the measure in t, $\mathrm{d}L_t^a(X.(\omega))$, is carried by the set $\{s : X_s(\omega) = a\}$.

4.2. Krylov's estimates and Itô–Krylov's formula for singular SDEs

Let us recall the Tanaka's formula. If $(X_t,\ t \geq 0)$ is a real–valued continuous semimartingale for each $t > 0$ and g is the difference between two convex functions, then, for each $x \in \mathbb{R}$, there exists an adapted process $(L_t^x(X.), t \geq 0)$ such that, for each $t \geq 0$, with probability 1 we have

$$g(X_t) = g(X_0) + \int_0^t g'_\ell(X_s)\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} L_t^x(X.) g''_\ell(\mathrm{d}x),$$

where g'_ℓ stands for the left first derivative of g and g''_ℓ is a signed measure which is the second derivative of g in the generalized function sense.

Proposition 4.3. *Let $(X_t)_{0\leq t\leq T,\ e\in E}$ be a solution to (1) in the sense of the Definition 2.1. Put*

$$\eta = 2 \sup_{0\leq t\leq T} |X_t| + 2\int_0^T \mathbb{1}_{\{X_s < x\}} |b(s, X_s)| \,\mathrm{d}s, \tag{8}$$

then, for any measurable and integrable function ϕ on $\mathbb{R}$, we have

$$\mathbb{E}\int_0^T \left|\phi\right|(X_s)\left\langle X\right\rangle_s \mathrm{d}s \leq 2\mathbb{E}\left[\eta\right]\left|\phi\right|_1 e^{2|f|_1}.$$

Proof: Let x be a real number, set for notational simplicity $\psi_x(y) = (y - x)^-$. By Tanaka's formula, we have

$$\begin{aligned}\psi_x(X_t) &= \psi_x(X_0) + \int_0^t \mathbb{1}_{\{X_s<x\}}\mathrm{d}X_s + \frac{1}{2}L_t^x(X.)\\ &= \psi_x(X_0) + \int_0^t \mathbb{1}_{\{X_s<x\}}\sigma(X_s)\mathrm{d}W_s\\ &\quad + \frac{1}{2}L_t^x(X.) + \int_0^t \mathbb{1}_{\{X_s<x\}} f(X_s)\left\langle X\right\rangle_s \mathrm{d}s + \int_0^t \mathbb{1}_{\{X_s<x\}} b(s, X_s)\mathrm{d}s.\end{aligned}$$

Since the map $y \longmapsto \psi_x(y)$ is one–Lipschitz, it follows that:

$$\begin{aligned}\frac{1}{2}L_t^x(X.) &\leq |X_t - X_0| + \int_0^t \mathbb{1}_{\{X_s<x\}}\left|f(X_s)\right|\left\langle X\right\rangle_s \mathrm{d}s\\ &\quad + \int_0^t \mathbb{1}_{\{X_s<x\}}\sigma(X_s)\mathrm{d}W_s + \int_0^T \mathbb{1}_{\{X_s<x\}}|b(s, X_s)|\,\mathrm{d}s.\end{aligned}$$

Hence

$$0 \leq L_t^x(X.) \leq 2\eta + 2\int_0^t \mathbb{1}_{\{X_s<x\}}\sigma(X_s)\mathrm{d}W_s + 2\int_{-\infty}^x L_t^y(X.)\left|f(y)\right|\mathrm{d}y, \tag{9}$$

where we have used the occupation density formula and the notation (8).
Taking the expectation in (9) we obtain

$$0 \leq \mathbb{E}\left[L_t^x(X.)\right] \leq 2\mathbb{E}\left[\eta\right] + 2\int_{-\infty}^x \mathbb{E}\left[L_t^y(X.)\right]\left|f(y)\right|\mathrm{d}y.$$

Now, Gronwall's Lemma gives

$$0 \leq \sup_x \mathbb{E}\left[L_t^x(X.)\right] \leq 2\mathbb{E}\left[\eta\right] e^{2|\mathrm{f}|_1}. \tag{10}$$

Now, let ϕ be measurable and integrable function on $\mathbb{R}$. The occupation density formula shows that

$$\begin{aligned}\mathbb{E}\int_0^T \left|\phi\right|(X_s)\left\langle X\right\rangle_s \mathrm{d}s &= \mathbb{E}\int_{-\infty}^{\infty} \left|\phi\right|(x) L_T^x(X.)\,\mathrm{d}x \\ &\leq \sup_x \mathbb{E}\left[L_t^x(X.)\right]\int_{-\infty}^{\infty}\left|\phi\right|(x)\mathrm{d}x \leq 2\left|\phi\right|_1 e^{2|\mathrm{f}|_1}\mathbb{E}\left[\eta\right].\end{aligned}$$

Proposition 4.3 is proved since $\mathbb{E}\left[\eta\right]$ is finite thanks to Definition 2.1.

Now, we shall establish an Itô–Krylov's change of variable formula for the solutions of one dimensional SDEs with measurable drifts.

Theorem 4.4. *Let $(X_t)_{0\leq t\leq T}$ be a solution to (1). Then, for any function g belonging to the space $\mathcal{W}^2_{1,loc}(\mathbb{R})$, we have with probability* 1

$$g(X_t) = g(X_0) + \int_0^t g'_\ell(X_s)\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X.) g''_\ell(\mathrm{d}a). \tag{11}$$

Proof. For $R > |X_0|$, and the fact that g and g'_ℓ are locally Lipschitz continuous functions.

Using Proposition 4.3, the term $\int_0^{t\wedge\tau_R} g''_\ell(X_s)\left\langle X\right\rangle_s \mathrm{d}s$ is well defined since

$$\mathbb{E}\left|\int_0^{t\wedge\tau_R} g''_\ell(X_s)\left\langle X\right\rangle_s \mathrm{d}s\right| \leq \mathbb{E}\left|\int_0^T g''_\ell(X_s)\left\langle X\right\rangle_s \mathrm{d}s\right| \leq \sup_x \mathbb{E}\left[L_t^x(X.)\right]\left|g''_\ell\right|_1.$$

Let g_n be a sequence of C^2–class functions obtained via a classical regularization by convolution. Indeed let ψ be an element of $C^\infty(\mathbb{R},\mathbb{R})$ with compact support and satisfies

$$\int_{\mathbb{R}} \psi(x)\mathrm{d}x = 1.$$

We define

$$g * \psi(n\cdot)(x) = \int_{\mathbb{R}} g(v)\psi(n(x-v))\mathrm{d}v,$$

and $\bar{\psi} \in C^\infty(\mathbb{R},\mathbb{R})$ such that, for a given $R > 0$

$$\bar{\psi}(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| > R. \end{cases}$$

Obviously, the sequence of measurable functions $\{g_n, n \geq 1\}$, such that $g_n(x) = n^2\bar{\psi}(\frac{x}{n})\left(g * \psi(n(\cdot))\right)(x)$ satisfy the following properties:

(i) g_n converges uniformly to g in the interval $[-R, R]$.

(ii) g'_n converges uniformly to g'_ℓ in the interval $[-R, R]$

(iii) g''_n converges in $L^1([-R, R])$ to g''_ℓ.

Classical Itô's formula applied to $g_n(X_{t\wedge\tau_R})$ gives

$$g_n(X_{t\wedge\tau_R}) = g_n(X_0) + \int_0^{t\wedge\tau_R} g'_n(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^{t\wedge\tau_R} g''_n(X_s)\left\langle X\right\rangle_s \mathrm{d}s. \tag{12}$$

Now, passing to the limit on n in (12) then use the above properties (*i*), (*ii*), (*iii*) and Proposition 4.3, to obtain

$$g(X_{t\wedge\tau_R}) = g(X_0) + \int_0^{t\wedge\tau_R} g'_\ell(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^{t\wedge\tau_R} g''_\ell(X_s)\langle X\rangle_s\,\mathrm{d}s,$$

which gives, by the occupation density formula,

$$\int_0^t g''_\ell(X_s)\langle X\rangle_s\,\mathrm{d}s = \int_{\mathbb{R}} L^a_t(X_\cdot)g''_\ell(a)\mathrm{d}a = \int_{\mathbb{R}} L^a_t(X_\cdot)g''_\ell(\mathrm{d}a),$$

the desired result.
Henceforth, the following types of Itô–Krylov's formula are used frequently throughout the paper.

Let a be a generator satisfying suitable conditions that guarantee the existence of a solution of the following SDE

$$X_t = x + \int_0^t a(s, X_s)\mathrm{d}s + \int_0^t \mathrm{f}(X_s)\sigma^2(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s.$$

Observe that $\int_0^t \mathrm{f}(X_s)\sigma^2(X_s)\,\mathrm{d}s = \int_0^t \mathrm{f}(X_s)\langle X\rangle_s\,\mathrm{d}s$ can be written using the local time $L^a_t(X_\cdot)$ of the semimartingale X as $\int_{\mathbb{R}} L^a_t(X_\cdot)\mathrm{f}(a)\mathrm{d}a$. Itô–Krylov's formula (11) applied to $\mathrm{F_f}(X_t)$ leads to

$$\mathrm{F_f}(X_t) = \mathrm{F_f}(x) + \int_0^t \left(\mathrm{F}'_\mathrm{f}(X_s)a(s, X_s) - \frac{1}{2}\mathrm{F}''_\mathrm{f}(X_s)\langle X\rangle_s\right)\mathrm{d}s + \int_0^t \mathrm{F}'_\mathrm{f}(X_s)\sigma(X_s)\mathrm{d}W_s.$$

In particular, if $a(s, x) = b(s, x) + \mathrm{f}(x)\sigma^2(x)$ and thanks to the ODE $\mathrm{F}''_\mathrm{f}(x) = 2\mathrm{f}(x)\mathrm{F}'_\mathrm{f}(x)$ we get

$$\mathrm{F_f}(X_t) = \mathrm{F_f}(x) + \int_0^t \mathrm{F}'_\mathrm{f}(X_s)b(s, X_s)\mathrm{d}s + \int_0^t \mathrm{F}'_\mathrm{f}(X_s)\sigma(X_s)\mathrm{d}W_s. \tag{13}$$

4.3. Existence and uniqueness of SDE with integrable generator

For each $0 \le s \le T$ we set $\overline{b}(s, x) = \mathrm{F}'_\mathrm{f}(\mathrm{F}^{-1}_\mathrm{f}(x))b(s, \mathrm{F}^{-1}_\mathrm{f}(x))$ and $\overline{\sigma}(x) = \mathrm{F}'_\mathrm{f}(\mathrm{F}^{-1}_\mathrm{f}(x))\sigma(\mathrm{F}^{-1}_\mathrm{f}(x))$. Remember that

$$\mathrm{F_f}(x) = \int_0^x \exp\left(2\int_0^y \mathrm{f}(t)\mathrm{d}t\right)\mathrm{d}y.$$

The main result of this section is given by Theorem 4.5 below. We shall refer the following simple equation

$$x_t = \mathrm{F_f}(x) + \int_0^t \overline{b}(s, x_s)\,\mathrm{d}s + \int_0^t \overline{\sigma}(x_s)\,\mathrm{d}W_s,$$

as $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$.

Theorem 4.5. *The equation* $\mathrm{Eq}(x, b + \mathrm{f}\sigma^2, \sigma)$ *has a unique strong solution if and only if* $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$ *has a unique strong solution.*

Proof: If $(X_t)_{0\le t\le T}$is a solution to $\mathrm{Eq}(x, b + \mathrm{f}\sigma^2, \sigma)$, then Itô–Krylov's formula (11) applied to $\mathrm{F_f}(X_t)$ leads to

$$\mathrm{F_f}(X_t) = \mathrm{F_f}(x) + \int_0^t \mathrm{F}'_\mathrm{f}(X_s)b(s, X_s)\,\mathrm{d}s + \int_0^t \mathrm{F}'_\mathrm{f}(X_s)\sigma(X_s)\,\mathrm{d}W_s.$$

Now, let us set $x_t = \mathrm{F_f}(X_t)$ with the notation $\overline{b}(s, \mathrm{F_f}(x)) := \mathrm{F}'_\mathrm{f}(x)b(s, x)$ and $\overline{\sigma}(\mathrm{F_f}(x)) := \mathrm{F}'_\mathrm{f}(x)\sigma(x)$. That is $x_\cdot$ is a solution to the equation $\mathrm{Eq}(\mathrm{F_f}(x), \overline{b}, \overline{\sigma})$. These notations will be used repeatedly along this paper. That is $(x_t)_{0\le t\le T}$ satisfies Eq$(\mathrm{F_f}(x),\overline{b},\overline{\sigma})$. Moreover, if $(X_t)_{0\le t\le T}$ is a solution of Eq$(x, b+\mathrm{f}\sigma^2,\sigma)$, in the sense of Definition 2.1 then from the inequality (4) we have

$$\int_0^T \left|\overline{b}\,(s,x_s)\right|^2 \mathrm{d}s = \int_0^T \left|\mathrm{F}'_{\mathrm{f}}(X_s)b\,(s,X_s)\right|^2 \mathrm{d}s \le e^{4|\mathrm{f}|_1}\int_0^T |b\,(s,X_s)|^2\,\mathrm{d}s.$$

similarly we have

$$\int_0^T \left|\mathrm{F}'_{\mathrm{f}}(X_s)\sigma\,(X_s)\right|^2 \mathrm{d}s \le e^{4|\mathrm{f}|_1}\int_0^T |\sigma\,(X_s)|^2\,\mathrm{d}s.$$

Consequently $x_. = \mathrm{F_f}(X_.) \in \mathcal{S}^2$, $\overline{b}(\cdot,x_.) = \mathrm{F}'_{\mathrm{f}}(X_.)b\,(s,X_.) \in \mathcal{M}^2_W$ and $\overline{\sigma}(x_.) = \mathrm{F}'_{\mathrm{f}}(X_.)\sigma(X_.) \in \mathcal{M}^2_W$ that is $(x_t)_{0\le t\le T}$, is a solution to Eq$(\mathrm{F_f}(x),\overline{b},\overline{\sigma})$ in the sense of the Definition 2.1.

Conversely: Let $(x_t)_{0\le t\le T}$ is a solution to Eq$(\mathrm{F_f}(x),\overline{b},\overline{\sigma})$, then Itô–Krylov's formula (11) applied to $X_t = \mathrm{F}_{\mathrm{f}}^{-1}(x_t)$ (since $\mathrm{F}_{\mathrm{f}}^{-1}$ belongs to $\mathcal{W}^2_{1,loc}(\mathbb{R})$) shows that

$$\mathrm{F}_{\mathrm{f}}^{-1}(x_t) = x + \int_0^t (\mathrm{F_f}^{-1})'(x_s)\mathrm{d}x_s + \frac{1}{2}\int_0^t (\mathrm{F}_{\mathrm{f}}^{-1})''(x_s)\overline{\sigma}^2\,(x_s)\,\mathrm{d}s,$$

then

$$X_t \;=\; x + \int_0^t (\mathrm{F_f}^{-1})'(x_s)\overline{b}(s,x_s)\mathrm{d}s + \frac{1}{2}\int_0^t (\mathrm{F}_{\mathrm{f}}^{-1})''(x_s)\overline{\sigma}^2\,(x_s)\,\mathrm{d}s + \int_0^t (\mathrm{F}_{\mathrm{f}}^{-1})'(x_s)\overline{\sigma}\,(x_s)\,\mathrm{d}W_s. \tag{14}$$

Notice that

$$(\mathrm{F}_{\mathrm{f}}^{-1})'(x) = \frac{1}{\mathrm{F_f}'(\mathrm{F}_{\mathrm{f}}^{-1}(x))} \text{ and } (\mathrm{F}_{\mathrm{f}}^{-1})''(x) = -\frac{\mathrm{F}''_{\mathrm{f}}(\mathrm{F_f}^{-1}(x))}{(\mathrm{F}'_{\mathrm{f}}(\mathrm{F_f}^{-1}(x)))^2}(\mathrm{F}_{\mathrm{f}}^{-1})'(x) = -\frac{\mathrm{F}''_{\mathrm{f}}(\mathrm{F}_{\mathrm{f}}^{-1}(x))}{(\mathrm{F}'_{\mathrm{f}}(\mathrm{F}_{\mathrm{f}}^{-1}(x)))^3}.$$

Set $\sigma\,(X_s) = (\mathrm{F}_{\mathrm{f}}^{-1})'(x_s)\overline{\sigma}\,(x_s)$, this implies that

$$\begin{aligned}(\mathrm{F}_{\mathrm{f}}^{-1})''(x_s)\overline{\sigma}^2\,(x_s) \;&=\; -\frac{1}{2}\frac{\mathrm{F_f}''(X_s)}{(\mathrm{F}'_{\mathrm{f}}(X_s))^3}\frac{\sigma^2\,(X_s)}{((\mathrm{F_f}^{-1})'(x_s))^2} \overset{\mathrm{d}s\text{ a.e.}}{=} \frac{1}{2}\frac{\mathrm{F}''_{\mathrm{f}}(X_s)}{\mathrm{F}'_{\mathrm{f}}(X_s)}\sigma^2\,(X_s) \\ &=\; \mathrm{f}(X_s)\sigma^2\,(X_s)\,.\end{aligned} \tag{15}$$

Substituting (15) in (14) and putting $b(s,X_s) = (\mathrm{F_f}^{-1})'(x_s)\overline{b}(s,x_s)$ we end up with

$$X_t = x + \int_0^t \left(b(s,X_s) + \mathrm{f}(X_s)\sigma^2\,(X_s)\right)\mathrm{d}s + \int_0^t \sigma\,(X_s)\,\mathrm{d}W_s.$$

It is easy to show thanks to the ineuqlities (4) and (5) that

$$|X_s| \le e^{2|\mathrm{f}|_1}\,|x_s| \text{ and } |\sigma\,(X_s)| \le e^{2|\mathrm{f}|_1}\,|x_s|\,.$$

Consequently $(X_s)_{0\le s\le T} = (\mathrm{F}_{\mathrm{f}}^{-1}(x_s))_{0\le s\le T}$ is a solution to Eq$(x, b+\mathrm{f}\sigma^2,\sigma)$, in the sense of Definition 2.1.

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